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DELIGNE-KATZ CORRESPONDENCE FOR OVERCONVERGENT ISOCRYSTALS

MASTER THESIS IN MATHEMATICS

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Prof. Dr. Phung Ho Hai

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Declaration

I declare that this thesis titled "Deligne-Katz correspondence for overconvergent isocrystals" is entirely my own work and has not been previously included in a thesis or dissertation submitted for a degree or any other qualification in this graduate university or any other institutions. I will take responsibility for the above declaration.

Hanoi, 10th October 2022

Signature of Student

Nguyen Khanh Hung

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Contents

Introduction

It is widely accepted that there are two different perspectives to study differential equations in general. The first and more widespread one is more analytic by directly solving the equations and studying some properties of their solutions such as their convergence or asymtotics. The second one, which studies properties and structure of the equations themselves, focuses on linear differential equations with polynomial coefficients. This direction, which has been studied for several decades by various mathematicians (Fuchs, Levelt, Turrittin, Katz, Deligne, ...), can be divided into several equivalent viewpoints: algebraic differential equations, differential modules (or $\mathcal{D}$-modules), modules with connection,.... Moreover, algebraic differential equations can be interpreted as local systems in complex geometry, locally constant sheaves in topology, lisse sheaves and $\mathbb{Q}_\ell$-local systems in étale cohomology,....

This dissertation's background focuses on the p-adic analogue of algebraic differential equations. Although p-adic differential equations were studied from the view point of p-adic analysis by Dwork, Robba, ... in 1960s-1980s, a breakthrough occured when they were formalized as crystals. This notion was firstly suggested by Grothendieck [1] and Berthelot [2] to construct a p-adic Weil cohomology theory, called the crystalline cohomology. Two decades later, Berthelot [3],[4] introduced the notion of overconvergent isocrystals for linking to solvable p-adic differential equations. In recent times, modern approaches have been applied in the literature of p-adic differential equations have been studied from several perspectives, for instance Tannakian formalism by André, p-adic analysis by Christol-Mebkhout, Berkovich's analytic spaces by Baldassarri-Pulita-Poineau,.... However, this thesis is faithful to the notions of p-adic geometry suggested by Berthelot.

The research topic of this dissertation is Deligne-Katz correspondence of differential modules in different settings. Specifically, Katz [5] established an interesting equivalence between the category of modules with regular connection at 0 over the affine line minus the origin and the category of modules with connection over the formal neighborhood of ∞. As a restriction of Katz correspondence, Deligne [6] established an equivalence between the category of modules with regular connection at both 0 and ∞ over the affine line minus the origin and the category of modules with regular connection at 0 over the formal neighborhood of 0. The main target of this thesis is to study the p-adic analogue of Deligne-Katz correspondence in the paper [7] of Matsuda and give some directions to extend this result.

This thesis is divided into three chapters.

1. Chapter 1 is an introduction of the theory of differential modules and modules with connection. We consider two important results including Turrittin-Levelt-Jordan decomposition and Deligne-Katz correspondence for differential modules in characteristic zero. Main references of this chapter is the book [8] and Katz's paper [5].

2. Chapter 2 is an overview of rigid geometry, which provides important notions and results for the next chapter. Although rigid geometry has been developed over several decades by perspectives of Tate curves, Raynaud's generic fiber, Berkovich's analytic spaces and Huber's adic spaces, this thesis only focuses on the first two viewpoints. The main reference of this chapter is the book [9].

3. In Chapter 3, we introduce the concept of overconvergent isocrystals and study the p-adic analogue of Deligne-Katz correspondence constructed by Matsuda. At the end of this chapter, we suggest some ideas to extend this result. The main reference of this chapter is Matsuda's paper [7].

For the reader's convenience, we suggest some additional references: [10] for basic notions and results in algebraic geometry, [11] for Galois descent, [12] for local fields, [13] and [14] for étale morphisms and étale fundamental groups, [15] for overconvergent isocrystals.

Chapter 1

Theory of differential modules

1.1 Kähler differentials

Let $A \to B$ be a homomorphism of commutative rings. For any B-module M, the A-module $\mathrm{End}_A(M)$ of A-linear endomorphisms of M is a B-bimodule. Specifically, for any $f \in \mathrm{End}_R(M)$, elements b, b' of B, m of M,

(i) the left B-module structure is given by $(bf)(m) = bf(m)$, and

(ii) the right B-module structure is given by $(fb')(m) = f(b'm)$.

Hence $\mathrm{End}_A(M)$ is endowed with a natural Lie algebra structure

$$[f, g] = f \circ g - g \circ f.$$

Definition 1.1.1. An A-**derivation** of B with values in a B-module M is defined as an A-linear map $d : B \to M$ satisfying the following conditions:

(i) $da = 0$ for any element a of A,

(ii) $d(b + b') = ds + ds'$ for elements b, b' of B,

(iii) $d(bb') = bdb' + b'db$ for $b.b'$ of B.

We denote by $\mathrm{Der}_{B/A} \subseteq \mathrm{End}_A(B)$ the left B-submodule and A-Lie subalgebra of A-derivation of B; similarly, $\mathrm{Der}_A(B, M) \subseteq \mathrm{Hom}_A(B, M)$ the left B-submodule of A-derivations of B with values in M.

Definition 1.1.2. The **module of differentials** (or **module of Kähler differentials**), denoted by $\Omega_{B/A}$, is a B-module equipped with an A-derivation $d : B \to \Omega_{B/A}$ satisfying the universal property: for any B-module M and an A-derivation $d' : B \to M$, there exists uniquely a homomorphism of B-modules $f : \Omega_{B/A} \to M$ making the diagram commutes:

$$\begin{tikzcd} B \arrow[r, "d"] \arrow[dr, "d'"'] & \Omega_{B/A} \arrow[d, "f"] \\ & M \end{tikzcd}$$

In other words, the derivation d induces a canonical and functorial identification

$$\mathrm{Hom}_B(\Omega_{B/A}, M) \cong \mathrm{Der}_A(B, M).$$

The existence of the module of differentials is clearly seen by its following direct construction: let F be the free B-module generated by symbols $\{db, b \in B\}$ and E be the quotient module of F by the submodule generated by the following elements:

(i) da for any element a of A,

(ii) $d(b+b') - db - db'$ for elements b, b' of B,

(iii) $d(bb') - bdb' - b'db$ for elements b, b' of B.

The following result is also a well-known and useful construction of module of differentials:

Proposition 1.1.3. [16, Proposition 6.1.3] *With the above notions, B is an A-algebra and we consider the diagonal homomorphism*

$$f : B \otimes_A B \to B, \qquad b \otimes b' \mapsto bb'$$

for elements b, b' of B. We denote the ideal $I = \mathrm{Ker}\, f$ of $B \otimes B$, then I/I^2 is also a B-module. Moreover, I/I^2 becomes a module of differentials of B/A by equipping the derivation

$$d : B \to I/I^2, \qquad b \mapsto db = 1 \otimes b - b \otimes 1 \mod I^2.$$

Example 1.1.4. Let B be the polynomial ring $B[x_1, \ldots x_n]$. Then $\Omega_{B/A}$ is the free B-module of rank n with a basis $\{dx_1 \ldots dx_n\}$.

1.2 Differential rings and differential modules

Let K be a field of characteristic zero.

Definition 1.2.1. A **differential ring** (F, ∂) over K is defined as a commutative K-algebra F equipped with a K-derivation $\partial \in \mathrm{Der}_{F/K}$. A **morphism** of differential rings $f : (F, \partial) \to (F', \partial')$ is defined as a K-algebra homomorphism which commutes with the derivations, that is $\partial' f = f \partial$.

Example 1.2.2. The following examples are the most useful, which are often rings of one-variable functions, which are equipped with the derivations $\partial_t = \frac{\partial}{\partial t}$ or $\vartheta_t = t\frac{\partial}{\partial t}$.

(i) the field of rational functions $K(t)$,

(ii) the ring $K[[t]]$ of formal power series or its fraction field $K((t))$,

(iii) for $K = \mathbb{C}$, the ring $\mathbb{C}\{t\}$ of convergent power series or its fraction field of meromorphic power series,

(iv) for $K = \mathbb{Q}_p$ or another p-adic field, the ring $K\{t\}$ of convergent power series or its fraction field $K(\{t\})$.

Definition 1.2.3. A **differential module** (M, ∇_∂) over (F, ∂) is defined as a projective F-module M of finite rank, endowed with a K-linear endomorphism ∇_∂ which satisfies the Leibniz rule:

$$\nabla_\partial(fm) = \partial(f)m + f\nabla_\partial(m),$$

for elements f of F,m of M. A **morphism** $(M, \nabla_\partial) \to (M', \nabla'_\partial)$ of differential modules over (F, ∂) is a F-linear morphism $M \to M'$ making the following diagram commutative:

$$\begin{tikzcd} M \arrow[r] \arrow[d, "\nabla_\partial"'] & M' \arrow[d, "\nabla'_\partial"] \\ M \arrow[r] & M' \end{tikzcd}$$

The kernel of ∇_∂ is a K-subspace of M, denoted by M^{∇_∂} and called the **module of horizontal elements**.

Remark 1.2.4. The rank of a differential module is the rank r of the underlying module, which is uniquely determined if the base differential ring has no zero-divisor.

For a differential extension (F', ∂') over (F, ∂) of differential rings over K and a differential module (M, ∇_∂) over (F, ∂), we can construct an extension of scalars $(M_{F'}, \nabla_{\partial'})$ over (F', ∂) of (M, ∇_∂) by setting $M_{F'} = M \otimes_F F'$. Its derivation is uniquely determined by Leibniz rule:

$$\nabla_{\partial'}(m \otimes f') := m \otimes \partial' f' + \nabla_\partial(m) \otimes f'$$

for elements f' of F', m of M.

Remark 1.2.5. In the classical setting of differential equations, we consider a linear differential equation of order r

$$\partial^r x + a_{r-1}\partial^{r-1}x + \cdots + a_1\partial x + a_0 x = 0$$

with elements a_i of F. It is clearly seen that the set of solutions forms a K-vector space of dimension not greater than r after extending the differential ring F to some F'. Indeed, the elements $x_1, x_2, \ldots, x_n \in F'$ are linearly independent if and only if its wronskian determinant is nonzero. If the dimension of solution space is exactly n, a basis of solutions is called the **fundamental system of solutions**.

The above differential equation can be converted into a system of linear differential equations of order one

$$\partial \mathbf{x} = G\mathbf{x},$$

where

$$\mathbf{x} = \begin{pmatrix} x \\ \partial x \\ \vdots \\ \partial^{r-1} x \end{pmatrix} \text{ and } G = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{r-1} \end{pmatrix}.$$

For some P invertible matrix of rank n with coefficients in F, $\mathbf{x}$ is a solution of this differential system if and only if $P\mathbf{x}$ is a solution of an "equivalent" system:

$$\partial \mathbf{x} = G_P \mathbf{x},$$

where $G_P = (\partial P)P^{-1} + PGP^{-1}$.

Definition 1.2.6. A differential module is called **trivial** if it is isomorphic to the direct sum of finitely many copies of (F, ∂).

Example 1.2.7. Differential modules over the differential ring $(K[[t]], \partial_t)$, $(\mathbb{C}\{t\}, \partial_t)$ or $(\mathbb{Q}_p\{t\}, \partial_t)$ are trivial. We will check that any differential module (M, ∇_{∂_t}) over $(K[[t]], \partial_t)$ are trivial. By contradiction, if M has some torsion elements, i.e. there exists $f \in K[[t]]$, $m \in M$ such that $fm = 0$. Let $\{m_1, \ldots, m_r\}$ be a generating set of M whose images in M/tM are linearly independent over K. Representing m linearly in terms of this set, we obtain a relation

$$f_1 m_1 + \ldots + f_r m_r = 0.$$

Because $m_1, \ldots, m_r$ are linearly independent modulo tM, $f_1, \ldots, f_r$ are elements of tM. This allows us to denote:

$$n := \min\{\mathrm{ord}_t(f_1), \ldots, \mathrm{ord}_t(f_r)\}$$

such that $n \geqslant 1$. Using Leibniz rule we obtain

$$\sum_{i=1}^{r} \partial_t(f_i)m_i + \sum_{i=1}^{r} f_i \nabla_{\partial_t}(m_i) = 0.$$

Let G be the matrix of size r with coefficients in $K[[t]]$ satisfying

$$\nabla_{\partial_t}((m_1, \ldots, m_r)) = G(m_1, \ldots, m_r),$$

we obtain a relation:

$$\sum_{i=1}^{r} g_i m_i = 0$$

such that

$$\min\{\mathrm{ord}_t(g_1), \ldots, \mathrm{ord}_t(g_r)\} = n - 1.$$

By repeating this step, we obtain a relation with $n = 0$, contradicting to the assumption that $m_1, \ldots, m_r$ are linearly independent modulo tM.

Example 1.2.8. In contrast to the previous example, differential modules over $(K((t)), \partial_t)$, $(\mathbb{C}(\{t\}), \partial_t)$ or $(\mathbb{Q}_p(\{t\}), \partial_t)$ are non-trivial, which will be discussed in next sections.

1.3 Modules with connections on algebraic varieties

This section is an introduction of differential modules and modules of differentials in terms of the geometric context of schemes.

Differential invariants

Let $f : X \to S$ be a morphism of schemes and $\Delta : X \to X \times_S X$ be the corresponding diagonal morphism, which is an immersion. The product $X \times_S X$ is equipped with two projections:

$$X \xrightarrow{\Delta} X \times_S X \underset{p_2}{\overset{p_1}{\rightrightarrows}} X,$$

which endow $\mathcal{O}_{X \times_S X} = \mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_X$ with two $\mathcal{O}_X$-module structures, given by

$$p_1^* = x \otimes 1, p_2^* = 1 \otimes x$$

for some section x of $\mathcal{O}_X$. The $\mathcal{O}_X$-structure induced by p_1^* is called the left structure, the $\mathcal{O}_X$-structure induced by p_2^* is called the right structure.

As usual, we denote by $\mathcal{I}$ the ideal sheaf of the diagonal Δ in $X \times_S X$. The **sheaf of principal parts of order** n on X/S is defined as the sheaf of rings

$$\mathcal{P}^n_{X/S} := \Delta^{-1}(\mathcal{O}_{X\times_S X}/\mathcal{I}^{n+1})$$

on X. For simplicity, $\mathcal{P}^n_{X/S}$ is regarded as an $\mathcal{O}_X$-module via the left structure and hence it is coherent as an $\mathcal{O}_X$-module. The n**th jet map** of X/S is defined as the right $\mathcal{O}_X$-linear map

$$j^n_{X/S} : \mathcal{O}_X \to \mathcal{P}^n_{X/S},$$

induced by p_2^*. The sheaves $\mathcal{P}^n_{X/S}$ for $n = 0, 1, \ldots$ form an inverse system via natural morphisms

$$\mathcal{P}^n_{X/S} \to \mathcal{P}^m_{X/S}, n \geqslant m.$$

Similarly as the affine case in Definition 1.1.2, the **sheaf of relative differentials** on X/S is defined by the kernel

$$\Omega^1_{X/S} := \mathcal{I}/\mathcal{I}^2$$

of the projection $p_{X/S} : \mathcal{P}^1_{X/S} \to \mathcal{P}^0_{X/S} = \mathcal{O}_X$. The left and right $\mathcal{O}_X$-structures coincide on $\Omega^1_{X/S}$. Moreover, we have $p_{X/S} \circ (p_2^* - p_1^*) = p_{X/S} \circ (j^1_{X/S} - p_1^*) = 0$, so that $j^1_{X/S} - p_1^* :$ $\mathcal{O}_X \to \mathcal{P}^1_{X/S}$ factors through an $\mathcal{O}_S$-linear map

$$d_{X/S} : \mathcal{O}_X \to \Omega^1_{X/S},$$

which is a derivation of $\mathcal{O}_X$ with values in $\Omega^1_{X/S}$.

By taking exterior products, we can construct for $n \in \mathbb{N}$ the $\mathcal{O}_X$**-module of relative differential** n**-forms** $\Omega^n_{X/S} = \bigwedge^n \Omega^1_{X/S}$.

The **relative tangent sheaf** $\mathcal{T}_{X/S}$ is defined as the dual $\mathcal{O}_X$-module of $\Omega^1_{X/S}$. Equivalently, it is the sheaf of $\mathcal{O}_S$-linear derivations of $\mathcal{O}_X$, that is $\mathcal{T}_{X/S} = (\Omega^1_{X/S})^\vee = \mathcal{D}er_{X/S}$.

We define **the sheaf of differential operators of order** $\leqslant n$ of X/S, denoting $\mathcal{D}iff^n_{X/S}$, by the subsheaf of $\mathcal{E}nd_{\mathcal{O}_S}(\mathcal{O}_X)$ whose sections locally factor through the $\mathcal{O}_X$-linear morphism $\bar{h} : \mathcal{P}^n_{X/S} \to \mathcal{O}_X$ as

$$\begin{tikzcd} \mathcal{O}_X \arrow[r, "h"] \arrow[d, "j^n_{X/S}"'] & \mathcal{O}_X \\ \mathcal{P}^n_{X/S} \arrow[ur, "\bar{h}"'] & \end{tikzcd}$$

Consequently, we obtain the isomorphism

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}^n_{X/S}, \mathcal{O}_X) \xrightarrow{\sim} \mathcal{D}iff^n_{X/S}, \qquad \bar{h} \mapsto \bar{h} \circ j^n_{X/S}$$

where the right structure of $\mathcal{P}^n_{X/S}$ induces on its $\mathcal{O}_X$-dual the right $\mathcal{O}_X$-module structure.

More generally, we can define the sheaf of differential operators of order $\leqslant n$ from an $\mathcal{O}_X$-module $\mathcal{F}$ to another $\mathcal{O}_X$-module $\mathcal{G}$, denoting $\mathcal{D}iff^n_{X/S}(\mathcal{F}, \mathcal{G})$, as the $\mathcal{O}_X$-module $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}^n_{X/S} \otimes_{\mathcal{O}_X} \mathcal{F}, \mathcal{G})$. Any differential operator can be identified with a $f^{-1}\mathcal{O}_S$-linear morphism by composition with the canonical map $j^n_{X/S}$.

Remark 1.3.1.

(i) If we assume that $f : X \to S$ is smooth, we can choose local étale coordinates of X, denoted by $\{x_1, \ldots, x_n\}$. Then as in Example 1.1.4, $\Omega^1_{X/S}$ is free generated by $dx_1, \ldots, dx_n$; $\mathcal{D}er_{X/S}$ is free generated by duals $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ and $\mathcal{D}iff^n_{X/S}$ is constructed inductively from $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$.

(ii) Some examples of differential operators:

(a) A differential operator of negative order is the zero morphism.

(b) A differential operator of order 0 is a morphism $D : \mathcal{F} \to \mathcal{G}$ of sheaves of $\mathcal{O}_X$-modules.

(c) For $\mathcal{F} = \mathcal{O}_X$, a differential operator $D : \mathcal{O}_X \to \mathcal{G}$ of order $\leqslant 1$ can be written as $v + d$, where $v : \mathcal{O}_X \to \mathcal{G}$ is a homomorphism of $\mathcal{O}_X$-modules; $d : \mathcal{O}_X \to \mathcal{G}$ is an S-derivation.

Connections

We consider the case that $f : X \to S$ is a smooth morphism, then $\Omega^1_{X/S}$ is a locally free sheaf of $\mathcal{O}_X$-modules.

Definition 1.3.2. For an $\mathcal{O}_X$-module $\mathcal{M}$, a **connection on $\mathcal{M}$ relative to** S is defined by one of the equivalent conditions:

(i) A left $\mathcal{O}_X$-linear map:

$$\begin{array}{cccc} \nabla & : & \mathcal{D}er_{X/S} & \to & \mathcal{E}nd(\mathcal{M}) \\ & & \partial & \mapsto & \nabla_\partial \end{array}$$

such that for every open subset U of X, $\partial \in \mathcal{D}er_{X/S}(U)$, $(\mathcal{M}(U), \nabla_\partial)$ is a differential module over the differential ring $(\mathcal{O}_X(U), \partial)$ satisfying the Leibniz's rule:

$$\nabla_\partial(ax) = \partial(a)x + a\nabla_\partial(x)$$

for a and x are sections of $\mathcal{O}_X(U)$ and $\mathcal{M}(U)$, respectively.

(ii) (Koszul's definition) An $\mathcal{O}_S$-linear map:

$$\nabla : \mathcal{M} \to \Omega^1_{X/S} \otimes_{\mathcal{O}_X} \mathcal{M}$$

satisfying the Leibniz's rule

$$\nabla(ax) = d_{X/S}(a) \otimes x + a\nabla(x)$$

for a and x are sections of $\mathcal{O}_X(U)$ and $\mathcal{M}(U)$ on some open subset U of X, respectively.

((iii) (Atiyah's definition) An $\mathcal{O}_X$-linear section

$$\Theta : \mathcal{M} \to \mathcal{P}^1_{X/S} \otimes_{\mathcal{O}_X} \mathcal{M}$$

of the projection $p : \mathcal{P}^1_{X/S} \otimes_{\mathcal{O}_X} \mathcal{M} \to \mathcal{M}$.

(iv) (Grothendieck's definition) An isomorphism of $\mathcal{P}^1_{X/S}$-modules

$$\varepsilon : \mathcal{M} \otimes_{\mathcal{O}_S} \mathcal{P}^1_{X/S} \to \mathcal{P}^1_{X/S} \otimes_{\mathcal{O}_S} \mathcal{M}$$

which modulo the kernel $\Omega^1_{X/S}$ of $\mathcal{P}^1_{X/S} \to \mathcal{O}_X$, reduces to the identity endomorphism of $\mathcal{M}$.

In this thesis, we only use the equivalence between (i) and (ii), therefore we check it here. For the equivalence of them with other two definitions, see the discussion after [8, Definition 4.2.1].

On the one hand, to prove (i)⇒(ii), we have to check via local étale coordinates. Specifically, we pick étale coordinates $(x_1, \ldots, x_n)$ on U/S for some Zariski open subset U of X. By Example 1.1.4, $\Omega^1_{U/S}$ is free generated by $dx_1, \ldots, dx_n$ and $\mathcal{D}er_{X/S}(U)$ is free on the dual basis $\partial_1, \ldots, \partial_n$ with $\partial_i = \partial/\partial x_i$. For such a map ∇_∂, we can define

$$\begin{aligned} \nabla : \mathcal{M} &\to \Omega^1_{X/S} \otimes_{\mathcal{O}_X} \mathcal{M} \\ m &\mapsto \sum_{i=1}^{n} dx_i \otimes \nabla_{\partial_i}(m) \end{aligned}$$

for any section m of $\mathcal{M}|_U$. This definition is clearly compatible with glueing open subsets of X, hence ∇ is defined globally. Since ∇_∂ satisfies Leibniz property, it also holds for ∇.

On the other hand, the implication (ii)$\Rightarrow$(i) is obvious when for each derivation ∂ on X/S, we take ∇_∂ by contraction:

$$\nabla_\partial = (\partial \otimes_{\mathcal{O}_X} \mathrm{id}_{\mathcal{M}}) \circ \nabla.$$

Since ∇ satisfies Leibniz property, it also holds for ∇_∂.

Remark 1.3.3. The **trivial connection** on $\mathcal{O}_X$ is given by the canonical derivation $d_{X/S} : \mathcal{O}_X \to \Omega^1_{X/S}$.

A connection ∇ can be extended to a homomorphism of sheaves of modules

$$\nabla_n : \Omega^n_{X/S} \otimes \mathcal{M} \to \Omega^{n+1}_{X/S} \otimes \mathcal{M}$$

by setting

$$\nabla_n(\omega \otimes m) = (-1)^n \omega \wedge \nabla(m) + d\omega \otimes m$$

where ω and m are sections of $\Omega^n_{X/S}(U)$ and $\mathcal{M}(U)$ respectively for some open subset U of X, and where $\omega \wedge \nabla(m)$ denotes the image of $\omega \times \nabla(m)$ under the canonical map

$$\Omega^n_{X/S} \otimes_{\mathcal{O}_X} \left(\Omega^1_{X/S} \otimes_{\mathcal{O}_X} \mathcal{M}\right) \to \Omega^{n+1}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{M}$$

which sends $\omega \otimes \tau \otimes m$ to $(\omega \wedge \tau) \otimes m$.

In general, although the fact that $\nabla_{n+1} \circ \nabla_n = 0$ is not satisfied, the following morphism is $\mathcal{O}_X$-linear:

$$K = \nabla_1 \circ \nabla : \mathcal{M} \to \Omega^2_{X/S} \otimes_{\mathcal{O}_X} \mathcal{M}.$$

We can easily verify that

$$(\nabla_{n+1} \circ \nabla_n)(\omega \otimes m) = \omega \otimes K(m)$$

where ω and m are sections of $\Omega^n_{X/S}(U)$ and $\mathcal{M}(U)$ for some open subset U of X.

The morphism K is called the **curvature** of ∇. In terms of local étale coordinates $\{x_1, \ldots x_n\}$, K can be expressed as follows. Let $\partial/\partial x_i \in \mathcal{D}er_{X/S}$ maps to $\vartheta_i \in \mathcal{D}i\!f\!f^1_{X/S}(\mathcal{M}, \mathcal{M})$ by ∇. Then the map $K : \mathcal{M} \to \Omega^2_{X/S} \otimes \mathcal{M}$ is given by

$$K = \sum_{i<j} dx_i \wedge dx_j \otimes [\vartheta_i, \vartheta_j].$$

In particular $K = 0$ if and only if the ϑ_i's mutually commute.

Definition 1.3.4. A connection whose curvature is zero is called **integrable** or **flat**.

Remark 1.3.5. For an integrable connection, we can construct the relative de Rham complex of $(\mathcal{M}, \nabla)$:

$$\mathcal{M} \to \Omega^1_{X/S} \otimes_{\mathcal{O}_X} \mathcal{M} \to \Omega^2_{X/S} \otimes_{\mathcal{O}_X} \mathcal{M} \to \cdots$$

1.4 Regular singularities

Let $\mathbb{C}\{\{t\}\}$ be the differential field of formal Laurent series whose 0 is the only isolated singularity, then the field $\mathbb{C}(\{t\})$ of meromorphic functions at 0 is actually a differential subfield of $\mathbb{C}\{\{t\}\}$. In this section we focus on the following system of linear differential equations:

$$\partial_t \mathbf{x} = G(t)\mathbf{x} \tag{1.4.1}$$

where $G(t) \in \mathrm{Mat}_r(\mathbb{C}\{\{t\}\})$ and $\mathbf{x}$ is a column vector of size r. Without losing the generalization, we can assume that 0 is also the isolated singularity of $G(t)$ in some neighborhood of 0.

Let z_0 be a point in the punctured unit disk and $\mathcal{O}_{z_0}$ the ring of germs of holomorphic functions at z_0. By Cauchy's theorem, as stated in Theorem 6.1.3,[8], the system admits a **fundamental solution matrix** at z_0, denoted by $X \in \mathrm{GL}_r(\mathcal{O}_{z_0})$. With initial condition $X(z_0) = I_r$, this matrix is uniquely determined.

By analytic continuation along a loop around 0:

$$T : \mathcal{O}_{z_0} \to \mathcal{O}_{z_0},$$

we have $TX_{z_0} = X_{z_0}C$, where $C \in \mathrm{GL}_r(\mathbb{C})$ depends on the chosen loop. This construction defines naturally a group anti-homomorphism from the fundamental group of the punctured disk to the group of invertible complex matrices of size r, called the **monodromy representation**.

Let A be a matrix in $\mathrm{Mat}_r(\mathbb{C})$, t^A the matrix

$$t^A = \exp{(A \ln t)} := \sum_{n=0}^{\infty} \frac{(A \ln t)^n}{n!},$$

for some fixed branch of $\ln t$ at z_0. Then

$$T(t^A) = \exp{(A \ln t + 2\pi i A)} = t^A \exp{(2\pi i A)}.$$

If A satisfies $\exp(2\pi i A) = C$ then

$$T(X_{z_0}t^{-A}) = X_{z_0}CC^{-1}t^{-A} = X_{z_0}t^{-A},$$

hence the following solution matrix is analytic in the punctured unit disk:

$$Z := X_{z_0}t^{-A}$$

The result that we have just proven is called the **complex monodromy theorem**.

Definition 1.4.1. Instead of considering the system (1.4.1) over $\mathbb{C}\{\{t\}\}$, we assume that it is determined over the differential subfield $\mathbb{C}(\{t\})$ (the field of germs of meromorphic functions at 0). In this situation, we can naturally divide into two cases:

(i) 0 is called a **regular singularity** if Z has at most a pole at 0, i.e. has coefficients in $\mathbb{C}(\{t\})$, or

(ii) 0 is called an **irregular singularity** if Z has an essential singularity at 0.

It is remarkable that these notions are dependent only on the $\mathbb{C}(\{t\})$-differential module associated to the system 1.4.1.

Turning to the formal setting, i.e. dealing with $\mathbb{C}((x))$-differential modules, the notion of regularity also makes sense. We will state this notion and some results in Fuchs-Frobenius theory.

By the construction of cyclic vectors, which is omitted in this text, the system (1.4.1) is equivalent to a scalar equation $Lx = 0$, where

$$L = \partial_t^r + a_{r-1}\partial_t^{r-1} + \cdots + a_1\partial_t + a_0$$

is a linear differential operator of order r, where for any i, a_i is a holomorphic function in some punctured disk and has at most a pole at 0.

Definition 1.4.2. L is said to satisfy the **Fuchs condition** at 0, or 0 is a **regular singularity** for L if

$$\mathrm{ord}_0(a_i) \geqslant i - r \text{ for } i = 0, \dots, r-1.$$

Theorem 1.4.3 (Frobenius-Fuchs, [8, Theorem 6.3.5])**.** *For the field $F = \mathbb{C}(\{t\})$ of germs of meromorphic functions on $\mathbb{C}$ at $t = 0$, the differential system (1.4.1) with coefficients in F is regular at 0 if and only if the differential operator obtained from (1.4.1), by application of cyclic vectors, satisfies the Fuchs condition at 0.*

Frobenius-Fuchs theory in the base field $\mathbb{C}$ was formalized to an arbitrary characteristic zero field by Fuchs as follows. For a field K of characteristic 0 and an algebraic closure $\overline{K}$ of K, we consider the differential field $F = K((t))$ of formal Laurent series endowed with usual derivations

$$\partial = \partial_t = \frac{d}{dt} \text{ or } \vartheta_t = t\frac{d}{dt}.$$

We consider a matrix A in $\mathrm{Mat}_r(K)$ and its associated differential system:

$$\vartheta_t \mathbf{x} = A\mathbf{x} \tag{1.4.2}$$

The solution matrix t^A in the complex case can be formalized as follows:

(i) if $A = \Delta = \mathrm{diag}(\Delta_1, \ldots, \Delta_\mu)$ is diagonal, then $t^\Delta = \mathrm{diag}(t^{\Delta_1}, \ldots, t^{\Delta_\mu})$;

(ii) if $A = N$ is nilpotent, then $t^N = \exp(N \ln t) = \sum_{k=0}^{\infty} \frac{N^k}{k!}(\ln t)^k$;

(iii) if $A = P(\Delta + N)P^{-1}$ give the Jordan canonical form (with $P \in \mathrm{GL}_r(\overline{K})$), then $t^A = Pt^\Delta t^N P^{-1}$.

Instead of dealing with system of constant coefficients (1.4.2), it is necessary to consider the general differential system

$$\vartheta_t \mathbf{x} = G\mathbf{x} \text{ with } G \in \mathrm{Mat}_r(K((t))). \tag{1.4.3}$$

We recall that, for a change of variables

$$\mathbf{x} \mapsto P\mathbf{x}$$

with $P \in \mathrm{GL}_r(K((t)))$, the system (1.4.3) is converted into

$$\vartheta_t \mathbf{x} = G_P \mathbf{x} \text{ with } G_P = (\vartheta_t P)P^{-1} + PGP^{-1}. \tag{1.4.4}$$

Definition 1.4.4. [8, Proposition 7.4.1] The differential system (1.4.3)

$$\vartheta_t \mathbf{x} = G\mathbf{x}, \text{ or equivalently, } \partial_t \mathbf{x} = t^{-1} G\mathbf{x}$$

is **regular**, or 0 is a **regular singularity**, if one of the following equivalent conditions holds:

(i) the system, after extending K to a finite Galois extension K', admits a solution matrix of the form $X = Zt^A$ with $A \in \mathrm{Mat}_r(K')$, $Z \in \mathrm{GL}_r(K'((t)))$;

(ii) we can choose a matrix P in $\mathrm{GL}_r(K((t)))$ such that $G_P \in \mathrm{Mat}_r(K[[t]])$;

(iii) there exists a matrix $P \in \mathrm{GL}_r(K'((t)))$ such that after extending K to a finite Galois extension K', $G_P \in \mathrm{Mat}_r(K')$ has the Jordan canonical form

$$G_P = \begin{pmatrix} \alpha_1 + N_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 + N_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_r + N_r \end{pmatrix}$$

where $\alpha_i - \alpha_j \notin \mathbb{Z}$ if $i \neq j$ and N_i is a standard upper-triangular nilpotent matrix with $\epsilon_i \in \{0, 1\}$ for all i as follows.

$$N_i = \begin{pmatrix} 0 & \epsilon_1 & & \cdots & 0 \\ & 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \\ & & & 0 & \epsilon_{n_1} \\ 0 & & \cdots & & 0 \end{pmatrix}.$$

Instead of giving the detailed proof, we make some discussions in this important result. The condition that any two eigenvalues of $G(0)$ (equivalently, G_P) have a non-integer difference is called **non-resonance** by [8, Definition 7.1.4]. In the case that the system (1.4.3) has $G \in \mathrm{GL}_r(K[[t]])$, non-resonance can be obtained via a shearing transformation [8, Lemma 7.2.4], i.e. there exists $P \in \mathrm{GL}_r(K'((t)))$ such that G_P is non-resonant with K' an extension of K. Moreover, after counted with multiplicity, the classes in $\overline{K}/\mathbb{Z}$ of eigenvalues of $G_P(0)$ are called the **exponents** of the differential system (1.4.3), by [8, Proposition-Definition 7.6.1]. This notion is well-defined due to their independence of the choice of P. In an analogue of Fuchs-Frobenius theory, we also have the regularity criterion in the formal setting:

Theorem 1.4.5. [8, Proposition 7.5.1] *The following conditions are equivalent:*

(i) the system (1.4.3) is regular;

(ii) we have $\mathrm{ord}_0(a_i) \geqslant i - r$, $i = 0, \ldots, r-1$ *for any differential equation* $Lx = 0$ *equivalent to (1.4.3), where*

$$L = \partial_t^r + a_{r-1}\partial_t^{r-1} + \cdots + a_1\partial_t + a_0, a_i \in K((t)).$$

(iii) *we have* $b_i \in K[[t]]$, $i = 0, \dots, r-1$ *for any differential equation* $\Gamma x = 0$ *equivalent to (1.4.3), where*

$$\Gamma = \vartheta_t^r + b_{r-1}\vartheta_t^{r-1} + \cdots + b_1\vartheta_t + b_0, b_i \in K((t)).$$

1.5 Turrittin-Levelt-Jordan decomposition

In this section we introduce Turrittin-Levelt-Jordan decomposition, which is the main tool to study both regular and irregular differential modules. This is a generalization of classical Jordan decomposition to the case of differential modules.

Jordan theory

Recall that we still consider a field K of characteristic 0 and $\vartheta_t = t\frac{d}{dt}$ a derivation on $K((t))$. For $(M, \nabla_{\vartheta_t})$ a differential module over $(K((t)), \vartheta_t)$, $\phi \in K((t))$ and $\nu \in \mathbb{N} \cup \{0\}$ we set

$$K_\phi^{(\nu)} = K_\phi^{(\nu)}(M) = \mathrm{Ker}_M(\nabla_{\vartheta_t} - \phi)^\nu,$$

and

$$M_\phi^{(\nu)} = \mathrm{Im}\left(K((t)) \otimes_K K_\phi^{(\nu)}(M) \to M\right).$$

Those $K_\phi^{(\nu)}$ are K-vector subspaces of M which are equipped with a nilpotent endomorphism $\nabla_{\vartheta_t} - \phi$ and those $M_\phi^{(\nu)}$ are differential submodules of M. We obtain natural inclusions

$$(0) = K_\phi^{(0)} \subseteq K_\phi^{(1)} \subseteq \cdots$$

and

$$(0) = M_\phi^{(0)} \subseteq M_\phi^{(1)} \subseteq \cdots.$$

Moreover, if $K_\phi^{(\nu)} = K_\phi^{(\nu+1)}$ for some ν, then $K_\phi^{(\nu)} = K_\phi^{(\nu+n)}$ for every $n \geqslant 0$.

Lemma 1.5.1. [8, Corollary 8.1.6] *For integers* $\lambda, \nu \geqslant 0$ *and* $\phi \in K((t))$ *we have*

$$(M/M_\phi^{(\lambda)})_\psi^{(\nu)} = M_\phi^{(\lambda+\nu)}/M_\phi^{(\lambda)}.$$

Proposition 1.5.2. [8, Proposition 8.1.9] *For a differential module* $(M, \nabla_{\vartheta_t})$ *over a differential field* $(K((t)), \vartheta_t)$ *and elements* $\phi, \psi \in K((t))$, *if* $\mathrm{Ker}_{K((t))}(\vartheta_t - \phi + \psi) = (0)$, *then* $M_\phi^{(\lambda)} \cap M_\psi^{(\nu)} = (0)$ *for any* λ, ν. *Otherwise,* $M_\phi^{(\nu)} = M_\phi^{(\psi)}$ *for any* ν.

Proof. Firstly we consider the case $\mathrm{Ker}_{K((t))}(\vartheta_t - \phi + \psi) = (0)$.

Claim 1. $M_\phi^{(1)} \cap M_\psi^{(1)} = (0)$. Let $m_1, \ldots, m_k$ (resp. $n_1, \ldots, n_\ell$) be linearly independent elements in $K_\phi^{(1)}$ (resp. $K_\psi^{(1)}$) and k is minimal in the sense that there is a relation

$$m_k = \sum_{i=1}^{k-1} a_i m_i$$

for $a_i \in K((t))$. Let m be an element in the intersection $M_\phi^{(1)} \cap M_\psi^{(1)}$. Then

$$m = \sum_{i=1}^{k} a_i m_i = \sum_{j=1}^{\ell} b_j n_j$$

and

$$\nabla_{\vartheta_t}(m) = \sum_{i=1}^{k} (\vartheta_t a_i) m_i + \phi m = \sum_{j=1}^{\ell} (\vartheta_t b_j) n_j + \psi m.$$

Combining these two equalities we get

$$\sum_{i=1}^{k-1} ((\vartheta_t a_k + \phi a_k) a_i - a_k(\vartheta_t a_i + \phi a_i)) m_i = \sum_{j=1}^{\ell} ((\vartheta_t a_k + \phi a_k) b_j - a_k(\vartheta_t b_j + \psi b_j)) n_j$$

By the minimality of k, all coefficients of the m_i's and n_j's of this equation must vanish. In particular,

$$(\vartheta_t a_k + \phi a_k) b_j - a_k(\vartheta_t b_j + \psi b_j) = 0 \forall j,$$

if $b_j \neq 0$, this implies

$$\vartheta_t(a_k/b_j) = (\psi - \phi) a_k/b_j,$$

hence $a_k = 0$ by the hypothesis of $\mathrm{Ker}(\vartheta_t - \phi + \psi)$. We conclude that for all j, $b_j = 0$ and $m = 0$; Claim 1 has been proven.

Claim 2. $M_\phi^{(\lambda)} \cap M_\psi^{(1)} = (0)$.

This statement can be checked by induction on λ. Let $N = M/M_\phi^{(1)}$ and $\pi : M \to N$ the natural projection. By Lemma 1.5.1, $\pi(M_\phi^{(\lambda+1)}/M_\phi^{(1)}) \cong N_\phi^{(\lambda)}$; and by Claim 1, $\pi(M_\psi^{(1)}) \cong N_\psi^{(1)}$. By the induction assumption,

$$N_\phi^{(1)} \cap N_\psi^{(1)} = (0),$$

equivalently,

$$M_\phi^{(\lambda+1)} \cap (M_\phi^{(1)} + M_\psi^{(1)}) = M_\phi^{(1)}.$$

We obtain

$$M_\phi^{(\lambda+1)} \cap M_\psi^{(1)} = M_\phi^{(1)} \cap M_\psi^{(1)} = (0).$$

Claim 3. $M_\phi^{(\lambda)} \cap M_\psi^{(\nu)} = (0)$.

This statement can also be checked by induction on ν. Let $N = M/M_\psi^{(1)}$ and $\pi : M \to N$ the natural projection. With a similar argument, $\pi(M_\psi^{(\nu+1)}/M_\psi^{(1)}) \cong N_\psi^{(\nu)}$ by Lemma 1.5.1, and $\pi(M_\phi^{(\lambda)}) \cong N_\psi^{(\lambda)}$ by Claim 2. By the induction assumption, $N_\phi^{(\lambda)} \cap N_\psi^{(\nu)} = (0)$, hence

$$M_\psi^{(\nu+1)} \cap (M_\phi^{(\lambda)} + M_\psi^{(1)}) = M_\psi^{(1)}$$

and

$$M_\phi^{(\lambda)} \cap M_\psi^{(\nu+1)} = (0).$$

Now we consider the case $\mathrm{Ker}_{K((t))}(\vartheta_t - \phi + \psi) \neq (0)$. For $\vartheta_t f = (tf_t'/f)f =: cf$ for $f \in K((t))^\times$, then for every $\phi \in K((x))$ and every ν,

$$K_{\phi+c}^{(\nu)} = fK_\phi^{(\nu)}.$$

Therefore if $\mathrm{Ker}_{K((t))}(\vartheta_t - \phi + \psi) = (0)$ for some $\phi, \psi \in K((t))$, then for every ν,

$$M_\phi^{(\nu)} = M_\psi^{(\nu)},$$

which we have to show. □

The above result allow us to classify the modules $M_\phi^{(\nu)}$ for $\phi \in K((x))$, $\nu = 0, 1, \ldots$ by the following definition:

Definition 1.5.3. Let $\vartheta_t \log K((t))^\times$ be the additive subgroup of $K((t))$ consisting of elements of the forms $f^{-1}\vartheta_t f$ for $f \in K((t))^\times$, which are called **logarithmic derivatives**.

Remark 1.5.4. As in the proof of Proposition 1.5.2, an element c of $K((t))$ is in $\vartheta_t \log K((t))^\times$ if and only if there exists $f \in K((t))^\times$ such that $\vartheta_t f = cf$, that is, if and only if $\mathrm{Ker}(\vartheta_t - c) \neq (0)$. Specifically, we have

$$\vartheta_t \log K((t))^\times \cong \mathbb{Z} + tK[[t]]$$

and the quotient

$$K((t))/\vartheta_t \log K((t))^\times \cong K\left[t^{-1}\right]/\mathbb{Z} \cong K/\mathbb{Z} \oplus t^{-1}K\left[t^{-1}\right].$$

Definition 1.5.5. A differential module $(M, \nabla_{\vartheta_t})$ of rank r over the differential field $(K((t)), \vartheta_t)$ is called **admitting a Jordan decomposition** if there exists elements $\phi_1, \dots, \phi_r$ of $K((t))$ such that

$$M = \bigoplus_{i=1}^{n} M_{\phi_1}^{(r)},$$

where the ϕ_i are pairwise distinct modulo $\vartheta_t \log K((t))^\times$. There classes $\overline{\phi}_i$ in $K((t))\vartheta_t \log K((t))^\times$ are called the characters of M.

For regular differential modules

Recall that we fixed a field K of characteristic 0 and the derivation $\vartheta_t = t\frac{d}{dt}$ of the field of formal Laurent series $K((t))$.

Definition 1.5.6. A differential module M over $K((t))$ is called **regular** if it contains a free $K[[t]]$-module $\widetilde{M}$ satisfying

$$M = \widetilde{M} \otimes_{K[[t]]} K((t))$$

and $\nabla_{\vartheta_t}(\widetilde{M}) \subset \widetilde{M}$ (stability under ∇_{ϑ_t}). This $\widetilde{M}$ is called a $K[[t]]$**-lattice**.

The notions in Section 1.3 of modules with connection and Section 1.4 of regular differential systems may be interpreted in the language of differential modules as follows.

Theorem 1.5.7. *For a differential module M of rank r over $K((t))$, the following conditions are equivalent:*

(i) M is regular;

(ii) the monic differential operator $\Gamma \in K((t))\langle\vartheta_t\rangle$ attached to some (resp. any) cyclic vector of M has coefficients in $K[[t]]$;

*(iii) the corresponding module with connection (M, ∇) admits a **logarithmic model**, i.e. a free $K[[t]]$-module $\widetilde{M}$ equipped with a logarithmic connection:*

$$\nabla^{\log} : \widetilde{M} \to \widetilde{M} \otimes_{K[[t]]} \Omega^{\log}_{K[[t]]/K}$$

with $\Omega^{\log}_{K[[t]]/K} := K[[t]]dt/t$.

Proof. The equivalence of (i) and (ii) follows from the Fuchs criterion (Proposition 1.4.5). We can prove (i)$\Leftrightarrow$(iii) by a similar argument as Definition 1.3.2. $\square$

Theorem 1.5.8 (**Jordan decomposition for regular differential modules**,[8, Theorem 8.3.4]). *For a regular differential module M of rank r over $K((t))$, there exists a finite extension $K' = K[\alpha_1, \dots, \alpha_n]$ of K such that $M_{K'((t))} = M \otimes_K K'$ admits a Jordan decomposition with respect to ϑ_t:*

$$M_{K'((t))} = \bigoplus_{i=1}^{n} M_{\alpha_i}^r$$

where

$$M_{\alpha_i}^{(r)} := \mathrm{Ker}_{K'((t))}(\nabla_{\vartheta_t} - \alpha_i)^r \otimes_{K'} K'((t))$$

and the $\alpha_i \in K'$ are pairwise distinct modulo $\mathbb{Z}$. The decomposition is independent of the choice of the α_i's modulo $\mathbb{Z}$ and is canonical.

Proof. The theorem follows from equivalent statements of regular differential systems in Definition 1.4.4 and Definition 1.5.6. Uniqueness holds since $m = \vartheta_t(t^m)/t^m$ for any integer m and

$$K/\mathbb{Z} \subseteq K((t))/\vartheta_t \log K((t))^\times,$$

therefore the only logarithmic derivatives which belong to K (and also K') are the integers, For canonicity, it is clearly seen that by Definition 1.5.5, the characters $\overline{\phi}_i$ change to $\overline{u\phi_i}$ with the same multiplicities after replacing ϑ_t by $u\vartheta_t$ for any $u \in K((t))^\times$. □

For irregular differential modules

Recall that we fixed a field K of characteristic 0 and the derivation $\vartheta_t = t\frac{d}{dt}$ of the field of formal Laurent series $F = K((t))$.

Any finite extension of $K((t))$ has the form $F' = K'((x^{1/N}))$, which is a complete valued field of ramification index N, with $t^{1/N}$ an Nth root of t and K'/K a finite extension. This extension $K'((t^{1/N}))/K((t))$ is Galois if and only if K'/K is a Galois extension containing the Nth roots of unity.

For such $F' = K'((t^{1/N}))$ an extension of $F = K((t))$ and the derivation ϑ_t, we have

$$\vartheta_t \log F'^\times = \frac{1}{N}\mathbb{Z} \oplus t^{1/N} K'[[t^{1/N}]].$$

It is remarkable that $\vartheta_t \log F'^\times \cap K'[[t^{-1/N}]] = \frac{1}{N}\mathbb{Z}$ and $\vartheta_t \log F'^\times + K'[t^{-1/N}] = F'$, therefore we obtain

$$F'/\vartheta_t \log F'^\times \cong K'[t^{-1/N}]/\frac{1}{N}\mathbb{Z} \cong \left(K'/\frac{1}{N}\mathbb{Z}\right) \oplus t^{-1/N} K'[t^{-1/N}].$$

Theorem 1.5.9 (**Turrittin-Jordan-Levelt decomposition**, [8, Theorem 16.1.2])**.** *For $(M, \nabla_{\vartheta_t})$ a differential module over $F = K((t))$ of rank r, there exists a finite extension $F' = F(\phi_1, \dots, \phi_n)$ of F over which $(M_{F'}, \nabla_{\vartheta_t})$ admits a Jordan decomposition of differential modules over F':*

$$M_{F'} = \bigoplus_{i=1}^{n} M_{\phi_i}^{(r)}$$

with characters $\overline{\phi}_i \in K'[t^{-1/N}]/\frac{1}{N}\mathbb{Z}$, where

$$M_{\phi_i}^{(r)} := \mathrm{Ker}_{M_{F'}}(\nabla_{\vartheta_t} - \phi_i)^r \otimes_{K'} F'.$$

Grouping together the direct summands $M_{\phi}^{(r)}$ for which the characters ϕ_i's only differ by constant terms, we obtain the Turrittin-Levelt decomposition of differential modules over F':

$$M_{F'} = \bigoplus_{j} L_{\psi_j} \otimes_{F'} U_j$$

where L_{ψ_j} is a differential module of rank one, $\psi_j \in t^{-1/N}K'[t^{-1/N}]$ and U_j is unipotent, i.e. a successive extension of the trivial differential module $(F', \nabla_{\vartheta_x})$ by itself.

Sketch of the proof. The proof proceeds by induction on pairs (r, ρ), where $r \in \mathbb{N}$ is the rank of M over $F = K((t))$ and $\rho = \rho(M) \in \frac{1}{r!}\mathbb{N}$ is the irregularity (or the Poincaré rank) of M, see [8, Section 15] for details. The set of (r, ρ) are lexicographically ordered, i.e. $(r, \rho) < (r'\rho')$ if $r < r'$ or $r = r'$ and $\rho < \rho'$.

The induction starts at $(r, 0)$ (it is the regular case, Theorem 1.5.8) and $(1, \rho)$ for any $\rho \in \mathbb{N}$ (it is the rank-one case).

Let $\mathbf{m} = (m, \dots, \vartheta_t^{r-1}m)$ be a cyclic basis and

$$\vartheta_t \mathbf{m} = \mathbf{m} \begin{pmatrix} 0 & \cdots & 0 & a_0 \\ 1 & \cdots & 0 & a_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & a_{r-1} \end{pmatrix}.$$

By [8, Corollary 15.3.6],

$$\rho = \max \left\{ 0, \max_{j=0,\dots,r-1} \left(-\frac{\mathrm{ord}_t(a_j)}{r-j} \right) \right\}$$

is a rational number with denominator bounded by r. Then $\rho = l/e, e \leqslant r$ is an irreducible fraction. The argument in the proof of [8, Theorem 15.2.2] allows us to consider the base change $(M_{K((t^{1/e}))}, \delta := t^\rho \vartheta_t)$ whose the basis $\mathbf{n} = \mathbf{m}\Xi$, Ξ is the diagonal matrix with entries $1, t^\rho, \dots, t^{(r-1)\rho}$. The matrix of $\delta = t^\rho \vartheta_t = t^{\rho+1}\frac{d}{dt} = e^{-1}(t^{1/e})^{e\rho+1}\frac{d}{dt^{1/e}} = e^{-1}(t^{1/e})^{e\rho}\vartheta_{t^{1/e}}$ in this new basis is

$$B_{-\rho} = \begin{pmatrix} 0 & \cdots & 0 & t^{r\rho}a_0 \\ 1 & \cdots & 0 & t^{(r-1)\rho}a_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & t^\rho a_{r-1} \end{pmatrix} + \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \rho t^\rho & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (r-1)\rho t^\rho \end{pmatrix} \in M_\rho(K[[t^{1/e}]]).$$

If $\rho = 0$, we are in the regular case, which is the beginning point of the induction.

If $\rho > 0$, we may apply the splitting lemma [8, Proposition 16.2.1]. Roughly speaking, for $\overline{M} := M_{K[[t^{1/e}]]} \otimes_{K[[t^{1/e}]]} K$ and $\overline{\nabla}_\delta$ the induced k-linear action of δ on $\overline{M}$, if

$$\overline{M} = \bigoplus \overline{M}_j$$

is the decomposition into k-spaces such that the sets of eigenvalues of $\overline{\nabla}_\delta$ are pairwise disjoint, it can be lifted to an unique decomposition of $M_{K[[t^{1/e}]]}$ into ∇_δ-stable $K[[t^{1/e}]]$-submodules.

If $B_{-\rho}(0)$ has at least two distinct eigenvalues in $\overline{K}$, this reduces to the case of rank $< r$, and the induction assumption applies.

If $B_{-\rho}(0)$ has only one eigenvalue $\zeta \in \overline{K}$, this eigenvalue cannot be 0, since we can use [8, 15.2.2] in the case $\rho > 0$. Tensoring M with the rank-one $K((t))/K$-differential module $L_{-\zeta t^{-\rho}}$ (with generator $\ell = \exp(\frac{\zeta}{\rho}t^{-\rho})$ and action $\vartheta_t \ell = -\zeta t^{-\rho}\ell$), we can prove that $\rho(M \otimes L) < \rho$ and the induction assumption applies. $\square$

Remark 1.5.10. From the Jordan decomposition in Theorem 1.5.9 of a differential module M, we can group the summands $M_{\phi_i}^{(\mu)}$ whose ϕ_i's have the same $t^{1/N}$-adic valuation to obtain a so-called slope decomposition as in [8, Proposition-Definition 17.1.3]. Although we do not discuss this decomposition in this thesis, it is convenient to study the structure of irregular differential modules.

1.6 Deligne-Katz correspondence in characteristic zero

In this section, we fix a field k of characteristic zero, the field $k((t))$ of formal Laurent series over k. and denote by $\mathbb{G}_{m,k}$ the multiplicative group over k with coordinate t:

$$\mathbb{G}_{m,k} = \operatorname{Spec} k[t, t^{-1}],$$

which can be geometrically interpreted as the affine line over k minus the origin, or the projective line over k minus two points 0 and ∞.

Fixing the derivation $\vartheta_t = t\frac{d}{dt}$, we denote by $\mathrm{MC}(k[t,t^{-1}]/k)$ and $\mathrm{MC}(k((t))/k))$ categories of differential modules over $k[t,t^{-1}]$ (resp. over $k((t))$), or categories of modules with integrable connection on $\mathbb{G}_{m,k}$ (resp. on $\operatorname{Spec} k((t))$).

By the k-linear embedding

$$k[t,t^{-1}] \hookrightarrow k((t)), \qquad t \mapsto t$$

we view $k((t))$ as the completion at 0 of the function field of $\mathbb{G}_{m,k}$. Hence we obtain a natural inverse image functor

$$\begin{aligned} \mathrm{MC}(k[t,t^{-1}]/k) &\to \quad \mathrm{MC}(k((t))/k) \\ M \qquad &\mapsto M \otimes_{k[t,t^{-1}]} k((t)). \end{aligned}$$

As an analogue of Definition 1.5.6 and the geometric interpretation of $\mathbb{G}_{m,k}$:

Definition 1.6.1.

(i) A differential module $(M, \nabla_{\vartheta_t})$ over $k[t,t^{-1}]$ is called **regular singular at** 0 if M admits a $k[t]$-lattice M_0, i.e. a $k[t]$-module M_0 stable under ∇_{ϑ_t} and satisfying $M = M_0 \otimes_{k[t]} k[t,t^{-1}]$.

(ii) A differential module $(M, \nabla_{\vartheta_t})$ over $k[t,t^{-1}]$ is called **regular singular at** ∞ if M admits a $k[t]$-lattice M_∞, i.e. a $k[t^{-1}]$-module M_∞ stable under ∇_{ϑ_t} and satisfying $M = M_\infty \otimes_{k[t^{-1}]} k[t,t^{-1}]$.

Example 1.6.2. Let M be a differential module over $k[t,t^{-1}]$ of rank one which is regular singular at 0. By a similar argument as in Example 1.2.7, M is free of rank one. Since M is regular singular at 0, $\nabla_{\vartheta_t}(k[t]) \subset k[t]$ and we can write

$$\nabla_{\vartheta_t}(1) = \sum_{i=0}^{n} a_i t^i \in k[t].$$

By Leibniz rule, we obtain

$$\nabla_{\vartheta_t}(f) = \vartheta_t(f) + f\nabla_{\vartheta_t}(1) = \left(t\frac{d}{dt} + \sum_{i=0}^{n} a_i t^i \right)(f).$$

Consequently, M has the explicit form

$$M = \left(k[t,t^{-1}], \nabla_{\vartheta_t} = t\frac{d}{dt} + \sum_{i=0}^{n} a_i t^i \right)$$

for some polynomial $\sum_{i=0}^{n} a_i t^i \in k[t]$. By a similar argument as in Remark 1.5.4, there exists an isomorphism between two such differential modules if and only if the difference of their derivations belongs to $\vartheta_t \log k[t,t^{-1}]^\times$, where

$$k[t,t^{-1}]^\times = \{ct^n, c \in k, n \in \mathbb{Z}\}$$

is the group of units of $k[t,t^{-1}]$ and thus

$$\vartheta_t \log k[t,t^{-1}]^\times = \mathbb{Z}.$$

Therefore the group of isomorphism classes of such an M is the additive group $k[t]/\mathbb{Z}$ via the map

$$M \mapsto \sum a_i t^i \mod \mathbb{Z}.$$

Similarly, a differential module M' over $k[t,t^{-1}]$ of rank one which is regular singular at ∞ has the explicit form

$$M' = \left(k[t,t^{-1}], \nabla_{\vartheta_t} = t\frac{d}{dt} + \sum_{i=0}^{n} a_i t^{-i} \right)$$

for some polynomial $\sum_{i=0}^{n} a_i t^{-i} \in k[t^{-1}]$ and the group of isomorphism classes of such M' is the additive group $k[t^{-1}]/\mathbb{Z}$ via the map

$$M \mapsto \sum a_i t^{-i} \mod \mathbb{Z}.$$

From the viewpoint of Remark 1.5.4, Theorem 1.5.8 and the discussion in Example 1.6.2, we obtain the following result:

Proposition 1.6.3. *The inverse image functor*

$$\mathrm{MC}_{rs\ at\ \infty}(k[t,t^{-1}]/k) \to \mathrm{MC}(k((t))/k)$$

between the category of differential modules over $k[t,t^{-1}]$ which are regular singular at infinity and the category of differential modules over $k((t))$ induces an equivalence between the full subcategories of rank-one objects.

In the next step, we recover Katz correspondence in [5, 2.4] to extend this result to an equivalence of larger categories, which is motivated by Turrittin-Levelt-Jordan decomposition (Theorem 1.5.9).

Definition 1.6.4. A differential module in $\mathrm{MC}(k[t,t^{-1}]/k)$ (resp. $\mathrm{MC}(k((t))/k)$) is called **unipotent** if it is isomorphic to a successive extension of the trivial object $\left(k[t,t^{-1}], t\frac{d}{dt}\right)$ (resp. $\left(k((t)), t\frac{d}{dt}\right)$) by itself, i.e. the matrix of the connection ∇ corresponding to some derivation (in the sense of Definition 1.3.2) is upper-triangular.

Let $\mathrm{NilpEnd}_k$ be the category of pairs (V,N) consisting of a finite-dimensional k-vector spaces V endowed with a k-linear nilpotent endomorphism N of V. The natural functors

$$\begin{aligned}\mathrm{NilpEnd}_K &\to \mathrm{MC}^{\mathrm{uni}}(k[t,t^{-1}]/k)\\ (V,N) &\mapsto \left(k[t,t^{-1}]\otimes_k V, D = t\frac{d}{dt}\otimes \mathrm{id} + \mathrm{id}\otimes N\right)\end{aligned}$$

and

$$\begin{aligned}\mathrm{NilpEnd}_K &\to \mathrm{MC}^{\mathrm{uni}}(k((t))/k)\\ (V,N) &\mapsto \left(k((t))\otimes_k V, D = t\frac{d}{dt}\otimes \mathrm{id} - \mathrm{id}\otimes N\right)\end{aligned}$$

are both equivalences, whose quasi-inverse functors are given by

$$(M,\nabla_{\vartheta_t}) \mapsto \left(\bigcup_{n\geqslant 1}\mathrm{Ker}(\nabla_{\vartheta_t})^n, \pm\nabla_{\vartheta_t}\right).$$

Therefore we obtain the following result.

Proposition 1.6.5. *The inverse image functor*

$$\mathrm{MC}^{uni}(k[t,t^{-1}]/k) \to \mathrm{MC}^{uni}(k((t))/k)$$

of full subcategories of unipotent objects is an equivalence.

Definition 1.6.6.

(i) A differential module M over $k[t,t^{-1}]$ is called **very special** if it admits a decomposition as follows:

$$M = \bigoplus_j L_j \otimes U_j$$

where L_j's are rank-one, regular singular at infinity and U_j's are unipotent.

(ii) A differential module M over $k((t))$ is called **very special** if it is a successive extension of one-dimensional objects.

By Proposition 1.6.3 and Proposition 1.6.5 we obtain:

Proposition 1.6.7. *The inverse image functor*

$$\mathrm{MC}(k[t,t^{-1}]/k) \to \mathrm{MC}(k((t))/k)$$

induces an equivalence between the full subcategories of very special objects.

Recall that any finite Galois extension of $k((t))$ has the form $k'((t^{1/N}))$ for some positive integer N and some finite Galois extension k' of k in $\overline{K}$ which contains all Nth roots of unity.

Definition 1.6.8. For such an integer N and an extension k' of k as above,

(i) a differential module M over $k[t,t^{-1}]$ is called (N,k')-**special** if its inverse image in $\mathrm{MC}(k'[t^{1/N},t^{-1/N}]/k')$, given by

$$M \mapsto M \otimes_{k[t,t^{-1}]} k'[t^{1/N},t^{-1/N}]$$

is very special.

(ii) a differential module M over $k((t))$ is called (N,k')-**special** if its inverse image in $\mathrm{MC}(k'((t^{1/N}))/k')$, given by

$$M \mapsto M \otimes_{k((t))} k'((t^{1/N}))$$

is a successive extension of one-dimensional objects.

For each (N,k') as above, the semidirect product

$$G = \mu_N(k') \ltimes \mathrm{Gal}(k'/k)$$

acts on $k'[t^{1/N},t^{-1/N}]$ and on $k'((t^{1/N}))$ by

$$g \cdot \sum a_n t^{n/N} = \sum \sigma(a_n)\zeta^n t^{n/N}, g = (\zeta,\sigma).$$

G also acts on $k'[t^{-1/N}]$ by

$$g \cdot \sum a_{-n} t^{-n/N} = \sum \sigma(a_{-n})\zeta^{-n} t^{-n/N}, g = (\zeta,\sigma).$$

By combining Proposition 1.6.7 with Galois descent of $\mathrm{MC}(k'[t^{1/N},t^{-1/N}]/k')$ (respectively $\mathrm{MC}(k'((t^{1/N}))/k')$ and $\mathrm{MC}(k'[t^{-1/N}]/k')$) with G-action, we obtain

Proposition 1.6.9. *For any (N, k') as above, the inverse image functor*

$$\mathrm{MC}(k[t, t^{-1}]/k) \to \mathrm{MC}(k((t))/k)$$

induces an equivalence between the full subcategories of (N, k')-special objects.

Definition 1.6.10. A differential module over $k[t, t^{-1}]$ (resp. over $k((t))$) is called **special** if it is (N, k')-special for some (N, k') as above.

By Theorem 1.5.9 of Turrittin-Levelt-Jordan decomposition, every differential module of $\mathrm{MC}(k((t))/k)$ is special. Taking the limit over (N, k') we obtain

Theorem 1.6.11 (Katz correspondence)**.** *The inverse image functor*

$$\mathrm{MC}(k[t, t^{-1}]/k) \to \mathrm{MC}(k((t))/k)$$

restricted to the full subcategory $\mathrm{MC}^{sp}(k[t, t^{-1}]/k)$ of special objects in $\mathrm{MC}(k[t, t^{-1}]/k)$, induces an equivalence of categories:

$$\mathrm{MC}^{sp}(k[t, t^{-1}]/k) \xrightarrow{\sim} \mathrm{MC}(k((t))/k).$$

By a different approach compared to Katz's, Deligne established the following correspondence, which can be considered a subcorrespondence of Katz correspondence:

Theorem 1.6.12 (Deligne correspondence, [6, 15.35])**.** *The inverse image functor*

$$\mathrm{MC}_{rs\ at\ 0,\infty}(k[t, t^{-1}]/k) \to \mathrm{MC}_{rs}(k((t))/k)$$

between the category of differential modules over $k[t, t^{-1}]$ which are regular singular at zero and at infinity and the category of regular differential modules over $k((t))$ is an equivalence.

Due to the fact that it is seemingly difficult to extend the approach to prove this result to p-adic setting, we do not present it here. Another proof, which can be extended to Deligne correspondence for formal regular-singular connections with parameters, is given in [17].

We will consider some p-adic analogues of both correspondences of Deligne and Katz in Chapter 3.

Chapter 2

An introduction to rigid geometry

2.1 Tate algebras and affinoid algebras

We start thís section by reviewing some notions of non-archimedean geometry.

Definition 2.1.1. For a field K, a map $|\cdot| : K \to [0, +\infty)$ is called a **non-archimedean absolute value** if for all $x, y \in K$ the following conditions hold:

(i) $|x| = 0$ if and only if $x = 0$,

(ii) $|xy| = |x||y|$,

(iii) $|x + y| \leqslant \max\{|x|, |y|\}$.

An absolute value $|\cdot|$ is called **trivial** if for all $x \in K$, $|x| \in \{0, 1\}$. We say an absolute value **discrete** if $|K^*|$ is a discrete subset of $[0, +\infty)$.

Obviously we can check that for all $x \in K$, $|-x| = |x|$ and $|1| = 1$.

Definition 2.1.2. For a field K, a map $v : K \to \mathbb{R} \cup \{\infty\}$ is called a **valuation** on K if for all $x, y \in K$, the following conditions satisfy:

(i) $v(x) = \infty$ if and only if $x = 0$,

(ii) $v(xy) = v(x) + v(y)$,

(iii) $v(x + y) \geqslant \min\{v(x), v(y)\}$.

Remark 2.1.3. There is a one-to-one correspondence between non-archimedean absolute values and valuations by setting

$$v(x) = -\log|x| \text{ for } x \in K$$

and

$$|x| = e^{-v(x)} \text{ for } x \in K.$$

On the other hand, we can equip a topology on K induced from the associated non-archimedean metric of an absolute value, i.e. for $x, y \in K$, set

$$d(x, y) = |x - y|.$$

Similarly as in archimedean fields, we also consider the convergence of sequences and infinite series of nonarchimedean counterparts. In particular, a nonarchimedean field K is called **complete** if every Cauchy sequence converges to an element of K.

Proposition 2.1.4. [9, 2.1, Lemma 3] *A series $\sum_{n=0}^{\infty} a_n$ of elements $a_n \in K$ determines a Cauchy sequence if and only if $\lim_{n\to\infty} |a_n| = 0$. In the case that K is complete, the series converges if and only if $\lim_{n\to\infty} |a_n| = 0$.*

Proof. Let $\varepsilon > 0$. By the condition $\lim_{n\to\infty} |a_n| = 0$, there exists an $N \in \mathbb{N}$ such that $|a_n| < \varepsilon$ for all $n \geqslant N$. Then for by nonarchimedean triangle inequality and for any integers $\ell \geqslant k > N$, we obtain

$$\left|\sum_{n=k}^{\ell} a_n\right| \leqslant \max_{k \leqslant n \leqslant \ell} |a_n| < \varepsilon,$$

which has to be shown. □

By the non-archimedean triangle inequality stated in Definition 2.1.1,

$$d(y, z) \leqslant \max\{d(x, y), d(z, x)\}, x, y, z \in K.$$

This obviously implies that any triangle in K is isoceles. Consequently, each point of a disk in K can serve as its center; any two disks with non-empty intersection are concentric.

For a center x in K and a radius $r \in (0, +\infty)$ we can consider the open disk

$$D^-(x, r) = \{z \in K, d(z, x) < r\},$$

the closed disk

$$D^+(x, r) = \{z \in K, d(z, x) \leqslant r\}.$$

However, both $D^-(x, r)$ and $D^+(x, r)$ are open and closed.

The preceding discussions lead us to a conclusion of the topology of K:

Proposition 2.1.5. [9, 2.1, Proposition 4] *The topology of K is totally disconnected, i.e. any subset in K except singleton sets is disconnected.*

For a complete nonarchimedean field K, let $\overline{K}$ be an algebraic closure of K. The absolute value of K can be extended uniquely to the one of $\overline{K}$ despite the non-completeness of $\overline{K}$.

For each integer $d \geqslant 1$, we consider the unit ball in $\overline{K}^d$:

$$\mathbb{B}^d(\overline{K}) = \{(x_1, \ldots, x_d) \in \overline{K}^d : |x_1| \leqslant 1, \ldots, |x_d| \leqslant 1\}.$$

As a consequence of Proposition 2.1.4, a formal power series

$$f = \sum_{\mathbf{n} \in \mathbb{N}^d} a_{\mathbf{n}} X^{\mathbf{n}} = \sum_{\mathbf{n} \in \mathbb{N}^d} a_{n_1 \ldots n_d} X_1^{n_1} \ldots X_d^{n_d} \in K[[X_1, \ldots, X_d]]$$

is convergent on the unit ball $\mathbb{B}^d(\overline{K})$ if and only if $\lim_{|\mathbf{n}| \to \infty} |a_{\mathbf{n}}| = 0$.

Definition 2.1.6. The K-algebra $T_d = K\{X_1, \ldots, X_d\}$ of all formal power series converging on $\mathbb{B}^d(\overline{K})$, i.e.

$$T_d = \left\{ \sum_{\mathbf{n} \in \mathbb{N}^d} a_{\mathbf{n}} X^{\mathbf{n}} \in K[[X_1, \ldots, X_d]], a_{\mathbf{n}} \in K, \lim_{|\mathbf{n}| \to \infty} |a_{\mathbf{n}}| = 0 \right\},$$

is called the **Tate algebra** of **restricted**, or **strictly convergent** power series. For simplicity, $T_0 = K$.

Proposition 2.1.7. *T_d is a normed K-algebra with the Gauss norm:*

$$|f| = \max_{\mathbf{n} \in \mathbb{N}^d} |a_{\mathbf{n}}|, \qquad f = \sum_{\mathbf{n}} a_{\mathbf{n}} X^{\mathbf{n}}.$$

Moreover, T_d is Banach with respect to Gauss norm.

Proof. Instead of giving the complete proof, we only prove the hardest part, which is to verify the multiplicativeness of this norm. Obviously we have $|gh| \leqslant |g||h|$ for $g, h \in T_n$. For its equality, we consider the valuation ring

$$R = \{x \in K, |x| \leqslant 1\},$$

its maximal ideal

$$\mathfrak{m} = \{x \in K, |x| < 1\}$$

and its residue field $k = R/\mathfrak{m}$. Let $R\{X_1, \ldots, X_d\}$ be the R-algebra of all restricted power series whose coefficients are in R (equivalently, the Gauss norm of those power series are not greater than 1). It is clear that the canonical epimorphism

$$R \to k, x \mapsto \tilde{x}$$

induces an epimorphism

$$\pi : R\{X_1, \ldots, X_d\} \to k[X_1, \ldots, X_d], \sum_{\mathbf{n}} a_{\mathbf{n}} X^{\mathbf{n}} \mapsto \sum_{\mathbf{n}} \tilde{a_{\mathbf{n}}} X^{\mathbf{n}}.$$

For any $g \in R\{X_1, \ldots, X_d\}$ we call $\tilde{g} = \pi(g)$ the **reduction** of g. We consider some cases of g and h as follows.

(i) If g or h is a constant element in T_d, $|gh| = |g||h|$ obviously holds.

(ii) If $g, h \in T_d$ satisfy $|g| = |h| = 1$, Then g, h and gh are elements of $R\{X_1, \ldots, X_d\}$ and

$$\pi(gh) = \tilde{g}\tilde{h} \neq 0,$$

since $k[\xi_1, \ldots, \xi_n]$ is an integral domain. Due to the fact that the reduction $\tilde{f} = 0$ if and only if $|f| < 1$ for any $f \in T_d$, we conclude $|gh| = 1$.

(iii) In the general case, there exists $c, d \in K$ such that $g = cg_0$, $h = dh_0$ such that $|g| = |c|$, $|h| = |d|$ and $|g_0| = |h_0| = 1$. Then

$$|gh| = |cdg_0h_0| = |cd||g_0h_0| = |c||d| = |g||h|$$

which we have to prove. □

With respect to the Gauss norm, the Tate algebra T_n is complete and hence a Banach K-algebra, by [9, 2.2, Proposition 3].

For a broader understanding of Tate algebras, we state some important analytic and algebraic properties of T_d without proofs in [9, 2.2-3].

(i) (Maximum principle) Let $f \in T_d$. then for all points x of the unit ball $\mathbb{B}^d(\overline{K})$, $|f(x)| \leqslant |f|$ and the equality holds.

(ii) Noether Normalization holds for Tate algebra. Specifically, for an ideal $\mathfrak{p} \subsetneq T_d$, we can choose a K-algebra monomorphism $T_{d_0} \to T_d$ for some $d_0 \in \mathbb{N}$ such that the following composition morphism is finite:

$$T_{d_0} \to T_d \to T_d/\mathfrak{p}.$$

e can be calculated uniquely as the Krull dimension of $T_d/\mathfrak{p}$.

(iii) T_d is Noetherian, Jacobson and factorial.

(iv) The Krull dimension of T_d is d.

Definition 2.1.8. A K-algebra A is called an **affinoid K-algebra** if there is a surjective homomorphism of K-algebras $\beta : T_d \to A$ for some d.

We can associate for each surjective homomorphism $\beta : T_d \to A$ a residue norm $|\cdot|_\beta$ on A given by

$$|\beta(f)|_\beta = \inf_{g \in \operatorname{Ker} \beta} |f - g|,$$

where $|\cdot|$ in the right-hand side is the Gauss norm. Indeed, the K-algebra norm $|\cdot|_\alpha$ induces the quotient topology of T_d on A. A is also a Banach K-algebra under $|\cdot|_\beta$ (see [9, 3.1, Proposition 5]). Moreover, by [9, 3.1, Proposition 20], all residue norms on A are equivalent.

Example 2.1.9. For an affinoid K-algebra K with the topology given by any residue norm. We consider for a tuple of variables $Y = (Y_1, \ldots, Y_n)$, the K-algebra of restricted power series in Y with coefficients in A as follows.

$$A\{Y\} = \left\{ \sum_{\mathbf{n} \in \mathbb{N}^d} a_{\mathbf{n}} Y^{\mathbf{n}} \in A[[Y]]; a_{\mathbf{n}} \in A, \lim_{\mathbf{n} \in \mathbb{N}^d} |a_{\mathbf{n}}| = 0 \right\}$$

This definition is clearly independent of the choice of residue norm.

Claim. $A\{Y\}$ is an affinoid K-algebra.

Indeed, for a tuple of variables $X = (X_1, \ldots, X_e)$ let $\beta : K\{X\} \to A$ be an epimorphism and it induces a morphism of K-algebras

$$\tilde{\alpha} : T_{d+e} = K\{X, Y\} \to A\{Y\}$$
$$\sum_{\mathbf{n} \in \mathbb{N}^d} \left(\sum_{\mathbf{m} \in \mathbb{N}^e} a_{\mathbf{m},\mathbf{n}} X^{\mathbf{m}} \right) Y^{\mathbf{n}} \mapsto \sum_{\mathbf{n} \in \mathbb{N}^d} \beta \left(\sum_{\mathbf{m} \in \mathbb{N}^e} a_{\mathbf{m},\mathbf{n}} X^{\mathbf{m}} \right) Y^{\mathbf{n}}$$

which is surjective. We can define the residue norm of $A\{Y\}$ induced from the residue norm of A by β:

$$\left| \sum_{\mathbf{n} \in \mathbb{N}^d} a_{\mathbf{n}} Y^{\mathbf{n}} \right|_{\widetilde{\beta}} = \max_{\mathbf{n} \in \mathbb{N}^d} |a_{\mathbf{n}}|_\beta \,.$$

Notice that this residue norm of $A\{Y\}$ coincides with its Gauss norm. Thus $A\{Y\}$ is an affinoid K-algebra.

2.2 Affinoid spaces

Definition 2.2.1. For an affinoid K-algebra A, we denote by $\operatorname{Sp} A = (\operatorname{Max} A, A)$ the **affinoid K-space** associated to A, where $\operatorname{Max} A$ is the set of maximal ideals.

Definition 2.2.2. On each affinoid K-space $\operatorname{Sp} A$, we can define its **Zariski topology** as follows: any Zariski closed subset of $\operatorname{Sp} A$ has the form:

$$V(\mathfrak{p}) = \{x \in \operatorname{Sp} A : f(x) = 0 \ \forall f \in \mathfrak{p}\} = \{x \in \operatorname{Sp} A; \mathfrak{p} \subset \mathfrak{m}_x\}$$

for some ideal $\mathfrak{p}$ of A.

Similarly as in algebraic geometry, we consider the ideal

$$I(X) = \{f \in A; f(x) = 0 \ \forall x \in X\} = \bigcap_{x \in X} \mathfrak{m}_x$$

for any subset X of $\operatorname{Sp} A$. From these definition, we can easily verify some relations between $V(\cdot)$ and $I(\cdot)$:

(i) For a subset X of $\operatorname{Sp} A$, $V(I(X)) = \overline{X}$, where $\overline{X}$ is the Zariski closure of X in $\operatorname{Sp} A$.

(ii) For an ideal $\mathfrak{p}$ of an affinoid algebra A, $I(V(\mathfrak{p})) = \operatorname{rad} \mathfrak{p}$ where $\operatorname{rad} \mathfrak{p}$ is the radical of $\mathfrak{p}$.

(iii) V and I determine inverse bijections between the set of closed subsets of $\operatorname{Sp} A$ and the set of reduced ideals in A for any affinoid K-algebra A.

The first assertion is proved similarly as in algebraic geometry, which is based on inclusion and exclusion properties of $V(\cdot)$ and $I(\cdot)$. The secone one is due to the Jacobsonness of an affinoid algebra, i.e. the nilradical of any ideal is the intersection of all maximal ideals containing it [9, 2.2, Proposition 16; 3.1, Proposition 3]. The third statement is a direct consequence of two above statements.

For any morphism $\phi : B \to A$ of affinoid K-algebras, we can construct the associated map of affinoid K-spaces:

$$\phi^{\#} : \operatorname{Sp} A \to \operatorname{Sp} B, \mathfrak{m} \mapsto \phi^{-1}(\mathfrak{m}).$$

Similarly as in algebraic geometry, we have an antiequivalence between the category of affinoid K-spaces and the category of affinoid K-algebras.

Canonical topology

Due to the coarseness of Zariski topology, we use a finer topology on affinoid K-spaces, which is derived from the topology of K. Specifically, for some $d \in \mathbb{N}$ and in the case that $\overline{K}$ is algebraically closed, $\operatorname{Sp} T_d$ can be identified with $\mathbb{B}^d(K)$; an arbitrary affinoid K-space $\operatorname{Sp} A$ can be considered as a closed subspace of this unit ball. In the case that $K \neq \overline{K}$, $\operatorname{Sp} T_d$ can be viewed as the quotient space of $\mathbb{B}^d(\overline{K})$ by the action of $\operatorname{Gal}(\overline{K}/K)$.

Notice that for an affinoid K-space A and a point $x \in \operatorname{Sp} A$, we denote for each $f \in A$ its image in $A/\mathfrak{m}_x$ by $f(x)$.

Definition 2.2.3. With the notion $X = \operatorname{Sp} A$ of an affinoid K-space, the **canonical topology** is defined as topology generated by all open sets of the following type:

$$X(g, \varepsilon) = \{x \in X; |g(x)| \leqslant \varepsilon\}.$$

for $g \in A$ and $\varepsilon > 0$. We also use the notations $X(g) = X(g, 1), g \in A$ for **basic open subsets** of X.

Proposition 2.2.4. [9, 3.3, Proposition 2] $\{X(g), g \in A\}$ *is a subbasis of the canonical topology on* $X = \operatorname{Sp} A$*, i.e. a subset* U *of* X *is open if and only if it can be written as a union of sets of type*

$$X(g_1, \ldots, f_r) := X(g_1) \cap \ldots \cap X(g_r)$$

for $g_1, \ldots, g_r \in A$.

Proof. It is clear that $\{|g(x)|, x \in A\} \subset |\overline{K}|$ for any $g \in A$. Consequently for $\varepsilon > 0$ we can write

$$X(g, \varepsilon) = \bigcup_{\substack{\varepsilon' \in |\overline{K}^*| \\ \varepsilon' \leqslant \varepsilon}} X(g, \varepsilon').$$

For $\varepsilon' \in |\overline{K}^*|$, let $N \in K$ be the norm of ε' in the splitting field over K. This implies $\varepsilon'^s = |N|$ for some positive integer s. We have

$$X(g, \varepsilon') = X(g^s, \varepsilon'^s) = X(N^{-1} g^s)$$

and we are done. ∎

Proposition 2.2.5. [9, 3.3, Proposition 4] *The following sets are open:*

- $\{x \in \operatorname{Sp} A, g(x) \neq 0\}$,

- $\{x \in \operatorname{Sp} A, g(x) \,\square\, \varepsilon\}$, $\square \in \{\leqslant, <, =, >, \geqslant\}$, $\varepsilon > 0$.

Proof. Instead of giving a complete proof, we only prove $\{x \in \operatorname{Sp} A, g(x) = \varepsilon\}$ is open; the openness of other sets is proved similarly.

With the notion $\mathfrak{m}_x$ of the maximal ideal corresponding to x in A and $g(x)$ the image of g in $A/\mathfrak{m}_x$. Furthermore, let $P(X) \in K[X]$ be the minimal polynomial of $g(x)$ over K and let

$$P(X) = \prod_{i=1}^{s}(X - \lambda_i), \lambda_i \in \overline{K}$$

be its factorization to linear terms with zeros $\lambda_i \in \overline{K}$. We have

$$\varepsilon = |g(x)| = |\lambda_i|$$

for all i, if we pick any embedding $A/\mathfrak{m}_x \hookrightarrow \overline{K}$.

Claim. For $h := P(g) \in A$, obviously $h(x) = P(g(x)) = 0$. If $y \in X$ satisfies $|h(y)| < \varepsilon^s$, then $|g(y)| = \varepsilon$.

Indeed, assume by contradiction that there exists an $y \in X$ satisfying $|h(y)| < \varepsilon^n$ such that $|g(y)| \neq \varepsilon$. Then picking an embedding $A/\mathfrak{m}_y \to \overline{K}$, we obtain for all i:

$$|g(y) - \lambda_i| = \max\left(|g(y)|, |\lambda_i|\right) \geqslant |\lambda_i| = \varepsilon$$

and this implies

$$|h(y)| = |P(g(y))| = \prod_{i=1}^{s} |g(y) - \lambda_i| \geqslant \varepsilon^s,$$

contradicting the choice of y. Consequently,

$$|g(y)| = \varepsilon \ \forall y \in X(c^{-1}h)$$

if $c \in K^*$ satisfies $|c| < \varepsilon^s$. In other words, x has an open neighborhood $X(c^{-1}h) \subset \{x \in \operatorname{Sp} A, g(x) = \varepsilon\}$, as desired. $\square$

Remark 2.2.6. As a direct consequence of Proposition 2.2.5, any morphism $\sigma : \operatorname{Sp} B \to \operatorname{Sp} A$ of affinoid spaces preserves the intersection of basic open sets $X(g), f \in A$. Specifically, for $g_1, \ldots, g_r \in A$ and $X(g_1, \ldots, f_r) = X(g_1) \cap \ldots \cap X(g_r)$ and we have

$$\sigma^{-1}\left((\operatorname{Sp} A)(g_1, \ldots, f_r)\right) = (\operatorname{Sp} B)\left(\sigma^*(g_1), \ldots, \sigma^*(g_r)\right).$$

In particular, σ is continuous with respect to the canonical topology.

Affinoid subdomains

Similarly as in algebraic geometry, we need the notion of affine open subschemes in rigid geometry. Recall that we have the following result in algebraic geometry:

Proposition 2.2.7. *For an affine scheme $X = \operatorname{Spec} A$ and an affine open subset $U = \operatorname{Spec} A_U \subseteq X$, we have*

$$\operatorname{Hom}(A_U, B) = \{f : A \to B : \operatorname{Im}(\operatorname{Spec} B \to \operatorname{Spec} A) \subseteq U\}$$

for every ring B. By Yoneda's lemma, this statement is equivalent to the fact that the algebra A_U is uniquely determined by the open set U.

Proof. On the one hand, let $g : A_U \to B$ is a ring homomorphism. $U = \operatorname{Spec} A_U$ is an affine open subset of $X = \operatorname{Spec} A$, thus there is a canonical ring homomorphism $i : A \to A_U$. We consider the composition homomorphism

$$g \circ i : A \to A_U \to B,$$

the induced morphism of affine schemes

$$(g \circ i)^* : \operatorname{Spec} B \to \operatorname{Spec} A$$

obviously has the image in U.

On the other hand, let $f : A \to B$ be a ring homomorphism such that the image of induced morphism of affine schemes is a subset of U. U is an affine subscheme of X, hence it is quasi-compact and satisfies the following decomposition

$$U = \bigcup_{i=1}^{r} D(f_i)$$

where $f_1, \ldots, f_r \in A$ and $D(f_i)$ are basic open subset of X. Because $U = \operatorname{Spec} A_U$, the ideal generated by $i(f_1), \ldots, i(f_r)$ in A_U is the whole A_U. Denote $\mathfrak{f} = (f(f_1), \ldots, f(f_r))$ an ideal of B. For any prime ideal $\mathfrak{q} \subset B$, since $\operatorname{im}(\operatorname{Spec} B \to \operatorname{Spec} A) \subseteq U$, there exists some i such that $f_i \neq f^{-1}\mathfrak{q}$, in other words, $(f_1, \ldots, f_r) \neq f^{-1}\mathfrak{q}$. Hence $\mathfrak{f} \nsubseteq \mathfrak{q}$ for every prime ideal $\mathfrak{q} \subset B$. We conclude that $\mathfrak{f} = B$ and the homomorphism $f : A \to B$ factorizes through $i : A \to A_U$. $\square$

Definition 2.2.8. For $X = \operatorname{Sp} A$ an affinoid K-space, a subset U of X is called an **affinoid subdomain** if the following functor:

$$\begin{array}{rccc} h_{A,U} & : & \{\text{affinoid } K\text{-algebras}\} & \to & \textbf{Sets} \\ & & B & \mapsto & \{f : A \to B : \mathrm{im}(f : \mathrm{Sp}\, B \to \mathrm{Sp}\, A) \subseteq U\} \end{array}$$

is representable by an affinoid K-algebra A_U.

Remark 2.2.9. By [9, 3.3, 12-19], the family of affinoid subdomains (in some affinoid space) is transitive and closed under finite intersections. Otherwise, any affinoid subdomain is open with respect to the canonical topology.

Example 2.2.10. For $X = \mathrm{Sp}\, A$ an affinoid K-space, there are some special affinoid subdomains of X, by constructing their associated affinoid algebras (cf. Definition 2.2.8).

(i) A **Weierstrass domain** is a subset of type

$$X(g_1, \dots, g_r) = \{x \in X, |g_i(x)| \leqslant 1\}$$

for $g_1, \dots, g_r \in A$. We associate

$$U = X(g_1, \dots, g_r) \rightsquigarrow A_U = A\{X_1, \dots, X_r\}/(X_i - g_i).$$

(ii) A **Laurent domain** is a subset of type

$$X(g_1, \dots, g_r, h_1^{-1}, \dots, h_s^{-1}) = \{x \in X, |g_i(x)| \leqslant 1, |h_j(x)| \geqslant 1\}$$

for $g_1, \dots, g_r, h_1, \dots, h_s \in A$. We associate

$$U = X(g_1, \dots, g_r, h_1^{-1}, \dots, h_s^{-1}) \rightsquigarrow A_U = A\{X_1, \dots, X_r, Y_1, \dots, Y_s\}/(X_i - g_i, h_j Y_j - 1).$$

(iii) A **rational domain** is a subset of type

$$X\left(\frac{g_1}{g_0}, \dots, \frac{g_r}{g_0}\right) = \{x \in X, |g_i(x)| \leqslant |g_0(x)|\}$$

for $g_0, g_1, \dots, g_r \in A$ without common zeros. We associate

$$U = X\left(\frac{g_1}{g_0}, \dots, \frac{g_r}{g_0}\right) \rightsquigarrow A_U = A\{X_1, \dots, X_r\}/(g_i - g_0 X_i).$$

Sheaf-theoretic view

We construct the sheaf theory of affinoid space in a similar way as in algebraic geometry.

In this part, for X a fixed affinoid K-space, we denote by $\mathcal{O}_X$ the contravariant functor from the category of affinoid subdomains of X to the category of K-algebras. Specifically, we associate for any affinoid subdomain $U \subset X$ the corresponding affinoid K-algebra $\mathcal{O}_X(U)$. In particular, if $X = \operatorname{Sp} A$ is an affinoid K-space, we define $\mathcal{O}_X(U) = A_U$, where A_U is the K-algebra in Definition 2.2.8.

We associate for any inclusion of affinoid subdomains $U \subset V$ of X a canonical morphism between the corresponding affinoid K-algebras

$$\begin{array}{cccc} \rho_U^V & : & \mathcal{O}_X(V) & \to & \mathcal{O}_X(U) \\ & & f & \mapsto & f|_U \end{array}$$

such that for $U \subset V \subset W$ inclusion of affinoid subdomains of X the following conditions hold:

(i) $\rho_U^U = \mathrm{id}_U$.

(ii) $\rho_U^W = \rho_U^V \circ \rho_V^W$.

Those maps ρ_U^V are considered as restrictions of affinoid functions on V to affinoid functions on U.

Definition 2.2.11. The presheaf $\mathcal{O}_X$ is called the presheaf of affinoid functions on X.

Recall that $\mathcal{O}_X$ is a sheaf if for all affinoid subdomains U and all covering of U by affinoid subdomains U_i the following conditions hold:

(**Id**) (Identity) If for all $i \in I$, $g \in \mathcal{O}_X(U)$ satisfies $g|_{U_i} = 0$, then $g = 0$.

(**G**) (Gluing) If for all $i, j \in I$, $\{g_i \in \mathcal{O}_X(U_i)\}$ satisfy $g_i|_{U_i \cap U_j} = g_j|_{U_i \cap U_j}$, there exists uniquely $g \in \mathcal{O}_X(U)$ such that for all $i \in I$, $g|_{U_i} = g_i$.

Condition (**Id**) holds by [9, 4.1, Corollary 4]. In contrast, condition (**G**) is generally not satisfied as the canonical topology on X is totally disconnected; therefore $\mathcal{O}_X$ is generally not a sheaf.

Similarly as in algebraic geometry, sheaf conditions (**Id**) and (**G**) are equivalent to the fact that the following sequence is exact:

$$\mathcal{O}_X(U) \to \prod_{i \in I} \mathcal{O}_X(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{O}_X(U_i \cap U_j), \qquad (*)$$

where

$$g \mapsto (g|_{U_i})_{i\in I}\,, (g_i)_{i\in I} \mapsto \begin{cases} \left(g_i|_{U_i\cap U_j}\right)_{i,j\in I} \\ \left(g_j|_{U_i\cap U_j}\right)_{i,j\in I} \end{cases}$$

for every affinoid subdomain U and every covering of U by affinoid subdomains U_i. By [9, 3.3, Theorem 20] of Geritzen and Grauert, we can only consider all rational subdomains instead of affinoid subdomains.

In fact, condition (**G**) holds with finite covers, which is called Tate's Acyclicity:

Theorem 2.2.12 (Tate's Acyclicity Theorem, [9, 4.3, Theorem 1])**.** *For an affinoid K-space X, the exactness of the sequence ($*$) holds for all finite coverings of X by rational subdomains.*

2.3 Rigid spaces

As we have already indicated at the end of previous section, the presheaf of affinoid functions will satisfy sheaf properties if we require some additional conditions for open coverings. To obtain a generalized Tate's Acylicity Theorem for arbitrary coverings, we have to use a more "general" topology. This topology also allows us to construct objects that serve as schemes in algebraic geometry, i.e. they are glued from local affinoid charts.

Admissible topology

Definition 2.3.1. A **Grothendieck topology** $\mathcal{T}$ is defined as a pair $(\operatorname{Cat}\mathcal{T}, \operatorname{Cov}\mathcal{T})$, with the first one a category and the second one a set of families of morphisms $(U_i \to U)_{i\in I}$ in $\operatorname{Cat}\mathcal{T}$, called **coverings**, such that $\mathcal{T}$ satisfies the following conditions:

(i) Any isomorphism $U \xrightarrow{\sim} V$ in $\operatorname{Cat}\mathcal{T}$ also belongs to $\operatorname{Cov}\mathcal{T}$.

(ii) If for $i \in I$, $(U_i \to U)_{i\in I}, (V_{ij} \to U_i)_{j\in J_i} \in \operatorname{Cov}\mathcal{T}$, then the composition

$$(V_{ij} \to U_i \to U)_{i\in I, j\in J_i} \in \operatorname{Cov}\mathcal{T}.$$

(iii) If $(U_i \to U)_{i\in I} \in \operatorname{Cov}\mathcal{T}$ and for every morphism $V \to U$ in $\operatorname{Cat}\mathcal{T}$, the fiber products $U_i \times_U V$ exist in $\operatorname{Cat}\mathcal{T}$ and $(U_i \times_U V \to V)_{i\in I} \in \operatorname{Cov}\mathcal{T}$.

Remark 2.3.2. In particular, we can naturally equip a Grothendieck topology on any topological space X. Indeed, let $\operatorname{Cat}\mathcal{T}$ be the category of open subsets of X whose morphisms are all inclusions and $\operatorname{Cov}\mathcal{T}$ all open covers of open subsets of X. Moreover, there are several useful examples of Grothendieck topology in algebraic geometry such as étale topology, $fppf$-topology and $fpqc$-topology.

Based on affinoid subdomains, we can canonically define a Grothendieck topology on an affinoid space, called the **weak Grothendieck topology**, as follows. This Grothendieck topology consists of the category of affinoid subdomains of X whose morphisms are inclusions and the set of finite coverings of inclusions $(U_i \hookrightarrow U)_{i\in I}$ in X. It is clear that the presheaf $\mathcal{O}_X$ of affinoid functions is also a sheaf by Tate's Acyclicity Theorem (Theorem 2.2.12).

We can naturally add open sets and coverings to weak Grothendieck topology in order that the presheaf $\mathcal{O}_X$ satisfies the gluing condition (**G**) as follows.

Definition 2.3.3. The **strong Grothendieck topology** (or **admissible topology**) on an affinoid K-space X is defined as follows.

(i) A subset U of X is called **admissible open** if we can choose a covering of U by (infinite) affinoid subdomains $U_i \subset X, i \in I$ such that for all $\phi : Z \to X$ morphisms of affinoid K-spaces such that $\phi(Z) \subset U$, the covering $\{\phi^{-1}(U_i)\}_{i\in I}$ of Z admits a finite subcovering of Z by affinoid subdomains.

(ii) A covering of some admissible open subset V of X by (infinite) admissible open sets $V_j, j \in J$ is called **admissible** if for each $\phi : Z \to X$ morphism of affinoid K-spaces such that $\phi(Z) \subset V$, the covering $\{\phi^{-1}(V_j)\}_{j\in J}$ of Z admits a finite subcovering of Z by affinoid subdomains.

Remark 2.3.4. We state without proofs some properties of admissible topology.

(i) The admissible topology is also a Grothendieck topology,

(ii) Morphisms of affinoid spaces are continuous with respect to admissible topology,

(iii) Any finite union of affinoid subdomains is admissible open,

(iv) The admissible topology on X is finer than the Zariski topology,

(v) Any sheaf on some affinoid space with respect to the weak Grothendieck topology can be extended uniquely to a sheaf with respect to the admissible one. This result is useful particularly for the presheaf $\mathcal{O}_X$, which is a sheaf with respect to the weak Grothendieck topology by Tate's Acyclicity Theorem (Theorem 2.2.12). We call the extended sheaf of $\mathcal{O}_X$ by the **sheaf of rigid analytic functions** on X and also denote it by $\mathcal{O}_X$.

We finish this part by proposing some examples of admissible open sets and admissible coverings.

Example 2.3.5. Consider $X = \operatorname{Sp} K\{T\} =$ "$\{z \in \widehat{\overline{K}} : |z| \leqslant 1\}$" the one-dimensional disk over K and we have $\mathcal{O}_X(X) = K\{T\}$. The "unit circle"

$$C := \{x \in X : |T(x)| = 1\} = \text{"}\{t \in \widehat{\overline{K}} : |t| = 1\}\text{"}$$

is a rational domain with

$$\mathcal{O}_X(V) = K\left\{T, T^{-1}\right\} = \left\{\sum_{\nu\in\mathbb{Z}} a_n T^n : |a_n| \to 0 \text{ if } |n| \to \infty\right\}.$$

Now we consider the "open unit disk"

$$U := X \setminus C = \{x \in X : |T(x)| < 1\} = \text{"}\{t \in \widehat{\overline{K}} : |t| < 1\}\text{"}.$$

Recall that $T(x)$ is the image of T via the canonical projection $K\{T\} \to K\{T\}\mathfrak{m}_x$, with $\mathfrak{m}_x$ the maximal ideal corresponding to x. We claim that U is admissible open in X. Let $\varepsilon \in |K^\times|$ with $0 < \varepsilon < 1$ and

$$U_n := \left\{x \in X : |T(x)| \leqslant \varepsilon^{1/n}\right\} = \text{"}\{t \in \widehat{\overline{K}} : |t| \leqslant \varepsilon^{1/n}\}\text{"}$$

for each $n \in \mathbb{N}$. We obtain a cover $U = \bigcup_{n\in\mathbb{N}} U_n$ by rational subdomains of X. Let $\phi : X' = \operatorname{Sp} A' \to X$ be a morphism of affinoid varieties such that $\operatorname{im}(\phi) \subseteq U$. By Maximum Principle,

$$|\phi^*(T)|_{\sup} := \sup_{y\in X'} |\phi^*(T)(y)| = \max_{y\in X'} |\phi^*(T)(y)| = \max_{y\in X'} |T(\phi(y))| < 1.$$

It follows that $\phi^{-1}(U_n) = X'$ for any sufficiently large n. Consequently, $X = U \cup C$ is not an admissible covering.

Example 2.3.6. For an affinoid K-space $X = \operatorname{Sp} A$, $f \in A$ and $c > 0$, the following sets are admissible open subsets of X:

$$\{f(x) \neq 0\}, \{x \in X : |g(x)| \,\square\, c\}, \square \in \{\leqslant, <, =, >, \geqslant\}.$$

Consequently, finite intersections and unions of sets of these types are admissible.

Example 2.3.7. There exists a open subset $U \subset \operatorname{Sp}\{T_1, T_2\}$ with respect to canonical topology but U is not admissible open. For detailed discussion, see [18, 2.2.12].

Rigid spaces

Definition 2.3.8. A **rigid (analytic)** K**-space** is a locally ringed K-space $(X, \mathcal{O}_X)$ (with respect to admissible topology) such that

(i) the admissible topology of X satisfies the complete conditions in [9, 5.1, Proposition 5], and

(ii) there exists an admissible covering $(U_i)_{i \in I}$ of X by affinoid K-spaces $(U_i, \mathcal{O}_X|_{U_i})$.

We call a morphism of locally ringed K-spaces with respect to the admissible topology by a **morphism** of rigid K-spaces.

Remark 2.3.9. Similarly as schemes in algebraic geometry, global rigid K-spaces are constructed by gluing local pieces, see [9, 5.3, Proposition 5]. As an obvious corollary, the fiber product of two rigid K-spaces exists.

Rigid analytification

In this part, we will construct the rigid analytification of any K-scheme Z of locally of finite type. This functor is considered as the p-adic analogue of Serre's GAGA functor in the complex case.

We begin by consider the affine d-space $\mathbb{A}^d_K$. For a fixed element c in K, $|c| > 1$ we denote

$$T_d^{(i)} := K\left\{c^{-i}X_1, \ldots, c^{-i}X_d\right\},$$

where $\operatorname{Sp} T_d^{(i)}$ can be regarded as the d-dimensional ball of radius $|c|^i$. We have canonical inclusions

$$T_d = T_d^{(0)} \hookleftarrow T_d^{(1)} \hookleftarrow \cdots \hookleftarrow K[X_1, \ldots, X_d]$$

and corresponding embeddings of affinoid subdomains

$$\mathbb{B}^d_K = \operatorname{Sp} T_d^{(0)} \hookrightarrow \operatorname{Sp} T_d^{(1)} \hookrightarrow \cdots$$

Using [9, 5.3, Proposition 5] as stated in Remark 2.3.9, we can glue these balls with respect to admissible covering to obtain their "union":

$$\mathbb{A}^{d,\mathrm{an}}_K := \bigcup_{i=0}^{\infty} \operatorname{Sp} T_d^{(i)}$$

and it is defined as the rigid analytification of the affine d-space $\mathbb{A}^d_K$. Moreover, this construction is canonical:

Lemma 2.3.10. [9, 5.4, Lemma 1] *The inclusions*

$$T_d = T_d^{(0)} \hookleftarrow T_d^{(1)} \hookleftarrow \cdots \hookleftarrow K[X_1, \ldots, X_d]$$

canonically induce inclusions

$$\operatorname{Max} T_d^{(0)} \subset \operatorname{Max} T_d^{(1)} \subset \ldots \subset \operatorname{Max} K[X_1, \ldots, X_d]$$

of maximal spectra such that $\operatorname{Max} K[X_1, \ldots, X_d]$ *is the union of subspectra* $\operatorname{Max} T_d^{(i)}$.

Proof. As we have inclusions of affinoid subdomains $\operatorname{Sp} T_d^{(i)} \hookrightarrow \operatorname{Sp} T_d^{(i+1)}$, the inclusions between maximal spectra of the above affinoid K-algebras are clear.

For simplicity, we first rewrite the tuple of variables $(X_1, \ldots, X_d) =: X$.

Claim 1. Let $\mathfrak{m}$ be a maximal ideal of $K\{X\}$. Then $\mathfrak{m}_0 := \mathfrak{m} \cap K[X]$ is a maximal ideal of $K[X]$ such that $\mathfrak{m} = \mathfrak{m}_0 K\{X\}$.

Firstly, there is a commutative diagram whose horizontal homomorphisms are injective:

$$\begin{tikzcd} K[X] \arrow[r] \arrow[d] & K\{X\} \arrow[d] \\ K[X]/\mathfrak{m}_0 \arrow[r] & K\{X\}/\mathfrak{m} \end{tikzcd}$$

By Noether's Normalization Theorem (Corollary 11, Section 2.2 and Proposition 3, Section 3.1, [9]), $K\{X\}/\mathfrak{m}$ is an affinoid K-algebra of Krull dimension $d_0 \leqslant d$; otherwise, $K\{X\}/\mathfrak{m}$ is a field, hence it is finite over K. By the injection in $K\{X\}/\mathfrak{m}$, $K[X]/\mathfrak{m}_0$ is also finite over K and it follows that $\mathfrak{m}_0$ is maximal in $K[X]$. To obtain $\mathfrak{m} = \mathfrak{m}_0 K\{X\}$, we consider two injections

$$K[X]/\mathfrak{m}_0 \hookrightarrow K\{X\}/\mathfrak{m}_0 K\{X\}, \qquad K[X]/\mathfrak{m}_0 \hookrightarrow K\{X\}/\mathfrak{m}.$$

On the other hand, $K[X]$ is dense in $K\{X\}$ with respect to canonical topology of K. It follows that both these injections are surjective, hence bijective, because K-vector spaces of finite dimension are complete. We conclude that $K\{X\}/\mathfrak{m}_0 K\{X\} \cong K\{X\}/\mathfrak{m}$, hence $\mathfrak{m} = \mathfrak{m}_0 K\{X\}$ and thus the canonical map

$$\operatorname{Max} T_d^{(i)} \to \operatorname{Max} K[X]$$

is a well-defined injective homomorphism for $i = 0$ and thus for all i.

Claim 2. For any maximal ideal $\mathfrak{m}_0$ of $K[X]$, there exists $N \in \mathbb{N}$ such that $\mathfrak{m}_0 T_d^{(i)}$ is maximal in $T_d^{(i)}$ for all $i \geqslant N$.

Noether Normalization theorem for polynomial rings implies that $K[X]/\mathfrak{m}_0$ is a finite extension of K. We pick an $N \in \mathbb{N}$ such that

$$|\overline{X_j}| \leqslant |c|^N$$

for $\overline{X_j}$ is the residue class of X_j in $K[X]/\mathfrak{m}_0$. Consequently, if $i \geqslant N$, the projection $K[X] \to K[X]/\mathfrak{m}_0$ factors through $T_d^{(i)} = K\{c^{-i}X\}$ via a unique K-homomorphism

$$T_d^{(i)} \to K[X]/\mathfrak{m}_0,\ X_j \mapsto \overline{X_j}, j = 1, \ldots, d$$

The kernel $\mathfrak{m}_1$ of this map is a maximal ideal in $T_d^{(i)}$ satisfying $\mathfrak{m}_1 \cap K[X] = \mathfrak{m}_0$. Consequently,

$$\operatorname{Max} K[X] = \bigcup_{i=0}^{\infty} \operatorname{Max} T_d^{(i)},$$

which we have to show. $\square$

Let $\operatorname{Spec} K[X]/\mathfrak{p}$ be an affine K-scheme of finite type with an ideal $\mathfrak{p} \subset K[X]$ and $X = (X_1, \ldots, X_d)$, its rigid analytification can be constructed similarly by considering the maps

$$T_d^{(0)}/\mathfrak{p}T_d^{(0)} \leftarrow T_d^{(1)}/\mathfrak{p}T_d^{(1)} \leftarrow T_d^{(2)}/\mathfrak{p}T_d^{(2)} \leftarrow \ldots \leftarrow K[X]/\mathfrak{p}$$

and

$$\operatorname{Max} T_d^{(0)}/\mathfrak{p}T_d^{(0)} \hookrightarrow \operatorname{Max} T_d^{(1)}/\mathfrak{p}T_d^{(1)} \hookrightarrow \operatorname{Max} T_d^{(2)}/\mathfrak{p}T_d^{(2)} \hookrightarrow \ldots \hookrightarrow \operatorname{Max} K[X]/\mathfrak{p}$$

where the first maps

$$\operatorname{Sp} T_d^{(i)}/\mathfrak{p}T_d^{(i)} \hookrightarrow \operatorname{Sp} T_d^{(i+1)}/\mathfrak{p}T_d^{(i+1)}$$

can be interpreted as inclusions of affinoid subdomains. By Lemma 2.3.10 that all maps into $\operatorname{Max} K[X]/\mathfrak{p}$ are injective and

$$\operatorname{Max} K[X]/\mathfrak{p} = \bigcup_{i=0}^{\infty} \operatorname{Max} T_d^{(i)}/\mathfrak{p}T_d^{(i)}.$$

Also by [9, 5.3, Proposition 5] as in Remark 2.3.9, we can glue $\operatorname{Sp} T_d^{(i)}/\mathfrak{p}T_d^{(i)}$ together to obtain a rigid K-space, called the rigid analytification of $\operatorname{Spec} K[X]/\mathfrak{p}$.

For a general scheme, its rigid analytification is characterized as follows.

Definition 2.3.11. [9, 5.3, Definition-Proposition 3 and Proposition 4] Let $(Z, \mathcal{O}_Z)$ be a K-scheme of locally of finite type. There exists a rigid K-space $(Z^{\mathrm{an}}, \mathcal{O}_{Z^{\mathrm{an}}})$, called the **rigid analytification** of $(Z, \mathcal{O}_Z)$, together with a morphism of locally ringed K-spaces

$$(\mathrm{an}, \mathrm{an}^*) : (Z^{\mathrm{an}}, \mathcal{O}_{Z^{\mathrm{an}}}) \to (Z, \mathcal{O}_Z)$$

satisfying the following universal property:

For $(Y, \mathcal{O}_Y)$ a rigid K-space and a morphism of locally ringed K-spaces (σ, σ^*) : $(Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$, there exists uniquely a morphism of rigid K-spaces $(\bar{\sigma}, \bar{\sigma}^*) : (Y, \mathcal{O}_Y) \to (Z^{\mathrm{an}}, \mathcal{O}_{Z^{\mathrm{an}}})$ makes the following diagram commute:

$$\begin{tikzcd}
(Y, \mathcal{O}_Y) \arrow[rr, "(\sigma,\sigma^*)"] \arrow[dr, "(\bar{\sigma},\bar{\sigma}^*)"'] & & (Z, \mathcal{O}_Z) \\
& (Z^{\mathrm{an}}, \mathcal{O}_{Z^{\mathrm{an}}}) \arrow[ur, "(\mathrm{an},\mathrm{an}^*)"'] &
\end{tikzcd}$$

Example 2.3.12. The analytification $\mathbb{A}_K^{1,\mathrm{an}}$ of the affine line $\mathbb{A}_K^1$ is constructed by gluing affinoid spaces

$$\operatorname{Sp} T_1^{(0)} \hookrightarrow \operatorname{Sp} T_1^{(1)} \hookrightarrow \cdots$$

where $\operatorname{Sp} T_1^{(d)} = K\{c^{-n}X\}$ can be refered as the disk with radius $|c|^d$ for some $c \in K$ with $|c| > 1$. We denote the annulus with radii $|c|^i$ and $|c|^{i+1}$ by

$$\mathcal{A}^{(i)} := \operatorname{Sp} K\{c^{-(i+1)}X, c^i X^{-1}\}.$$

Then for any i, we obtain an admissible covering:

$$\operatorname{Sp} T_1^{(i+1)} = \operatorname{Sp} T_1^{(i)} \cup \mathcal{A}^{(i)}.$$

As a result, the analytification $\mathbb{A}_K^{1,\mathrm{an}}$ admits an admissible cover by affinoid subdomains:

$$\mathbb{A}_K^{1,\mathrm{an}} = \operatorname{Sp} T_1^{(0)} \cup \bigcup_{i\in\mathbb{N}} \mathcal{A}^{(i)}.$$

Example 2.3.13. Denote by $\mathbb{P}_K^d = \operatorname{Proj} K[X_0, \ldots, X_d]$ the projective d-space with variables $X_0, \ldots, X_d$ and K is not necessarily algebraically closed. Denote

$$A_i = K\left[\frac{X_0}{X_i}, \ldots, \frac{X_d}{X_i}\right]$$

for the homogeneous localization by X_0 of $K[X_0, \ldots, X_d]$. Then $\mathbb{P}_K^d$ is covered by $U_i = \operatorname{Spec} A_i \cong \mathbb{A}_K^d$. Applying the construction of rigid analytifications of affine K-schemes of finite type, we obtain an admissible cover of each U_i^{an}:

$$U_i^{\mathrm{an}} = \bigcup_{j\in\mathbb{N}} \operatorname{Sp} K\left\{c^{-j}\frac{X_0}{X_i}, \ldots, c^{-j}\frac{X_d}{X_i}\right\} \cong \mathbb{A}_K^{d,\mathrm{an}}, i = 0, \ldots, d,$$

for $c \in K$, $|c| > 1$. We claim that $\mathbb{P}_K^{d,\mathrm{an}}$ is covered by the unit balls

$$\operatorname{Sp} K\left\{\frac{X_0}{X_i}, \ldots, \frac{X_d}{X_i}\right\} \subset U_i^{\mathrm{an}}, i = 0, \ldots, d.$$

Indeed, let x be a closed point in $\mathbb{P}_K^d$ and $\kappa(x)$ its residue field, which is a finite extension of K. Obviously x can be refered as an $\kappa(x)$-point in $\mathbb{P}_K^d(\kappa(x))$; thus $x = (x_0 : \ldots : x_d)$ by homogeneous coordinates $x_i \in \kappa(x)$. Let $i \in \{0, \ldots, d\}$ such that

$$|x_i| = \max\{|x_0|, \ldots, |x_d|\},$$

with the extended absolute value on $\kappa(x)$ from the one of K. This implies

$$x \in \operatorname{Sp} K\left\{\frac{X_0}{X_i}, \ldots, \frac{X_d}{X_i}\right\}$$

and

$$\mathbb{P}_K^{d,\mathrm{an}} = \bigcup_{i=0}^{d} \operatorname{Sp} K\left\{\frac{X_0}{X_i}, \ldots, \frac{X_d}{X_i}\right\},$$

which is also an admissible cover of $\mathbb{P}_K^{d,\mathrm{an}}$.

2.4 Relation with formal geometry

In classical rigid geometry, we deal with a nonarchimedean field K. The idea of Raynaud is to extend our structures (over K) in rigid geometry to those over its valuation ring R. Firstly, we consider R-algebras $R\langle X_1, \ldots, X_n\rangle$ of restricted power series whose coefficients are in R and their quotients with respect to finitely generated ideals. Those R-algebras can be viewed as R-models of affinoid K-algebras. Specifically, for such an R-model, its generic fiber is indeed an affinoid K-algebra.

Although we can consider general adic rings as in [9, 7] in formal geometry, it is enough to work with the valuation ring R of a field K with a complete non-Archimedean absolute value $|\cdot|$, with the ideal of definition $I = (t)$ for some uniformizer $t \in R$, $0 < |t| < 1$.

Formal schemes

Firstly we recall the construction of formal schemes. Similarly as in algebraic geometry, we have to define affine formal schemes and the global ones are glued from local affine charts.

Definition 2.4.1. For an I-adically complete and separated R-algebra A, with ideal of definition IA, let $\mathfrak{X} = \operatorname{Spf} A$ be the set of all open prime ideals $\mathfrak{p} \subset A$, equivalently, each $\mathfrak{p} \in \operatorname{Spf} A$ contains some $I^n A$. Then we can naturally identify $\mathfrak{X}$ with the closed subset $\operatorname{Spec} A/IA \subset \operatorname{Spec} A$. The topology on $\operatorname{Spf} A$ is induced from the Zariski topology of $\operatorname{Spec} A$. We can define $\mathcal{O}_{\mathfrak{X}}$ be the sheaf of (topological) R-algebras based on basic open subsets:

$$\mathcal{O}_{\mathfrak{X}} : D(g) \mapsto A\{g^{-1}\} = \varprojlim_n \left((A/I^n A)\,[g^{-1}]\right)$$

The **affine formal scheme** of A is defined as the locally ringed space $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$.

Definition 2.4.2. We call a **formal R-scheme** for a locally (topologically) ringed space $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ such that $\mathfrak{X}$ admits an open (formal) affine covering $(\mathfrak{X}_i, \mathcal{O}_{\mathfrak{X}}|_{\mathfrak{X}_i})$.

Example 2.4.3.

(i) For a scheme X and a quasi-coherent ideal sheaf $\mathcal{I}$ of $\mathcal{O}_X$, let Y be the closed subscheme corresponding to $\mathcal{I}$. Then

$$(\widehat{X}, \mathcal{O}_{\widehat{X}}) := \left(Y, \varprojlim_n (\mathcal{O}_X/\mathcal{I}^n)\right)$$

is a locally ringed space, called **the formal completion of X along Y**. In the affine case let $X = \operatorname{Spec} A$ and the ideal $\mathfrak{a} \subset A$ associates for $\mathcal{I}$. We obtain

$$(\widehat{X}, \mathcal{O}_{\widehat{X}}) = \operatorname{Spf}(\varprojlim_n A/\mathfrak{a}^n) = \operatorname{Spf} \hat{A}$$

where $\hat{A}$ is the $\mathfrak{a}$-adic completion of A.

(ii) In particular, we consider the case $A = R[X]$ with a complete discrete valuation ring R with residue field k and fraction field K, $X = (X_1, \ldots, X_d)$, $\mathfrak{a} = (t)$ for some non-unit $t \in R \setminus \{0\}$. Therefore $X \cong \mathbb{A}_R^d$ and there is a bijection between Y and the special fiber $\mathbb{A}_k^d$ of X. The formal completion of X along Y is

$$\widehat{X} = \operatorname{Spf} R\{X\}.$$

The generic fiber of this space

$$\mathbb{B}_K^d = \operatorname{Sp} K\{X\} = \operatorname{Sp}(R\{X\} \otimes_R K)$$

is exactly the affinoid unit ball and it admits a canonical open immersion into the rigid analytification:

$$\mathbb{B}_K^d \hookrightarrow \mathbb{A}_K^{d,\mathrm{an}}.$$

Raynaud's functor

In this part, we use a slightly modified setting of Example 2.4.3(ii), i.e. a complete (non-archimedean) valuation ring R with residue field k and fraction field K. With the above notation $|\cdot|$ for the absolute value, we denote the ideal of definition by $I = (t)$ for some uniformizer $t \in R$, $0 < |t| < 1$.

Definition 2.4.4. An R-algebra A is called **of topologically finite type** if it is isomorphic to an R-algebra of type $R\{X_1, \ldots X_d\}/\mathfrak{a}$ that is endowed with the I-adic topology and where $\mathfrak{a}$ is an ideal in $R\{X_1, \ldots, X_d\}$.

The condition being of topologically finite type can be checked on complete localizations, i.e. $A\{f^{-1}\}$ for some A and $f \in A$, which allows us to extend these notions to formal R-schemes.

Definition 2.4.5. A formal R-scheme $\mathfrak{X}$ is called **locally of topologically finite type** if there exists an open covering $(\mathfrak{X}_i)_{i\in I}$ of $\mathfrak{X}$ by affine formal spectra $\mathfrak{X}_i = \operatorname{Spf} A_i$ of R-algebras of topologically finite type.

For the "generic fiber" functor from the category of formal R-schemes to the category of rigid K-spaces, we first consider the affine case:

$$\mathtt{an} : \mathfrak{X} = \operatorname{Spf} A \mapsto \mathfrak{X}^{\mathrm{an}} = \operatorname{Sp}(A \otimes_R K)$$

where we claim that $A \otimes_R K$ is an affinoid K-algebra. Indeed, for an R-algebra of topologically finite type A, it can be written as a quotient $R\{X_1, \dots, X_d\}/\mathfrak{a}$. Localizing by $S = R \setminus \{0\}$, we obtain

$$A \otimes_R K = S^{-1}(R\{X_1, \dots, X_d\})/(\mathfrak{a}),$$

We only have to check $S^{-1}(R\{X_1, \dots, X_d\}) = K\{X_1, \dots, X_d\}$. Firstly we have obvious inclusions

$$R\{X_1, \dots, X_d\} \subset S^{-1}\left(R\{X_1, \dots, X_d\}\right) \subset K\{X_1, \dots, X_d\}.$$

It is clearly seen that, we can choose an element $s \in S$ for each $f \in K\{X_1, \dots, X_d\}$ such that $s^{-1}f \in R\{X_1, \dots, X_d\}$ because the limit of coefficients of f tends to zero. As a result, $A \otimes_R K$ is an affinoid K-algebra and the rigid K-space $\mathfrak{X}^{\mathrm{an}} = \operatorname{Sp}(A \otimes_R K)$ is well-defined.

On the other hand, a morphism of affine formal R-schemes $\phi : \operatorname{Spf} A \to \operatorname{Spf} B$ corresponds to the R-homomorphism $\phi^* : B \to A$. Then the "generic fiber"

$$\phi^{*,\mathrm{an}} : B \otimes_R K \to A \otimes_R K$$

naturally induces a morphism

$$\phi^{\mathrm{an}} : \operatorname{Sp}(A \otimes_R K) \to \operatorname{Sp}(B \otimes_R K)$$

of affinoid K-spaces as the image of ϕ under the functor $\mathtt{an}$.

Moreover, we can prove that this functor commutes with complete localization, hence preserves basic open subsets. This implies that $\mathtt{an}$ preserves open immersions, which is the key to extend the functor $\mathtt{an}$ to global formal R-schemes. For detailed discussion, see the part before [9, 7.4, Proposition 3].

The above discussion is the proof of the following result.

Proposition 2.4.6. [9, 7.4, Proposition 3] *With the notions R, K as above, the "generic fiber" functor*

$$A \mapsto A \otimes_R K$$

induces a functor from the category of formal R-schemes that are locally of topologically finite type, to the category of rigid K-spaces

$$\mathfrak{X} \mapsto \mathfrak{X}^{\mathrm{an}}.$$

In general, there is a classification of all formal schemes whose generic fiber can be regarded as a rigid analytic space, called the **Raynaud's generic fiber**, see [9, 8.4, Theorem 3].

Chapter 3

Deligne-Katz correspondence for overconvergent isocrystals

In this chapter, we will consider isocrystals, particularly overconvergent isocrystals, which plays as a special class of p-adic differential modules. Roughly speaking, crystals (or more properly isocrystals) are the p-adic analogues of locally constant sheaves in topology, locally free sheaves in sheaf cohomology, lisse sheaves in étale cohomology or local systems in de Rham cohomology.

In this chapter, we use the following notations.

- k be a perfect field of characteristic $p > 0$.
- K be a field of characteristic 0, complete with respect to a discrete valuation, with residue field k.
- $\mathcal{O}_K$ is the ring of integers of K.
- $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_K$.
- $\varphi : K \to K$ is a continuous automorphism induced from the absolute Frobenius endomorphism on k, which is usually called the Frobenius lift on K.
- K_n is the subfield of K consisting of the elements fixed by φ^n.

3.1 Convergent and overconvergent isocrystals

This notion of overconvergent isocrystals, defined by Berthelot [3], [4] is motivated by the notion of crystals or isocrystals given by Grothendieck [1] or Berthelot [2] in the construction of infinitesimal cohomology and crystalline cohomology and Monsky-Washnitzer cohomology [19].

On smooth affine schemes

Let $X = \operatorname{Spec} A$ be a smooth affine scheme of finite type over k. By [20, 4, Theorem 6], there exists a smooth affine scheme $\widetilde{X}$ of finite type over $\mathcal{O}_K$ such that $\widetilde{X} \times_{\mathcal{O}_K} k = X$. Instead of the coordinate ring of $\widetilde{X}$ due to its dependence of the lift, we consider its unique p-adic completion $\widehat{A}$, called the **complete lift** of A. We can write $\widehat{A} = \mathcal{O}_K\{x_1, \dots, x_d\}/(f_1, \dots, f_m)$, where $\mathcal{O}_K\{x_1, \dots, x_d\}$ is the $\mathcal{O}_K$-algebra of restricted power series. Specifically, the latter $\mathcal{O}_K$-algebra consists of power series converging in the unit ball $\{|x_1| \leqslant 1, \dots, |x_d| \leqslant 1\}$, as analogous to Definition 2.1.6. This $\mathcal{O}_K$-algebra is also considered as the p-adic completion of $\mathcal{O}_K[x_1, \dots, x_d]$.

We will follow the construction of the module of Kähler differentials (Proposition 1.1.3) and the module with connection (Definition 1.3.2).

Denote by $\widehat{A}_K$ the affinoid algebra $\widehat{A} \otimes_{\mathcal{O}_K} K$ (cf. Definition 2.1.8). Let $\widehat{I}$ be the ideal of the complete tensor product $\widehat{A}_K \widehat{\otimes}_K \widehat{O}_K$ which is the kernel of the multiplication map $a \otimes b \mapsto ab$. Denote $\widehat{\Omega}^1_{A_K/K} = \widehat{I}/\widehat{I}^2$, which is also a $\widehat{A}_K$-module. In terms of finiteness of A, if $\widehat{A} \cong \mathcal{O}_K\{x_1, \dots, x_d\}/(f_1, \dots, f_m)$, $\widehat{\Omega}^1_{A_K/K}$ is the quotient of the free $\widehat{A}_K$-module generated by $dx_1, \dots, dx_d$ by the submodule generated by $df_1, \dots, df_m$. Let $\widehat{\Omega}^i_{A_K/K}$ be the i-th exterior power of $\widehat{\Omega}^1_{A_K/K}$.

Definition 3.1.1. A **module with formal connection** over X is a finite locally free $\widehat{A}_K$-module M endowed with an integrable connection

$$\nabla : M \to M \otimes_{\widehat{A}_K} \widehat{\Omega}^1_{A_K/K}.$$

As an analogue of Definition 1.3.4, an integrable connection means that ∇ is a K-linear homomorphism satisfying the Leibniz rule

$$\nabla(am) = m \otimes da + a\nabla m, a \in \widehat{A}_K, m \in M$$

such that the homomorphisms

$$0 \to M \to M \otimes \widehat{\Omega}^1_{A_K/K} \to M \otimes \widehat{\Omega}^2_{A_K/K} \to \cdots$$

induced by ∇ forms a complex of K-vector spaces.

Definition 3.1.2. With local étale coordinates $(x_1, \dots, x_d)$ on X as above, denote by

$$\nabla_{\partial_{x_i}} : M \to M$$

the corresponding derivation of $\frac{\partial}{\partial x_i}$. Obviously $\nabla_{\partial_{x_i}}$'s mutually commute due to the integrability of ∇. Then ∇ is called **convergent** if for $m \in M, a_1, \dots, a_d \in \widehat{A}$ with $|a_i| < 1$ and $c_{\mathbf{n}} \in \widehat{A}$ for each n-tuple $\mathbf{n} = (n_1, \dots, n_d)$ of nonnegative integers, the series

$$\sum c_{\mathbf{n}} a_1^{n_1} \cdots a_d^{n_d} \frac{\nabla_{\partial_{x_1}}^{n_1} \cdots \nabla_{\partial_{x_d}}^{n_d}(M)}{n_1! \cdots n_d!}$$

converges to an element of M.

Definition 3.1.3. A **convergent isocrystal** over X is a module with formal connection (M, ∇) such that ∇ is convergent.

Remark 3.1.4.

(i) This definition depends on the choice of local coordinates. Although convergent isocrystals can be defined regardless to this choice, it is unnecessary to refer it in this thesis.

(ii) The complex

$$0 \to M \to M \otimes \widehat{\Omega}^1_{A_K/K} \to M \otimes \widehat{\Omega}^2_{A_K/K} \to \cdots$$

is the **de Rham complex** of M and its cohomology is the **convergent cohomology** of X with coefficients in M.

(iii) However, finite-dimensionality of convergent cohomology does not hold. For instance, we consider $(K\{t\}, d)$ the trivial isocrystal on $\mathbb{A}^1$, i.e. $K\{t\}$ is a module with trivial connection. Obviously the map $d : K\{t\} \to \widehat{\Omega}^1_{K\{t\}/K}$ is not surjective. Indeed, for a sequence $(a_i) \subset k$ of which infinitely many are zero, the differential form $\sum_i a_i p^i t^{p^i - 1} dt$ is not exact.

By the above remark, one of the reasons why convergent isocrystals is quite inconvenient is the infinite-dimensionality of cohomology modules. We consider a refined notion, which plays a central role in this thesis, as follows.

Definition 3.1.5. A K-algebra A equipped with a nonarchimedean absolute value $|\cdot|$ is called **weakly complete** (or **of Monsky-Washnitzer type**) if for any $f_1, \dots, f_d \in A$ with $|f_i| < 1$ and any $c_{\mathbf{n}} \in A$ for each d-tuple $\mathbf{n} = (n_1, \dots, n_d)$ of nonnegative integers with $|c_{\mathbf{n}}| \geqslant 1$, the sum

$$\sum_{\mathbf{n}} c_{\mathbf{n}} f_1^{n_1} \cdots f_d^{n_d}$$

converges under $|\cdot|$ to an element of A.

Remark 3.1.6.

(i) A complete algebra is weakly complete, but the converse is generally not true.

(ii) We denote by $\mathcal{O}_K[x_1, \ldots, x_d]^\dagger$ the weakly complete lift of $\mathcal{O}_K[x_1, \ldots, x_d]$, for the algebra of formal power series that there exists an $\eta > 1$ so that the series converges for $|x_1|, \ldots, |x_d| \leqslant \eta$. This algebra is weakly complete but not complete.

The weakly complete lift of an algebra of finite type can be defined in a similar way as follows. Let $X = \operatorname{Spec} A$ be a smooth affine scheme of finite type over k. By [20, 4, Theorem 6], there is a smooth affine scheme $\widetilde{X}$ of finite type over $\mathcal{O}_K$ with $\widetilde{X} \times_{\mathcal{O}_K} k = X$. Thus the weakly complete lift $A^\dagger$ of A exists and is also unique up to a noncanonical isomorphism. Specifically, if $A = k[x_1, \ldots, x_d]/(f_1, \ldots, f_m)$, $A^\dagger = \mathcal{O}_K[x_1, \ldots, x_d]^\dagger/(f_1, \ldots, f_m)$.

We also follow the construction of the module of Kähler differentials (Proposition 1.1.3) and the module with connection (Definition 1.3.2).

Denote $A_K^\dagger$ by $A^\dagger \otimes_{\mathcal{O}_K} K$. Let $I^\dagger$ be the ideal of weakly complete tensor product $A_K^\dagger \otimes_K^\dagger A_K^\dagger$, which is the kernel of the multiplication map $a \otimes b \mapsto ab$ and put $\Omega^{1,\dagger}_{A_K/K} = I^\dagger/(I^\dagger)^2$.

Definition 3.1.7. An **overconvergent isocrystal** over X is a finite locally free $A_K^\dagger$-module M equipped with an overconvergent integrable connection

$$\nabla : M \to M \otimes_{A_K^\dagger} \Omega^{1,\dagger}_{A_K/K},$$

where the overconvergence of ∇ is defined similarly as in Definition 3.1.2 unless using the weakly completeness of ∇ on some étale coordinates.

Remark 3.1.8.

(i) Similarly as with convergent isocrystals, we can define de Rham cohomology of an overconvergent isocrystal M, which is called the overconvergent cohomology, or more commonly the **rigid cohomology** of X with coefficients in M.

(ii) The rigid cohomology of $\mathbb{A}^1$ with coefficients in the trivial isocrystal is finite dimensional, specifically, $H^0_{\mathrm{rig}}(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1})$ is one-dimensional and other cohomology modules are zero-dimensional.

On smooth curves

For X/k a smooth scheme, let $\mathfrak{X}/\mathcal{O}_K$ be a formally smooth formal $\mathcal{O}_K$-model of X and $\mathfrak{X}^{\mathrm{an}}$ the rigid analytic space corresponding to $\mathfrak{X}$ (which is constructed in Proposition 2.4.6).

To define the notion of overconvergent isocrystal on a general scheme, we need to recall some concepts of rigid geometry, which is too abstract in our setting of this thesis and has not been discussed in Chapter 2.

Definition 3.1.9 (Section 1.1, [3]; Chapter 2, [15])**.** If $Z \subseteq X$ is a locally closed subscheme, then the **tube** $]Z[_{\mathfrak{X}^{\mathrm{an}}}$ is the set of points of $\mathfrak{X}^{\mathrm{an}}$ whose specialization lies in Z. Specifically, the specialization morphism

$$\mathtt{sp} : \mathfrak{X}^{\mathrm{an}} \to \mathfrak{X}$$

maps the points of the rigid analytic space $\mathfrak{X}^{\mathrm{an}}$ to the closed points of the formal scheme $\mathfrak{X}$, Then the tube is defined as

$$]Z[_{\mathfrak{X}^{\mathrm{an}}} := \mathtt{sp}^{-1}(Z).$$

Suppose there is a formally smooth compactification $\overline{\mathfrak{X}}$ of $\mathfrak{X}$, i.e. a formally smooth proper formal $\mathcal{O}_K$-scheme such that $\mathfrak{X} \subseteq \overline{\mathfrak{X}}$ (notice that this assumption holds in the case that X is a smooth curve). Denote by $\overline{\mathfrak{X}}^{\mathrm{an}}$ the rigid analytic space corresponding to $\overline{\mathfrak{X}}$ and $\overline{X} = \overline{\mathfrak{X}} \otimes k$ be the corresponding smooth compactification of X.

Definition 3.1.10 ([3, 1.2.1], [15, 3])**.** A **strict neighborhood** of $\mathfrak{X}^{\mathrm{an}}$ in $\overline{\mathfrak{X}}^{\mathrm{an}}$ is an admissible open subset $V \subseteq \overline{\mathfrak{X}}^{\mathrm{an}}$ such that $\{V, \overline{\mathfrak{X}}^{\mathrm{an}} - \mathfrak{X}^{\mathrm{an}}\}$ is an admissible cover of $\overline{\mathfrak{X}}^{\mathrm{an}}$.

Definition 3.1.11. An **overconvergent isocrystal** on X is an $\mathcal{O}_{\overline{\mathfrak{X}}^{\mathrm{an}}}$-module $\mathcal{M}$ endowed with an overconvergent connection ∇ such that on some strict neighborhood V, $\mathcal{M}|_V$ is a locally free module and

$$\nabla|_V : \mathcal{M}|_V \to \mathcal{M}|_V \otimes_{\mathcal{O}_V} \Omega^1_V$$

is an integrable connection on $\mathcal{M}|_V$.

Remark 3.1.12. Explicitly, in the case that X is a smooth affine curve, let $\mathfrak{X}/\mathcal{O}_K$ be a formally smooth $\mathcal{O}_K$-model and an embedding $\mathfrak{X} \hookrightarrow \overline{\mathfrak{X}}$ with $\overline{\mathfrak{X}}$ formally smooth and proper over $\mathcal{O}_K$; $\mathfrak{X}^{\mathrm{an}}$ and $\overline{\mathfrak{X}}^{\mathrm{an}}$ their corresponding rigid analytic spaces (via Raynaud generic fibers). For $x \in \overline{X}$, we denote by t a local section of $\mathcal{O}_{\overline{\mathfrak{X}},x}$. Hence the completion of $\overline{\mathfrak{X}}$ at x is the formal spectrum $\operatorname{Spf} \mathcal{O}_K[[t]]$.

Then the tube $]x[_{\overline{\mathfrak{X}}^{\mathrm{an}}}$ can be interpreted as the "open unit disk" in $\overline{\mathfrak{X}}^{\mathrm{an}}$, whose the ring of global sections can be identified with $K[[t]]$ of formal power series:

$$]x[_{\overline{\mathfrak{X}}^{\mathrm{an}}}= \{y \in \overline{\mathfrak{X}}^{\mathrm{an}}, |t(y)| < 1\}.$$

We recall that $t(y)$ the image of t via the projection

$$K\{t\} \to K\{t\}/\mathfrak{m}_y,$$

for $\mathfrak{m}_y$ the maximal ideal corresponding to y.

An admissible open subspace V of $\overline{\mathfrak{X}}^{\mathrm{an}}$ containing $\mathfrak{X}^{\mathrm{an}}$ is a strict neighborhood of $\mathfrak{X}^{\mathrm{an}}$ if and only if for every $x \in \overline{X} - X$, V contains some annulus $\lambda \leqslant |t| < 1$ in the "unit disk" $]x[_{\overline{\mathfrak{X}}^{\mathrm{an}}}$. In fact, it is enough to consider the family of open subsets:

$$V_\lambda := \{y \in \mathfrak{X}^{\mathrm{an}}, |t(y)| \geqslant \lambda\}$$

for $\lambda < 1$.

For a locally free sheaf $\mathcal{M}$ on some strict neighborhood V of $\mathfrak{X}^{\mathrm{an}}$, a connection on $\mathcal{M}$ is overconvergent if it satisfies the following properties:

(i) For any $x \in X$, the restriction $(\mathcal{M}, \nabla)|_{]x[}$ has a full set of horizontal sections, i.e. the kernel of ∇ has a basis whose the number of elements equals to the rank of $\mathcal{M}$.

(ii) For $x \in \overline{X} - X$, the restriction $(\mathcal{M}, \nabla)|_{]x[\cap V}$ satisfies the property: for any $y \in]x[\cap V$, $(\mathcal{M}, \nabla)|_{]x[\cap V}$ has a full set of horizontal sections converging in a disk of radius $r(y) < 1$, and $r(y) \to 1$ as y goes to the boundary of $]x[$.

It is remarkable that this description of overconvergent connection is equivalent to the description stated in the previous subsection for smooth affine schemes, where it is defined based on local coordinates.

Definition 3.1.13.

(i) For $\mathcal{M}, \mathcal{N}$ overconvergent isocrystals on X/K, a morphism $\mathcal{M} \to \mathcal{N}$ is a morphism, which is compatible with connections, of locally free modules over some strict neighborhood on which the restrictions of $\mathcal{M}$ and $\mathcal{N}$ are locally free. We obtain a category $\mathrm{Isoc}^\dagger(X/K)$ of overconvergent isocrystals.

(ii) Let X_K^{an} be the rigid analytification of the K-scheme X_K (cf. Definition 2.3.11). We denote by $\mathrm{MC}^\dagger(X_K/K)$ the full subcategory of $\mathrm{MC}(X_K/K)$ of modules with overconvergent connection, i.e. locally free sheaves of $\mathcal{O}_{X_K}$-modules with connection satisfying conditions (i) and (ii) in Remark 3.1.12.

Remark 3.1.14.

(i) In fact, overconvergent isocrystals over X/K can be defined as a family of modules with overconvergent connection on frames, i.e. tuples (X, Y, P) where $X \hookrightarrow Y$ is the open immersion of k-varieties, $Y \hookrightarrow P$ is a closed immersion of formal $\mathcal{O}_K$-schemes. This thesis focuses on the Monsky-Washnitzer frame $(X, \overline{X}, \widehat{\mathbb{P}}^1_{\mathcal{O}_K})$ for X a smooth curve over k. For details, see [15, Section 8.1]. As a result, the category $\mathrm{Isoc}^\dagger(X/K)$ of overconvergent isocrystals depend only on X and K, not on the formal model $\mathfrak{X}$ and its compactification $\overline{\mathfrak{X}}$.

(ii) The category $\mathrm{MC}^\dagger(X_K/K)$ is well-defined because X_K^{an} is a subspace of $\overline{\mathfrak{X}}^{\mathrm{an}}$ and is a strict neighborhood of $\mathfrak{X}^{\mathrm{an}}$. We can check this explicitly, in particular, for our case $X = \mathbb{G}_{m,k} \hookrightarrow \mathbb{P}^1_k$ and a local section t of $\mathcal{O}_{\mathfrak{X},0}$, we have strict inclusions:

$$\mathfrak{X}_K^{\mathrm{an}} \subsetneq X_K^{\mathrm{an}} \subsetneq \overline{\mathfrak{X}}_K^{\mathrm{an}}$$

with $\mathfrak{X}_K^{\mathrm{an}}$ "=" $\{|t| = 1\}$ the unit circle, the rigid analytification X_K^{an} "=" $\bigcup_{\lambda<1}\{\lambda \leqslant |t| \leqslant 1/\lambda\}$ by Definition 2.3.11 and $\mathbb{P}_K^{1,\mathrm{an}} = \mathrm{Sp}\, K\{t\} \cup \mathrm{Sp}\, K\{1/t\}$ by Example 2.3.13.

By the above notations, we obtain the following result:

Proposition 3.1.15. *There is a canonical functor*

$$\mathrm{MC}^\dagger(X_K/K) \to \mathrm{Isoc}^\dagger(X/K)$$

given by $M \mapsto M^\dagger$ the weakly completion.

In general, this functor is not fully faithful. For counterexamples, see [21, Examples 2.3-4].

Definition 3.1.16. With the above notions K, k, we recall $\varphi : K \to K$ a Frobenius lift of the Frobenius $F : k \to k$. For a smooth curve over k, we denote by $\varphi : X \to X$ the absolute Frobenius morphism. Then there is a φ-linear functor:

$$\varphi^* : \mathrm{Isoc}^\dagger(X/K) \to \mathrm{Isoc}^\dagger(X/K).$$

An **overconvergent F-isocrystal** is defined as an overconvergent isocrystal $\mathcal{M}$ endowed with an isomorphism

$$\Phi : \varphi^*\mathcal{M} \xrightarrow{\sim} \mathcal{M}.$$

The morphism Φ is called a **Frobenius structure** of $\mathcal{M}$. Denote the category of overconvergent F-isocrystals by $F - \mathrm{Isoc}^\dagger(X/K)$.

Example 3.1.17. In this thesis, we restrict to the case $X = \mathbb{G}_{m,k}$ the affine line over k minus the origin and the smooth compactification

$$X = \operatorname{Spec} k[t, t^{-1}] \hookrightarrow \overline{X} = \mathbb{P}^1_k.$$

We also take a formally smooth model $\mathfrak{X} = \operatorname{Spf} \mathcal{O}_K\langle t, t^{-1}\rangle$ and its compactification $\overline{\mathfrak{X}} = \mathbb{P}^1_{\mathcal{O}_K}$. Let $A^\dagger$ be the weak completion of $A = \mathcal{O}_K[t, t^{-1}]$ and $\Omega_{A^\dagger} := \Omega^{1,\dagger}_{A^\dagger}$ the differential module of $A^\dagger$ in the sense of Monsky-Washnitzer, which is discussed in the part before Definition 3.1.7. In terms of strict neighborhoods of the tube $]X[_{\overline{\mathfrak{X}}^{\mathrm{an}}}$, we denote

$$A^\dagger_K := A^\dagger \otimes_{\mathcal{O}_K} K \cong \varinjlim_V \Gamma(V, \mathcal{O}_{\overline{\mathfrak{X}}^{\mathrm{an}}})$$

$$\Omega_{A^\dagger_K/K} := \Omega_{A^\dagger} \otimes_{\mathcal{O}_K} K \cong \varinjlim_V \Gamma(V, \Omega^1_V),$$

where V runs through the cofinal set of strict neighborhoods of $\mathfrak{X}^{\mathrm{an}}$ in $\overline{\mathfrak{X}}^{\mathrm{an}}$.

We denote by $\mathrm{MC}(A^\dagger_K/K)$ (resp. $\mathrm{MC}^\dagger(A^\dagger_K/K)$) the category of $A^\dagger_K$-module projective of finite type (resp. the full subcategory of $\mathrm{MC}(A^\dagger_K/K)$ of objects with connection satisfying conditions (i) and (ii) in Remark 3.1.12). Moreover, we have the following equivalence of categories, also by the construction of overconvergent isocrystals in Remark 3.1.12:

$$\mathrm{Isoc}^\dagger(X/K) \xrightarrow{\sim} \mathrm{MC}^\dagger(A^\dagger_K/K).$$

Recall that we have denoted by φ the Frobenius lift on K of the Frobenius endomorphism on k. We also denote by φ the Frobenius lift on $A^\dagger_K$ and $\mathrm{MCF}_n(A^\dagger_K/K)$ the category of $A^\dagger_K$-modules projective of finite type M with an integrable connection ∇ and a φ^n-linear endomorphism φ_n of M which commutes with ∇, i.e. the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "\nabla"] \arrow[d, "\varphi_n"'] & M \otimes_{A^\dagger_K} \Omega_{A^\dagger_K/K} \arrow[d, "\varphi_n \otimes d(\varphi^n)"] \\ M \arrow[r, "\nabla"'] & M \otimes_{A^\dagger_K} \Omega_{A^\dagger_K/K} \end{tikzcd}$$

and the linearization

$$\Phi_n : (\varphi^{n*}M, \varphi^{n*}\nabla) \to (M, \nabla)$$

is an isomorphism. Notice that for the Frobenius endomorphism φ of $A_K^\dagger$ and all $n \in \mathbb{N}$, we define

$$d(\varphi^n) : \Omega_{A_K^\dagger/K} \to \Omega_{A_K^\dagger/K}, fdt/t \mapsto \mu\varphi^n(f)dt/t$$

with $\mu = \vartheta(\varphi^n(t))/\varphi^n(t)$, $\vartheta = td/dt$.

Moreover, by [4, Theorem 2.5.7], the connection of every object in $\mathrm{MCF}_n(A_K^\dagger/K)$ is overconvergent. Consequently, we have the following equivalence of categories:

$$F - \mathrm{Isoc}^\dagger(X/K) \to \mathrm{MCF}_1(A_K^\dagger/K).$$

3.2 The Robba ring

Recall that we always consider a perfect field k of characteristic $p > 0$, a complete discrete valuation field K of characteristic 0 whose the residue field is k and $\mathcal{O}_K$ its ring of integers. We choose the normalized valuation on K, denoted by $|\cdot|$, such that $|p| = p^{-1}$.

For an interval $I \subset [0, \infty]$, we denote by $\mathcal{A}(I)$ the K-algebra of formal Laurent series:

$$\mathcal{A}(I) = \left\{ \sum_{n\in\mathbb{Z}} a_n\rho^n; a_n \in K, \forall\rho \in I, \lim_{n\to\pm\infty} |a_n|t^n = 0 \right\}.$$

In other words, $\mathcal{A}(I)$ is the set of analytic functions with coefficients in K, which are convergent in the annulus $\{|x| \in I\}$. If I is a closed interval, the $\mathcal{A}(I)$ have obvious topologies induced by the canonical topology of K. If I is open or half-open, $\mathcal{A}(I)$ is endowed by the inductive limit topology inducing from the formula

$$\mathcal{A}(I) = \bigcap_{J\subset I, J \text{ closed}} \mathcal{A}(J).$$

Definition 3.2.1. The Robba ring, denoted by $\mathcal{R}_{K,t}$ is defined to be

$$\mathcal{R}_{K,t} := \varinjlim_{\lambda<1} \mathcal{A}([\lambda, 1))$$

and regarded it as topological K-algebra given the inverse limit topology. By definition of $\mathcal{A}([\lambda, 1))$, we observe that

$$\mathcal{R}_{K,t} = \left\{ \sum_{n=-\infty}^{\infty} a_nt^n \;\middle|\; \begin{array}{l} a_n \in K \\ \forall\rho \in (0,1), |a_n|\rho^n \to 0, n \to \infty \\ \exists\lambda \in (0,1), |a_n|\lambda^n \to 0, n \to -\infty \end{array} \right\}$$

We denote by $K\langle t\rangle^\dagger$ and $\mathcal{O}_K\langle t\rangle^\dagger$ subrings of $\mathcal{R}_{K,t}$ by

$$K\langle t\rangle^\dagger = \left\{ \sum_{n=-\infty}^{\infty} a_n t^n \;\middle|\; \begin{array}{l} a_n \in K \\ \exists C > 0, \forall n > 0, |a_n| < C, \\ \exists \lambda \in (0,1), |a_n|\lambda^n \to 0, n \to -\infty \end{array} \right\},$$

$$\mathcal{O}_K\langle t\rangle^\dagger = \left\{ \sum_{n=-\infty}^{\infty} a_n t^n \;\middle|\; \begin{array}{l} a_n \in \mathcal{O}_K \\ \exists C > 0, \forall n > 0, |a_n| < C, \\ \exists \lambda \in (0,1), |a_n|\lambda^n \to 0, n \to -\infty \end{array} \right\},$$

The following result is necessary for not only this section:

Lemma 3.2.2. [22, Proposition 2.2] *Let R be a discrete valuation ring over $\mathbb{Z}_p$ whose the valuation is nonarchimedean, induced by that of $\mathbb{Z}_p$ and π its uniformizer. Assume that R satifies the condition: $\sum_{n\geqslant 0} a_n$ converges in R for any sequence $(a_n)_{n\geqslant 0}$ of elements in R satisfying $|a_n| \leqslant C\eta^n$ for some $C > 0$ and $\eta \in (0,1)$. Then (R, π) is a Henselian pair.*

Proof. By [23, Chapter XI, §2, Proposition 1], we only need to verify that for a monic polynomial $f(X) \in R[X]$ whose $\bar{f}(X)$ factors as $\bar{f}(X) = X^d(X-1)^d$, then f factors as $f = FG$ with monic polynomials G, H such that $\bar{G} = X^d$, $\bar{H} = (X-1)^d$. Let

$$f = g + X^d h, g, h \in R[X], \bar{g} = 0, \bar{h} = (X-1)^d.$$

Then h is invertible in $R[[X]]$. Let $F = -g/h = \sum_{n\geqslant 0} F_n X^n$. Since $g \in IR[X]$, $|F_n| < \eta$ for some $\eta < 1$. We claim that there exists $Q \in R[[X]]$ and $r \in R[[X]]$ such that

$$X^d - (X^d - F)Q = r, \deg(r) < d. \tag{$*$}$$

We can define $Q^{(u)} \in I^u R[[X]]$ and $r^{(u)} \in I^u R[X]$ inductively for $n = 0, 1, 2, \ldots$ by

$$Q^{(0)} = 1, r^{(0)} = 0, Q^{(u)} F = X^d Q^{(u+1)} + r^{(u+1)}, \deg(r^{(u)}) < d,$$

and we write

$$Q^{(u)} = \sum_{n\geqslant 0} Q_n^{(u)} X^n, r^{(u)} = \sum_{n\geqslant 0} r_n^{(u)} X^n.$$

Then $|Q_n^{(u)}| < \eta^u$ and $|r_n^{(u)}| < \eta^u$. Let $Q = \sum_{u\geqslant 0} Q^{(u)}$ and $r = \sum_{u\geqslant 0} r^{(u)}$. For the assumptions of R, these series converge and Q and r satisfies the equation $(*)$. Furthermore $\bar{Q} = \overline{Q^{(0)}} = 1$, so that Q is invertible in $R[[X]]$. Let $G = X^d - r$, $H = h/Q$, and we have $H \in R[X]$. This completes the proof. $\square$

By Lemma 3.2.2, $K\langle t\rangle^\dagger$ is a Henselian discrete valuation field with the ring of integers $\mathcal{O}_K\langle t\rangle^\dagger$ (with respect to Gauss norm), which is also a Henselian discrete valuation ring.

Let $E = k((t))$ be the residue field of $\mathcal{O}_K\langle t\rangle^\dagger$ and F/E a finite Galois extension, i.e. $F \cong k'((u))$ for some finite Galois extension k'/k. Since $K\langle t\rangle^\dagger$ is Henselian, there exists a finite étale extension $\mathcal{O}_\mathcal{F}$ of $\mathcal{O}_K\langle t\rangle^\dagger$ with residue field F. We denote the fraction field of $\mathcal{O}_\mathcal{F}$ by $\mathcal{F}$ and by K'/K an unramified extension of with residue field k'. Matsuda [22, Proposition 3.4] and [7, Lemma 2.2] proved that

$$\mathcal{F} \cong K'\langle u\rangle^\dagger \ \text{ and } \ \mathcal{F} \otimes_{K\langle t\rangle^\dagger} \mathcal{R}_{K,t} \cong \mathcal{R}_{K',u},$$

respectively. We use Matsuda's notations $\mathcal{F} \otimes_{K\langle t\rangle^\dagger} \mathcal{R}_{K,t}$ by $\mathcal{R}_{K,t}(F)$ or $\mathcal{R}_K(F)$ for simplicity.

Frobenius structure

With the above notion, φ is a Frobenius endomorphism of K. We also denote by φ the Frobenius lift to $\mathcal{O}_K\langle t\rangle^\dagger$ of the Frobenius endomorphism $x \mapsto x^p$ of the residue field $E = k((t))$. By Lemma 2.5, [7], φ extends uniquely to the continuous endomorphism of $\mathcal{R}_K$:

$$\varphi : \mathcal{R}_K \to \mathcal{R}_K, \varphi\left(\sum_{n\in\mathbb{Z}} a_n t^n\right) = \sum_{n\in\mathbb{Z}} \varphi(a_n)\varphi(t)^n,$$

where $\varphi^n(t) - t^q \in \pi\mathcal{R}_{K,t}$, for $q = p^n$ and π the uniformizer of K.

For a finite separable extension F of $E = k((t))$, there exists a finite extension $\mathcal{O}_\mathcal{F}$ of $\mathcal{O}_K\langle t\rangle^\dagger$ with residue field F and $\mathcal{F}$ its fraction field, as discussed in the previous section. The Henselian property of $\mathcal{O}_K\langle t\rangle^\dagger$ allows us to extends uniquely a Frobenius endomorphism of $K\langle t\rangle^\dagger$ to that of $\mathcal{F}$. Consequently, we also obtain the unique extension of φ to $\mathcal{R}_{K,t}(F) = \mathcal{F} \otimes_{K\langle t\rangle^\dagger} \mathcal{R}_{K,t}$ given by $\varphi(f \otimes g) = \varphi(f) \otimes \varphi(g)$. We can also regard this homomorphism as the Frobenius endomorphism of $\mathcal{R}_{K',u}$.

Let $\mathrm{MC}(\mathcal{R}_K/K)$ be the category of free $\mathcal{R}_K$-modules M of finite type with connection $\nabla : M \to M \otimes \Omega_{\mathcal{R}_K/K}$, where $\Omega_{\mathcal{R}_K/K}$ is defined to be $\mathcal{R}_K dt/t$.

For the unique Frobenius endomorphism φ of $\mathcal{R}_K$ and all $n \in \mathbb{N}$, we define

$$d(\varphi^n) : \Omega_{\mathcal{R}_K/K} \to \Omega_{\mathcal{R}_K/K}, f dt/t \mapsto \mu\varphi^n(f) dt/t$$

with $\mu = \vartheta_t(\varphi^n(t))/\varphi^n(t)$, $\vartheta_t = td/dt$.

Definition 3.2.3. A φ^n-structure φ_n on (M, ∇) is a φ^n-linear map $\varphi_n : M \to M$ which commutes with ∇, i.e. the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\nabla} & M \otimes_{\mathcal{R}_K} \Omega_{\mathcal{R}_K/K} \\ \varphi_n \big\downarrow & & \big\downarrow \varphi_n \otimes d(\varphi^n) \\ M & \xrightarrow[\nabla]{} & M \otimes_{\mathcal{R}_K} \Omega_{\mathcal{R}_K/K} \end{array}$$

and the linearization

$$\Phi_n = \mathrm{id} \otimes_{\varphi^n} \varphi_n : \mathcal{R}_K \otimes_{\varphi^n} M \to M$$

is an isomorphism of $\mathcal{R}_K$-modules.

Definition 3.2.4. A triple (M, ∇, φ_n) is called a (φ^n, ∇)**-module** over $\mathcal{R}_K$ if (M, ∇) is an object in $\mathrm{MC}(\mathcal{R}_K/K)$ and φ_n is a φ^n-structure on it. A morphism

$$f : (M, \nabla, \varphi_n) \to (M', \nabla', \varphi'_n)$$

is an $\mathcal{R}_K$-linear map which commutes with connections and φ^n-structures. We denote the category of (φ^n, ∇)-modules by $\mathrm{MCF}_n(\mathcal{R}_K/K)$.

Remark 3.2.5.

(i) Back to the setting of Example 3.1.17, the Robba ring can be represented as

$$\mathcal{R}_K = \varinjlim_{V} \Gamma(V \cap]0[_{\overline{\mathfrak{X}}^{\mathrm{an}}}, \mathcal{O}_{\overline{\mathfrak{X}}^{\mathrm{an}}}),$$

where V runs through the set of strict neighborhoods of $\mathfrak{X}^{\mathrm{an}}$ in $\overline{\mathfrak{X}}^{\mathrm{an}}$. Here $]0[_{\overline{\mathfrak{X}}^{\mathrm{an}}}$ denotes the tube of the point $0 \in \overline{X} - X$, which is also considered as the open unit disk at 0 in $\overline{\mathfrak{X}}^{\mathrm{an}}$ as stated in Remark 3.1.12. If we choose V runs through the strict neighborhoods V_λ stated in Remark 3.1.12, we obtain Definition 3.2.1 of Robba ring.

(ii) In terms of the ring $\mathcal{A}(I)$ of analytic functions on the annulus defined by the interval $I \subset [0, \infty]$, the algebra $A_K^\dagger$ in Example 3.1.17 can be represented as

$$A_K^\dagger = \varinjlim_{\lambda < 1} \mathcal{A}\left(\left[\lambda, \frac{1}{\lambda}\right]\right)$$

also by choosing the strict neighborhoods V_λ stated in Remark 3.1.12. Explicitly, we have

$$A_K^\dagger = \left\{ \sum_{n \in \mathbb{Z}} a_n t^n \;\middle|\; \begin{array}{l} a_n \in K \\ \exists \lambda \in (0,1), |a_n| \left(\frac{1}{\lambda}\right)^n \to 0, n \to \infty \\ \text{and } |a_n| \lambda^n \to 0, n \to -\infty \end{array} \right\}.$$

3.3 Matsuda's version of Katz correspondence for overconvergent isocrystals

Fix a parameter t of $\mathfrak{X}^{\mathrm{an}}$, we have a canonical injection $A_K^\dagger \hookrightarrow \mathcal{R}_K$ by the discussion in Example 3.1.17, Definition 3.2.1 and Remark 3.2.5. We also fix a Frobenius φ of $\mathcal{R}_K$ such that $\varphi(A_K^\dagger) \subset A_K^\dagger$. Then we have canonical functors

$$\mathrm{MC}(A_K^\dagger/K) \to \mathrm{MC}(\mathcal{R}_{\mathcal{K}}/\mathcal{K}),$$
$$\mathrm{MCF}_n(A_K^\dagger/K) \to \mathrm{MCF}_n(\mathcal{R}_{\mathcal{K}}/\mathcal{K})$$

We denote (M, ∇, φ_n) (or (M, ∇)) the object of $\mathrm{MCF}_n(R/K)$ (resp. $\mathrm{MC}(R/K)$) by M if no confusion occurs, for R denotes either $\mathcal{R}_K$ or $A_K^\dagger$.

Unipotent objects

As in above section, R denotes either $\mathcal{R}_K$ or $A_K^\dagger$. The purpose of this section is to study unipotent objects over R.

Definition 3.3.1. A free R-module (M, ∇) of finite rank with connection is called **unipotent** if (M, ∇) is a successive extension of the trivial object (R, d) by itself, i.e. the matrix of the connection ∇ corresponding to some derivation (in the sense of Definition 1.3.2) is upper-triangular.

Let $\mathrm{MC}^{\mathrm{uni}}(R/K)$ be the full subcategory of unipotent objects of $\mathrm{MC}(R/K)$. The next result is considered as an analogue of Proposition 1.6.5.

Theorem 3.3.2. [7, Theorem 4.1] *We denote by* $(\mathrm{NilpEnd}\,/K)$ *the category of pairs* (V, N) *consisting of a finite-dimensional* K*-vector space* V *endowed with a* K*-linear nilpotent endomorphism* N *of* V*. Then the functor*

$$\mathrm{NilpEnd}\,/K \to \mathrm{MC}^{\mathrm{uni}}(R/K)$$
$$(V, N) \mapsto (V \otimes_K R, \nabla_N),$$

induces an equivalence of categories, where ∇_N *is given by*

$$\nabla_N : v \otimes f \mapsto Nv \otimes \frac{dt}{t} + 1 \otimes \vartheta_t(f)\frac{dt}{t},$$

with the derivation $\vartheta_t = td/dt$.

Let $R_0 := \{\sum_{n\in\mathbb{Z}} a_n t^n, a_0 = 0\}$. Then $\vartheta_t : R_0 \to R_0$ is a bijection and similarly as in Proposition 1.6.5, the inverse functor is given by

$$(M, \nabla) \mapsto \left(\bigcup_{n\geqslant 1} \operatorname{Ker} \vartheta_t^n, \vartheta_t \right),$$

where ϑ_t in the right hand side denotes the endomorphism $\nabla(td/dt)$ of M (here M is considered as a differential module by equivalence in Definition 1.3.2).

Corollary 3.3.3. [7, Corollary 4.2] *There exists an equivalence of categories*

$$\mathrm{MC}^{\mathrm{uni}}(A_K^\dagger/K) \to \mathrm{MC}^{\mathrm{uni}}(\mathcal{R}_K/K)$$

given by $(M, \nabla) \mapsto (M \otimes_{A_K^\dagger} \mathcal{R}_K, \nabla \otimes 1)$.

Lemma 3.3.4. [7, Lemma 4.3] *Any unipotent object* (M, ∇) *admits a* φ*-structure on it.*

Proof. By Theorem 3.3.2, it is enough to consider the case $(V \otimes R, \nabla_N)$ for an r-dimensional vector space V over K and the matrix of N

$$N = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & 0 & 1 \\ 0 & & & 0 \end{pmatrix}$$

for some basis $\mathbf{v} = (v_1, \dots, v_r)$ of V. Consider φ-linear morphism φ_1 on (M, ∇) determined by $\varphi_1(\mathbf{v} \otimes 1) = (\mathbf{v} \otimes 1)A$ with

$$A = \begin{pmatrix} f_0 & f_1 & f_2 & f_3 & \cdots \\ & pf_0 & pf_1 & pf_2 & \ddots \\ & & p^2 f_0 & p^2 f_1 & \ddots \\ & & & \ddots & \ddots \\ 0 & & & & \ddots \end{pmatrix} \tag{3.3.1}$$

For $\mathbf{v} = (v_1, \dots, v_r)$ we compute:

$$\begin{aligned} \nabla_N(\varphi_1(\mathbf{v} \otimes 1)) &= \nabla_N((\mathbf{v} \otimes 1)A) \\ &= \left(\mathbf{v}N \otimes \frac{dt}{t} \right) A + \left(\mathbf{v} \otimes \frac{dt}{t} \right) \vartheta_t(A) \\ &= \left(\mathbf{v} \otimes \frac{dt}{t} \right) (\vartheta_t(A) + NA) \end{aligned}$$

and

$$\begin{aligned}(\varphi_1 \otimes d\varphi)(\nabla_N(\mathbf{v} \otimes 1)) &= (\varphi_1 \otimes d\varphi)\left(\mathbf{v}N \otimes \frac{dt}{t}\right)\\ &= \left(\mathbf{v}A \otimes \mu\frac{dt}{t}\right)\varphi(N)\\ &= \left(\mathbf{v} \otimes \frac{dt}{t}\right)\mu A\varphi(N),\end{aligned}$$

for $\mu = \vartheta_t(\varphi(t))/\varphi(t)$.

Since ∇_N and φ_1 commutes (Definition 3.2.3), we have $\vartheta_t(A) = -NA + \mu A\varphi(N)$. By [7, Lemma 2.7], $\mu = \sum_{n\in\mathbb{Z}} c_n t^n \in \mathcal{R}_K$ has $c_0 = p$. Denote $\mu' = \mu - p$, by computation on matrices, we obtain

$$\vartheta_t(f_0) = 0, \qquad \vartheta_t(f_i) = \mu' f_{i-1},\ i = 1, 2, \ldots, r-1. \tag{3.3.2}$$

We only have to show the existence of such $f_0, f_1, \ldots$ in 3.3.1 satisfying 3.3.2. Recall the subring $R_0 := \{\sum_{n\in\mathbb{Z}} c_n t^n \in \mathcal{R}_K, c_0 = 0\}$ and $\vartheta_t : R_0 \to R_0$ is an automorphism. Let I be the inverse homomorphism of ϑ_t. We claim that there exists inductively $g_i \in \mathcal{R}_K$ for $i = 1, 2, \ldots, r-1$ by

$$g_1 = \mu', \qquad g_i = \mu' I(g_{i-1}), i = 2, \ldots$$

and $g_i \in R_0$ such that for any n-tuple $\alpha_0, \ldots, \alpha_{r-1}$ of elements of K, $f_i = \alpha_0 I(g_i) + \ldots + \alpha_{i-1} I(g_1) + \alpha_i$ satisfy 3.3.2, and if $\alpha_0 \neq 0$, φ_1 is a φ-structure. For detailed computation, see the last part of the proof of this result [7, Lemma 4.3]. □

Corollary 3.3.5. [7, Corollary 4.4] *The category* $\mathrm{MC}^{\mathrm{uni}}(A_K^\dagger/K)$ *can be considered canonically as a full subcategory of* $\mathrm{MC}^\dagger(A_K^\dagger/K)$.

Remark 3.3.6. Let (P, ∇, ψ_n) be a (φ^n, ∇)-module over $\mathcal{R}_K$ and ψ_n' another φ^n-structure on (P, ∇). We have their linearizations:

$$\Psi_n := \mathrm{id}_{\mathcal{R}_K} \otimes_{\varphi^n} \psi_n, \Psi_n' := \mathrm{id}_{\mathcal{R}_K} \otimes_{\varphi^n} \psi_n'.$$

Then $\Psi_n' \circ \Psi_n^{-1}$ gives an automorphism of (P, ∇). Consequently, we have a one-to-one correspondence between the set of φ^n-structures on (P, ∇) and the set of automorphism of (P, ∇) if it is equipped a φ^n-structure.

By [7, Lemma 4.6], we have $\dim \mathrm{Hom}_\nabla(M, M) = r$ for any indecomposable unipotent R-modules M of rank r with connection (here we still denote that R is either $A_K^\dagger$ or $\mathcal{R}_K$).

Consequently, as in the proof of Lemma 3.3.4, for each choice of $(\alpha_0, \ldots, \alpha_{r-1}) \in K^r$, the above matrices 3.3.1 give all the φ^n-linear morphisms on (M, ∇).

Corollary 3.3.7. [7, Corollary 4.7] *The following subfunctor is an equivalence of categories*

$$\mathrm{MCF}_n^{\mathrm{uni}}(A_K^\dagger/K) \to \mathrm{MCF}_n^{\mathrm{uni}}(\mathcal{R}_K/K).$$

Proof. We fix (M, ∇) an unipotent object in $\mathrm{MC}(\mathcal{R}_K/K)$. By Corollary 3.3.3, there exists an $A_K^\dagger$-module with connection $(M^\dagger, \nabla^\dagger)$ in $\mathrm{MC}^{\mathrm{uni}}(A_K^\dagger/K)$ whose inverse image is (M, ∇). We are reduced to check that every φ^n-structure φ_n on M extends to $M^\dagger$. By Remark 3.3.6, such φ^n-structures on M (resp. $M^\dagger$) correspond bijectively to automorphisms of M (resp. $M^\dagger$). Also by Corollary 3.3.3, the following natural map is isomorphism:

$$\mathrm{Hom}_\nabla(M^\dagger, M^\dagger) \to \mathrm{Hom}_\nabla(M, M),$$

our assertion is clear. $\square$

Special étale covers

For a field k of characteristic zero, we recall that there is an equivalence of categories between the category of finite étale covers of $\mathbb{A}_k^1 \setminus \{0\}$ and the category of finite étale covers of the formal neighborhood at 0 (for example, [6, 15.23]). Katz[24] extended this equivalence for a general field k by introducing the notion of special étale covers.

Definition 3.3.8. For an arbitrary field k, let $\mathbb{A}_0 = \operatorname{Spec} k[t]$ the affine chart containing 0, $\mathbb{A}_\infty = \operatorname{Spec} k[t^{-1}]$ the affine chart containing ∞, $\mathbb{G}_{m,k} = \operatorname{Spec} k[t, t^{-1}]$ the multiplicative group (or the affine line minus the origin). Let $E \to \mathbb{G}_{m,k}$ be a finite étale cover. For every natural number N invertible in k, we consider

$$[\mathrm{N}] : \mathbb{G}_{m,k} \to \mathbb{G}_{m,k}, t \mapsto t^N.$$

Then the finite étale cover $U \to \mathbb{G}_{m,k}$ is called **N-tame at** 0 (resp. **N-tame at** ∞) if the pullback $[\mathrm{N}]^*U$ of U by $[\mathrm{N}]$ extends to a finite étale cover $\widetilde{U}_N$ of $\mathbb{A}_0$ (resp. of $\mathbb{A}_\infty$). Specifically, this extended cover is the normalization of $\mathbb{A}_0$ (the case for $\mathbb{A}_\infty$ is similar) in $[\mathrm{N}]^*E$:

$$\begin{array}{ccccc} U & \longleftarrow & [\mathrm{N}]^*U & \hookrightarrow & \widetilde{U}_N \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{G}_{m,k} & \xleftarrow{[\mathrm{N}]} & \mathbb{G}_{m,k} & \hookrightarrow & \mathbb{A}_0. \end{array}$$

The finite étale cover $E \to \mathbb{G}_{m,k}$ is called **tame at** 0 (resp. **tame at** ∞) if it is N-tame at 0 (resp. N-tame at ∞) for some invertible N in k.

Definition 3.3.9.

(i) For a connected scheme X and a geometric point $\bar{x}$ of X, let $U \to X$ be a finite étale cover of X. It is clearly seen that the étale fundamental group $\pi_1^{\text{ét}}(X, \bar{x})$ acts continuously on the finite set $U(\bar{x})$, which is the fiber of $\bar{x}$. The **monodromy group** of $U \to X$ at $\bar{x}$ is defined as the image of $\pi_1^{\text{ét}}(X, \bar{x})$ in $\mathrm{Aut}(U(\bar{x})$.

(ii) For a field k of arbitrary characteristic, let X be a geometrically connected scheme over k. The **geometric monodromy group** of $E \to X$ is defined as the image of the étale fundamental group $\pi_1^{\text{ét}}(X \otimes_k K, \bar{x})$ in $\mathrm{Aut}(U(\bar{x}))$ for some separably closed field extension K of k and any geometric point $\bar{x}$ of $X \otimes_k K$.

Remark 3.3.10.

(i) This definition is independent of the choice of $\bar{x}$, since we always have canonical bijections between fibers of geometric points on U, which induced the isomorphism of their monodromy groups (or geometric monodromy groups).

(ii) The definition of geometric monodromy group is also independent of the choice of K. Indeed, let k^{sep} be the separable closure of k in K. With the notion $Z \to X \otimes_k k^{\text{sep}}$ of a connected finite étale covering, then Z is a connected scheme over k^{sep}. By [25, 4.5.21], Z remains to be connected after any field extension of this separable closure k^{sep}. Then if we denote by $\bar{y}$ the image of $\bar{x}$ in $X \otimes_k k^{\text{sep}}$, the natural map of étale fundamental groups

$$\pi_1^{\text{ét}}(X \otimes_k k', \bar{x}) \to \pi_1^{\text{ét}}(X \otimes_k k^{\text{sep}}, \bar{y})$$

is surjective, which is the result we need.

(iii) For U a finite étale cover of X, there is a finite Galois extension k' of k such that the geometric monodromy group of $U \to X$ equals to the monodromy group of $U \otimes_k k' \to X \otimes_k k'$. Indeed, let G be the geometric monodromy group of this cover, then we can choose a finite étale connected G-torsor $Z \to X \otimes_k k^{\text{sep}}$ over which $U \to X$ split completely. Then we can descend this G-torsor to a finite étale connected G-torsor $Z_0 \to X \otimes_k k'$ over a finite Galois extension k' of k which still splits $U \to X$.

Definition 3.3.11. For a field k of characteristic p, the finite étale cover $U \to \mathbb{G}_{m,k}$ is called **special at** 0 (resp. **special at** ∞) if it is tame at 0 (resp. tame at ∞) and its geometric monodromy group has a unique p-Sylow subgroup.

Proposition 3.3.12. [24, Lemma 1.3.2] *With the above notions that k is a field of characteristic p and $U \to \mathbb{G}_{m,k}$ is a finite étale covering, the following statements are equivalent:*

(i) $U \to \mathbb{G}_{m,k}$ *is special at* 0*,*

(ii) there exists an integer $N \geqslant 1$*,* $\gcd(N,p) = 1$ *such that the inverse image* $[N]^*E$ *of* E *by the* N*th-power map* $[N] : \mathbb{G}_{m,k} \to \mathbb{G}_{m,k}$ *extends to a finite étale covering* $\widetilde{U}_N \to \mathbb{A}_0$ *whose geometric monodromy group is a* p*-group:*

$$\begin{tikzcd} U \arrow[d] & {[N]^*U} \arrow[l] \arrow[r, hook] \arrow[d] & \widetilde{U}_N \arrow[d] \\ \mathbb{G}_{m,k} & \mathbb{G}_{m,k} \arrow[l, "{[N]}"'] \arrow[r, hook] & \mathbb{A}_0. \end{tikzcd}$$

(iii) There exists an integer $N \geqslant 1$*,* $\gcd(N,p) = 1$ *and a finite Galois extension* k'/k *containing* N *distinct* N*th roots of unity, such that the inverse image* $[N]^*(U \otimes_k k')$ *of* $U \to \mathbb{G}_{m,k}$ *by the composition*

$$\mathbb{G}_{m,k'} \xrightarrow{[N]} \mathbb{G}_{m,k'} \to \mathbb{G}_{m,k}$$

extends to a finite étale covering $\widetilde{U}_N \otimes_k k' \to \mathbb{A}_0$ *whose monodromy group is a* p*-group.*

Similarly, the equivalence of corresponding statements for special étale covers at ∞ *of* $\mathbb{G}_{m,k}$ *also holds.*

Proof. First of all, the implication (ii)$\Rightarrow$(i) is obvious, and (i)$\Rightarrow$(ii) holds because over an algebraically closed field L, for any integer $N \geqslant 1$ prime to p, the unique open normal subgroup of $\pi_1^{\text{ét}}(\mathbb{G}_{m.L}, \bar{x})$ of index N is the one corresponding to the Nth power covering $[N] : \mathbb{G}_{m,L} \to \mathbb{G}_{m.L}$.

The equivalence (ii)$\Leftrightarrow$(iii) follows from Remark 3.3.10(iii). $\square$

By this definition, Katz established the following equivalence

Theorem 3.3.13. [24, Theorem 1.4.1] *For any field k, we denote by $k((t))$ (resp. $k((t^{-1}))$) the field of formal Laurent series over k in the variable t (resp. in the variable t^{-1}).*

(i) Via the canonical injection $k[t, t^{-1}] \hookrightarrow k((t))$, there is an equivalence between the category of special étale covers at ∞ of $\mathbb{G}_{m,k}$ and the category of finite étale covers of $\operatorname{Spec} k((t))$.

(ii) Via the canonical injection $k[t, t^{-1}] \hookrightarrow k((t^{-1}))$, there is an equivalence between the category of special étale covers at 0 *of $\mathbb{G}_{m,k}$ and the category of finite étale covers of* $\operatorname{Spec} k((t^{-1}))$.

Étale objects

Global case

The purpose of this section is to study étale objects over the multiplicative group scheme.

Definition 3.3.14. An object (M, ∇, φ_n) in $\mathrm{MCF}_n(A_K^\dagger/K)$ is called **unit-root** if there exists a sub-$A^\dagger$-module L of M, free of finite rank such that

(i) $M \cong A_K^\dagger \otimes_{A^\dagger} L$,

(ii) $\varphi_n(L) \subset L$,

(iii) $\Phi_n : \mathrm{id} \otimes \varphi_n : A^\dagger \otimes_{\varphi_n} L \to L$ is an isomorphism of $A^\dagger$-modules.

It is remarkable that, since we are using the injection from the formal neighborhood of the point 0 to $\mathbb{G}_{m,k}$, i.e.

$$\operatorname{Spec} k((t)) \hookrightarrow \mathbb{G}_{m,k}, t \mapsto t,$$

we will use the notion of special étale covers at ∞ in Definition 3.3.11.

Let $U \to X = \operatorname{Spec}(A^\dagger \otimes_{\mathcal{O}_K} k)$ be a special finite étale cover at ∞. By Lemma 3.2.2, $(A^\dagger, (\pi))$ is a Henselian pair and thus there exists a finite étale Galois extension $B^\dagger$ of $A^\dagger$ such that $U \cong \operatorname{Spec}(B^\dagger \otimes_{\mathcal{O}_K} k)$ uniquely up to isomorphism.

Recall from Section 3.2 that K_n is the subfield of K fixed by φ^n the nth-power of Frobenius endomorphism of K. We denote by $\mathrm{Rep}_{K_n}^{\mathrm{sp}}(\pi_1^{\text{ét}}(X, \bar{x}))$ the full subcategory of the category of finite-dimensional continuous representations of $\pi_1^{\text{ét}}(X, \bar{x})$ on which this étale fundamental group acts through a finite quotient corresponding to some special Galois cover of X.

For an object V in $\mathrm{Rep}^{\mathrm{sp}}_{K_n}(\pi_1^{\text{ét}}(X,\bar{x}))$, let $B^\dagger$ be the finite étale Galois extension of $A^\dagger$ corresponding to a special étale cover $U \to X$ such that $\pi_1^{\text{ét}}(X,\bar{x})$ acts on V through $\mathrm{Gal}(B^\dagger/A^\dagger)$. Let $B^\dagger_K = B^\dagger \otimes_{\mathcal{O}_K} K$. We define

$$D_{A^\dagger_K,n}(V) = (V \otimes_{K_n} B^\dagger_K)^{\pi_1^{\text{ét}}(X,\bar{x})},$$

where $\sigma \in \pi_1^{\text{ét}}(X,\bar{x})$ acts on $V \otimes_{K_n} B^\dagger_K$ by $\sigma \otimes \sigma$. Because we take the stabilizer, $D_{A^\dagger_K,n}(V)$ is independent of the choice of $B^\dagger$. We endow $D_{A^\dagger_K,n}(V)$ with φ^n-structure $\varphi_n = \mathrm{id} \otimes \varphi^n$. Here φ^n is the Frobenius endomorphism of $A^\dagger_K$ extended uniquely to $B^\dagger_K$.

Lemma 3.3.15. [7, Lemma 5.1] *$M = D_{A^\dagger_K,n}(V)$ admits uniquely a connection ∇ which commutes with φ_n. Moreover, (M, ∇, φ_n) is a unit-root object in $\mathrm{MCF}_n(A^\dagger_K/K)$.*

Proof. We first observe that M is a projective $A^\dagger_K$-module of finite type by Galois descent and hence it is free by [26, Proposition 6.1]. We also have

$$M \otimes_{A^\dagger_K} B^\dagger_K \cong V \otimes_{K_n} B^\dagger_K.$$

We can use the argument in [27, A.2.2.4] as follows to prove the existence, uniqueness and commutativity with φ_n of ∇. Every connection on M can be extended uniquely to a connection on $M \otimes_{A^\dagger_K} (A^\dagger_K)^{\mathrm{unr}}$, and this extended connection commutes with φ_n if and only if the commutativity also holds with the original connection on M. We also have

$$M \otimes_{A^\dagger_K} (A^\dagger_K)^{\mathrm{unr}} \cong V \otimes_K (A^\dagger_K)^{\mathrm{unr}}.$$

Let ∇ be such an above connection. We notice that for each $v \in V$, assume $\nabla(v \otimes 1) = \omega$, then by commutativity with φ, we have $\varphi\omega = \omega$. Thus $\omega = 0$ by similar argument as in [27, A.2.2.4]. Consequently, using Leibniz's rule,

$$\nabla(v \otimes \lambda) = v \otimes d\lambda, \forall \lambda \in (A^\dagger_K)^{\mathrm{unr}},$$

which is our desired (and unique) connection.

To check that (M, ∇, φ_n) is unit-root, as in Definition 3.3.14 we can choose $L = (V_0 \otimes_{\mathcal{O}_{K_n}} B^\dagger)^{\pi_1^{\text{ét}}(X,\bar{x})}$ for V_0 the $\mathcal{O}_{K_n}$-lattice of V. □

Definition 3.3.16.

(i) An object (M, ∇, φ_n) in $\mathrm{MCF}_n(A^\dagger_K/K)$ is called **special unit-root** if it arises from a representation of the Galois group of a special étale Galois cover of X in the above way. We denote the full subcategory of $\mathrm{MCF}_n(A^\dagger_K/K)$ of special unit-root objects by $\mathrm{MCF}_n^{\mathrm{spur}}(A^\dagger_K/K)$.

(ii) An object (M, ∇) in $\mathrm{MC}^\dagger(A_K^\dagger/K)$ is called **special étale** if it there exists a φ^n-structure φ_n on (M, ∇) for some n such that (M, ∇, φ_n) is special unit-root. We denote the full subcategory of $\mathrm{MC}^\dagger(A_K^\dagger/K)$ of special étale objects by $\mathrm{MC}^{\mathrm{se}}(A_K^\dagger/K)$.

Remark 3.3.17. By Lemma 3.3.15, we obtain a functor

$$D_{A_K^\dagger,n} : \mathrm{Rep}_{K_n}^{\mathrm{sp}}\left(\pi_1^{\text{ét}}(X, \bar{x})\right) \to \mathrm{MCF}_n^{\mathrm{spur}}(A_K^\dagger/K).$$

Lemma 3.3.18. [7, Lemma 5.2] *$D_{A_K^\dagger,n}$ is an equivalence of categories if $\mathbb{F}_{p^n} \subset k$.*

The quasi-inverse functor of $D_{A_K^\dagger,n}$ is constructed as follows. For an object (M, ∇, φ_n) in $\mathrm{MCF}_n^{\mathrm{spur}}(A_K^\dagger/K)$, we denote by V the finite representation of $\pi_1^{\text{ét}}(X, \bar{x})$ over K_n such that $M \cong D_{A_K^\dagger,n}(V)$ in $\mathrm{MCF}_n(A_K^\dagger/K)$. Take a special étale cover $U \to X$ such that $\pi_1^{\text{ét}}(X, \bar{x})$ acts on V through $\mathrm{Gal}(U/X)$ and $B^\dagger$ the finite étale extension of $A^\dagger$ corresponding to $U \to X$. We can pick U such that $W(\mathbb{F}_{p^n}) \subset B^\dagger$, where $W(\mathbb{F}_{p^n})$ is the Witt ring with residue field $\mathbb{F}_{p^n}$. Let $B_K^\dagger = B^\dagger \otimes K$. The quasi-inverse functor is defined as

$$V_{A_K^\dagger,n}(M) = (M \otimes_{A_K^\dagger} B_K^\dagger)^{\varphi_n = 1} := \left\{x \in M \otimes_{A_K^\dagger} B_K^\dagger; \varphi_n(x) = x\right\}$$

Here φ_n acts on $M \otimes B_K^\dagger$ by $\varphi_n \otimes \varphi^n$. In fact, $V_{A_K^\dagger,n}(M)$ is independent of the choice of $B^\dagger$. By definition of the action of φ_n on $V_{A_K^\dagger,n}(M)$, we have

$$V_{A_K^\dagger,n}(M) \cong V \otimes_{K_n} (B_K^\dagger)^{\varphi^n=1}.$$

It is clearly seen that $(B_K^\dagger)^{\varphi^n=1} = \{x \in B_K^\dagger, \varphi^n(x) = x\} = K_n$. Indeed, if $x \in (B_K^\dagger)^{\varphi^n=1}$, then $dx = d\varphi^n(x) = \varphi^n(dx)$. This implies $dx = 0$, as stated in the proof of Lemma 3.3.15, by a similar argument as in [27, A.2.2.4]. Thus $x \in K^{\mathrm{unr}}$ the maximal unramified extension of K in $B_K^\dagger$. Since $\mathbb{F}_{p^n} \subset k$, we have $K_n^{\mathrm{unr}} = K_n$.

Local case

In this section, we consider étale objects over the Robba ring $\mathcal{R}_K$. We recall that, for a fixed parameter t of $\mathcal{R}_K$, its residue field is $E = K((t))$ the field of formal Laurent series over K. It is also remarkable that, as in Section 3.2, we denoted by $\mathcal{R}_K(F)$ the unique finite étale extension of $\mathcal{R}_K$ for each finite separable extension F of E by [22, Proposition 3.4].

Definition 3.3.19.

(i) An object (M, ∇) in $\mathrm{MC}(\mathcal{R}_K/K)$ is called **étale** if there exists a finite separable extension F of the residue field E of $\mathcal{R}_K$ such that $(M \otimes_{\mathcal{R}_K} \mathcal{R}_K(F), \nabla \otimes \mathcal{R}_K(F))$ is trivial in $\mathrm{MC}(\mathcal{R}_K(F)/K_F)$, where K_F is the algebraic closure of K in $\mathcal{R}_K(F)$. We denote by $\mathrm{MC}^{\text{ét}}(\mathcal{R}_K/K)$ the full subcategory of $\mathrm{MC}(\mathcal{R}_K/K)$ of étale objects.

(ii) An object (M, ∇, φ_n) in $\mathrm{MCF}_n(\mathcal{R}_K/K)$ is called **unit-root** if there exists a free sub-$\mathcal{O}_K\langle t\rangle^\dagger$-module L of M such that

  (i) $M \cong L \otimes_{\mathcal{O}_K\langle t\rangle^\dagger} \mathcal{R}_K$,

  (ii) $\varphi_n(L) \subset L$,

  (iii) $1 \otimes \varphi_n : \mathcal{O}_K\langle t\rangle^\dagger \otimes_{\varphi_n} L \to L$ is an isomorphism of $\mathcal{O}_K\langle t\rangle^\dagger$-modules.

We denote by $\mathrm{MCF}_n^{\mathrm{ur}}(\mathcal{R}_K/K)$ the full subcategory of $\mathrm{MCF}_n(\mathcal{R}_K/K)$ of unit-root objects.

Lemma 3.3.20. [7, Lemma 5.3] *If $(M, \nabla) \in \mathrm{MC}(\mathcal{R}_K/K)$ is an étale object, then there exists a sufficiently large n such that (M, ∇) has a unit-root φ^n-structure.*

Proof. As (M, ∇) is an étale object, there exists a finite Galois extension F of E trivializing (M, ∇) and $G = \mathrm{Gal}(F/E)$. Let V_1 be the kernel of $\nabla \otimes \mathcal{R}_K(F)$ on $M \otimes \mathcal{R}_K(F)$. Since $(M \otimes \mathcal{R}_K(F), \nabla \otimes \mathcal{R}_K(F))$ is a module with trivial connection, V_1 is stable under the action of G and it induces a representation:

$$\rho : G \to \mathrm{GL}(V_1).$$

By Brauer's theorem [28, Theorem 24], if for some n, K_n contains some mth roots of unity for sufficiently large m, there is an equivalence between the category of representations of G over K_n and the category of representations of G over $\overline{K_n}$. Consequently, there is an equivalence between the category of representations of G over K_n and the category of representations of G over K, since K also contains that mth root of unity. If we denote by V the V_1-lattice which is stable under the action of G, we obtain

$$M \cong (V \otimes_{K_n} \mathcal{R}_K(F))^G,$$

as desired.

For the general case, let K' be an abelian extension of K such that K' contains some mth roots of unity for sufficiently large m, thus $M \otimes K'$ contains a unit-root φ^n-structure

φ_n for some n. Consequently, φ_n commutes with the action of $\mathrm{Gal}(K'/K)$ and M also has a unit-root φ^n-structure by Galois descent. □

Now we consider the case $G = \mathrm{Gal}(E^{\mathrm{sep}}/E)$. We denote by $\mathrm{Rep}^{\mathrm{fin}}_{K_n}(G)$ the category of finite-dimensional continuous representations of G on which G acts through finite quotients. Specifically, for an object V in $\mathrm{Rep}^{\mathrm{fin}}_{K_n}(G)$, take a finite Galois extension F of E such that G acts on V through $\mathrm{Gal}(F/E)$. We define

$$D_{\mathcal{R}_K,n}(V) = (V \otimes_{K_n} \mathcal{R}_K(F))^G.$$

Here $\sigma \in G$ acts on $V \otimes \mathcal{R}_K(F)$ by $\sigma \otimes \sigma$. The φ^n-structure of $D_{\mathcal{R}_K,n}(V)$ is given by $\varphi_n = \mathrm{id} \otimes \varphi^n$. Obviously $D_{\mathcal{R}_K,n}$ is independent on the choice of F. Moreover, similarly as in Lemma 3.3.15, there exists a unique connection ∇ on $M = D_{\mathcal{R}_K,n}(V)$ which commutes with φ_n; (M, ∇, φ_n) is unit-root in $\mathrm{MCF}_n(\mathcal{R}_K/K)$. For details, [7, Lemma 5.4].

Definition 3.3.21. An object (M, ∇, φ_n) in $\mathrm{MCF}^{\mathrm{ur}}_n(\mathcal{R}_K/K)$ is called **finite unit-root** if it is isomorphic to $D_{\mathcal{R}_K,n}(V)$ for some V in $\mathrm{Rep}^{\mathrm{fin}}_{K_n}(G)$. We denote the full subcategory of $\mathrm{MCF}^{\mathrm{ur}}_n(\mathcal{R}_K/K)$ by $\mathrm{MCF}^{\mathrm{fur}}_n(\mathcal{R}_K/K)$.

By Lemma 3.3.20, we can regard $D_{\mathcal{R}_K,n}$ as a functor

$$D_{\mathcal{R}_K,n} : \mathrm{Rep}^{\mathrm{fin}}_{K_n}(G) \to \mathrm{MCF}^{\mathrm{fur}}_n(\mathcal{R}_K/K).$$

Moreover, similarly as in Lemma 3.3.18:

Lemma 3.3.22. [7, Lemma 5.7] *If $\mathbb{F}_{p^n} \subset k$, $D_{\mathcal{R}_K,n}$ is an equivalence of categories.*

Specifically, if (M, ∇, φ_n) be an object in $\mathrm{MCF}^{\mathrm{fur}}_n(\mathcal{R}_K/K)$; i.e. there exists an object V in $\mathrm{Rep}^{\mathrm{fin}}_{K_n}(G)$ such that $M \cong D_{\mathcal{R}_K,n}(V)$. Let F/E be a finite Galois extension such that G acts on V by $\mathrm{Gal}(F/E)$ and the residue field of F contains $\mathbb{F}_{p^n}$. We denote

$$V_{\mathcal{R}_K,n}(M) := (M \otimes_{\mathcal{R}_K} \mathcal{R}_K(F))^{\varphi_n=1} := \{x \in M \otimes_{\mathcal{R}_K} \mathcal{R}_K(F), \varphi_n(x) = x\}.$$

Here φ_n acts on $M \otimes_{\mathcal{R}_K} \mathcal{R}_K(F)$ by $\varphi_n \otimes \varphi^n$. We endow $V_{\mathcal{R}_K,n}(M)$ with Galois action by $\mathrm{id} \otimes \sigma$ for $\sigma \in G$. It is clearly seen that $V_{\mathcal{R}_K,n}(M)$ is independent of the choice of F.

Katz correspondence for étale objects

Proposition 3.3.23. [7, [Proposition 5.10] *If $\mathbb{F}_{p^n} \subset k$, the inverse image functor*

$$\mathrm{MCF}^{spur}_n(A^\dagger_K/K) \to \mathrm{MCF}^{fur}_n(\mathcal{R}_K/K)$$

is an equivalence of categories.

This result is a corollary of Lemma 3.3.18, Lemma 3.3.22 and the following corollary of Theorem 3.3.13:

Proposition 3.3.24. [24, Corollary 1.4.7] *The induced map of fundamental groups*

$$\mathrm{Gal}(E^{sep}/E) \to \pi_1^{\acute{e}t}(X, \bar{x})$$

is an isomorphism; recall that $E = k((t))$ is the residue field of $\mathcal{R}_K$.

We want to establish Katz correspondence for étale objects, specifically the lower inverse image functor in the following commutative diagram is an equivalence of categories:

$$\begin{tikzcd} \mathrm{MCF}_n^{\mathrm{spur}}(A_K^\dagger/K) \arrow[r, "\iota'"] \arrow[d, "\text{forgetful}"'] & \mathrm{MCF}_n^{\mathrm{fur}}(\mathcal{R}_K/K) \arrow[d, "\text{forgetful}"] \\ \mathrm{MC}^{\mathrm{se}}(A_K^\dagger/K) \arrow[r, "\iota"'] & \mathrm{MC}^{\mathrm{et}}(\mathcal{R}_K/K) \end{tikzcd}$$

However, the forgetful functors

$$\mathrm{MCF}_n^{\mathrm{spur}}(A_K^\dagger/K) \to \mathrm{MC}^{\mathrm{se}}(A_K^\dagger/K),$$

$$\mathrm{MCF}_n^{\mathrm{fur}}(\mathcal{R}_K/K) \to \mathrm{MC}^{\mathrm{et}}(\mathcal{R}_K/K)$$

are faithful but not full if $K_n \subset K$, since morphisms in $\mathrm{MCF}_n^{\mathrm{spur}}(A_K^\dagger/K)$ and $\mathrm{MCF}_n^{\mathrm{fur}}(\mathcal{R}_K/K)$ cannot be "extended" to K. However, this "extension" exists via a finite unramified extension of K.

Lemma 3.3.25. [7, Lemma 5.11] *Let M and N be objects in $\mathrm{MCF}_n^{spur}(A_K^\dagger/K)$ (resp. $\mathrm{MCF}_n^{fur}(\mathcal{R}_K/K)$). Then for some finite unramified extension K' of K we have*

$$\mathrm{Hom}_{\mathrm{MCF}_n^{spur}(A_{K'}^\dagger/K')}(M', N') \otimes_{K_n'} K' \cong \mathrm{Hom}_{\mathrm{MC}^{se}(A_{K'}^\dagger/K')}(M', N')$$

$$\textit{(resp.}\ \mathrm{Hom}_{\mathrm{MCF}_n^{fur}(\mathcal{R}_{K'}/K')}(M', N') \otimes_{K_n'} K' \cong \mathrm{Hom}_{\mathrm{MC}^{se}(A_{K'}^\dagger/K')}(M', N')).$$

Here $M' = M \otimes_K K'$ and $N' = N \otimes_K K'$.

This result implies the full faithfulness of Katz correspondence for étale objects. For essential surjectivity, [7, Corollary 5.9] shows that, any object in $\mathrm{MC}^{\mathrm{et}}(\mathcal{R}_K/K)$ is isomorphic to the image under the forgetful functor of an object in $\mathrm{MCF}_n^{\mathrm{fur}}(\mathcal{R}_K/K)$ for some n. This is a corollary of Lemma 3.3.20 and [7, Theorem 5.8], which is a result of Tsuzuki [29, Theorem 4.2.6]. Combining this with the commutativity of the above diagram and the equivalence in 3.3.23, we obtain Katz correspondence for étale objects:

Corollary 3.3.26. [7, Corollary 5.12] *The inverse image functor*

$$\iota : \mathrm{MC}^{se}(A_K^\dagger/K) \to \mathrm{MC}^{et}(\mathcal{R}_K/K)$$

is an equivalence of categories.

Special objects

The notion of special objects and their Katz equivalence is the combination of results of the last two subsections.

Definition 3.3.27.

(i) An object M in $\mathrm{MC}(A_K^\dagger/K)$ is called **special** if it admits a decomposition as

$$M = \bigoplus_i P_i \otimes U_i$$

where P_i is special étale and U_i is unipotent. An object (M, ∇, φ_n) in $\mathrm{MCF}_n(A_K^\dagger/K)$ is called **special** if (M, ∇) is special as an object in $\mathrm{MC}(A_K^\dagger/K)$. The full subcategory of $\mathrm{MC}(A_K^\dagger/K)$ (resp. $\mathrm{MCF}_n(A_K^\dagger/K)$) of special objects is denoted by $\mathrm{MC}^{\mathrm{sp}}(A_K^\dagger/K)$ (resp. $\mathrm{MCF}_n^{\mathrm{sp}}(A_K^\dagger/K)$).

(ii) An object M in $\mathrm{MC}(\mathcal{R}_K/K)$ is called **special** if it admits a decomposition as

$$M = \bigoplus_j P_j \otimes U_j$$

where P_j is étale and U_j is unipotent. An object (M, ∇, φ_n) in $\mathrm{MCF}_n(\mathcal{R}_K/K)$ is called **special** if (M, ∇) is special as an object in $\mathrm{MC}(\mathcal{R}_K/K)$. The full subcategory of $\mathrm{MC}(\mathcal{R}_K/K)$ of special objects is denoted by $\mathrm{MC}^{\mathrm{sp}}(\mathcal{R}_K/K)$.

Proposition 3.3.28. *There exists an equivalence of categories*

$$\mathrm{MC}^{sp}(A_K^\dagger/K) \to \mathrm{MC}^{sp}(\mathcal{R}_K/K).$$

Quasi-unipotent objects

Definition 3.3.29. An object (M, ∇) in $\mathrm{MC}(\mathcal{R}_K/K)$ is called **quasi-unipotent** if via a finite separable extension F of the residue field $E = k((t))$ of $\mathcal{R}_K$, $(M \otimes_{\mathcal{R}_K} \mathcal{R}_K(F), \nabla \otimes$

$\mathcal{R}_K(F)$) is an unipotent object in $\mathrm{MC}(\mathcal{R}_K(F)/K_F)$ with K_F the algebraic closure of K in $\mathcal{R}_K(F)$. An object (M, ∇, φ_n) in $\mathrm{MCF}_n(\mathcal{R}_K/K)$ is called **quasi-unipotent** if (M, ∇) is quasi-unipotent as an object in $\mathrm{MC}(\mathcal{R}_K/K)$. The full subcategory of $\mathrm{MC}(\mathcal{R}_K/K)$ (resp. $\mathrm{MCF}_n(\mathcal{R}_K/K)$ of quasi-unipotent objects is denoted by $\mathrm{MC}^{\mathrm{qu}}(\mathcal{R}_K/K)$ (resp. $\mathrm{MCF}_n^{\mathrm{qu}}(\mathcal{R}_K/K)$)).

It is clearly seen that any special object in $\mathrm{MC}(\mathcal{R}_K/K)$ is quasi-unipotent. Indeed, by definition of étale objects in $\mathrm{MC}(\mathcal{R}_K/K)$ (Definition 3.3.19), for any special object in $\mathrm{MC}(\mathcal{R}_K/K)$:

$$M = \bigoplus_{i=1}^{\ell} P_i \otimes U_i$$

with P_i étale and U_i unipotent, there exists some field extension F of the residue field $E = k((t))$ such that $P_i \otimes \mathcal{R}_K(F)$ are trivial. Consequently, $M \otimes \mathcal{R}_K(F)$ is unipotent.

Following [7, 7], we will show that the converse is also true.

Proposition 3.3.30. [7, Proposition 7.4] *Suppose (M, ∇) is quasi-unipotent in* $\mathrm{MC}(\mathcal{R}_K/K)$. *If M is irreducible (i.e. M has no proper subobjects), then M is étale.*

Proof. Let (M, ∇) be a quasi-unipotent $\mathcal{R}_K$-module over $\mathcal{R}_K$ of rank n. The quasi-unipotence of M allows us to choose a finite Galois extension F of E such that $M' = M \otimes \mathcal{R}_K(F)$ is unipotent. G is denoted for the Galois group $\mathrm{Gal}(F/E)$. Then we can choose a basis $f = (f_1, \dots, f_n)$ of M' such that

$$\nabla f = fN \otimes \frac{dt}{t}$$

where for $n_1 > n_2 > \ldots > n_r$

$$N = \begin{pmatrix} N_{m_1,n_1} & & & 0 \\ & N_{m_2,n_2} & & \\ & & \ddots & \\ 0 & & & N_{m_r,n_r} \end{pmatrix}$$

and for I_{m_i} the identity matrix, N_{m_i,n_i} is the following nilpotent matrix of rank $m_i n_i$

$$N_{m_i,n_i} = \begin{pmatrix} 0 & I_{m_i} & & & 0 \\ & 0 & I_{m_i} & & \\ & & \ddots & \ddots & \\ & & & 0 & I_{m_i} \\ 0 & & & & 0 \end{pmatrix}$$

For $\sigma \in G$, let $\sigma(f) = fQ_\sigma$ with $Q_\sigma \in M_n(\mathcal{R}_K(F))$. For $\vartheta_t = t\frac{d}{dt}$ We have

$$\begin{aligned}\sigma(\nabla(f)) &= \sigma\left(fN \otimes \frac{dt}{t}\right)\\ &= \sigma(fN) \otimes \frac{dt}{t}\\ &= \sigma(f)N \otimes \frac{dt}{t}\\ &= fQ_\sigma N \otimes \frac{dt}{t}.\end{aligned}$$

and

$$\begin{aligned}\nabla(\sigma(f)) &= \nabla(fQ_\sigma)\\ &= fNQ_\sigma \otimes \frac{dt}{t} + f\vartheta_t Q_\sigma \otimes \frac{dt}{t}.\end{aligned}$$

Since σ and ∇ commute, we have

$$\vartheta_t Q_\sigma = Q_\sigma N - NQ_\sigma.$$

By computation as in [7, Lemma 7.2, Corollary 7.3], if $Q_\sigma = (q_{ij}) \in M_n(\mathcal{R}_K(F))$, we have $q_{ij} = 0$ for (i, j) such that $m_1 < i$ and $1 \leqslant j \leqslant m_1$. Let M' be the sub-$\mathcal{R}_K(F)$-module of M generated by $f_1, \ldots, f_{m_1}$, then M'_1 is stable under the action of G and the connection ∇ (since the first m_1 columns of both N and Q_σ are all zero). By Galois descent, there exists ∇-module M_1 over $\mathcal{R}_K$ such that $M'_1 = M_1 \otimes_{\mathcal{R}_K} \mathcal{R}_K(F)$. Therefore if M is irreducible, $M_1 = 0$ and $n_1 = 0$. This implies $N = 0$ and M is étale. $\square$

In the next step, we recall the following notions in [7, 7], which is analogous to the notions in [5, 2]. The strategy to prove this result follows on from the proof of classical Katz correspondence (Section 2, [5]).

For an object (M, ∇) in $\mathrm{MC}(\mathcal{R}_K/K)$, we denote its cohomology groups by

$$H^0_\nabla(M) = \operatorname{Ker} \nabla, H^1_\nabla(M) = \operatorname{Coker} \nabla.$$

For objects M, M' in $\mathrm{MC}(\mathcal{R}_K/K)$, we denote

$$\mathrm{Ext}^i_\nabla(M, M') = H^i_\nabla(\mathrm{Hom}(M, M')).$$

Consequently, we can identify $\mathrm{Ext}^0_\nabla(M, M')$ with $\mathrm{Hom}_\nabla(M, M') \cong \mathrm{Hom}_{\mathrm{MC}(\mathcal{R}_K/K)}(M, M')$ and $\mathrm{Ext}^1_\nabla(M, M')$ with the group of classes of extensions of M' by M in $\mathrm{MC}(\mathcal{R}_K/K)$.

An object (M, ∇) in $\mathrm{MC}(\mathcal{R}_K/K)$ is called **geometrically irreducible** if for any finite extension K' of K, $M \otimes_K K'$ is irreducible in $\mathrm{MC}(\mathcal{R}_K/K)$.

Lemma 3.3.31. [7, Lemma 7.7] *For* (M, ∇) *a quasi-unipotent object in* $\mathrm{MC}(\mathcal{R}_K/K)$*, there exists a finite extension* K'/K *such that* $M \otimes_K K'$ *is special.*

Sketch of the proof. We only have to check that $M \otimes_K K'$ for some extension K'/K has the form $P \otimes U$ where P is geometrically irreducible étale and U is indecomposable unipotent in $\mathrm{MC}(\mathcal{R}_{K'}/K')$. By the quasi-unipotence of M, we can choose a finite separable extension F of $E = k((t))$ such that $M \otimes_{\mathcal{R}_K} \mathcal{R}_K(F)$ is unipotent. We only have to consider the case that every irreducible subobject of M is geometrically irreducible by replacing K by an extension K', which is constructed from the extension $\mathcal{R}_K(F)$ and the existence of irreducible decomposition of the regular representation of $\mathrm{Gal}(F/E)$, as in the proof of [7, Lemma 7.7].

In the next step, we use induction on the rank of M. We denote $M' = M/P$ for some irreducible subobject P of M. The induction step is based on two following results.

Lemma 3.3.32. [7, Lemma 7.5] *For* (M, ∇) *and* (M', ∇') *quasi-unipotent geometrically irreducible objects in* $\mathrm{MC}(\mathcal{R}_K/K)$*, we have*

$$\mathrm{Ext}^i_{\nabla}(M, M') = \begin{cases} K, & \text{if } M \cong M', \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 3.3.33. [7, Lemma 7.6] *For geometrically irreducible objects* P *and* P'*, indecomposable unipotent objects* U *and* U' *in* $\mathrm{MC}^{qu}(\mathcal{R}_K/K)$*. We have*

$$\dim_K \mathrm{Ext}^i_{\nabla}(P \otimes U, P' \otimes U') = \begin{cases} \min(\mathrm{rank}\, U, \mathrm{rank}\, U'), & \text{if } P' \cong P, \\ 0, & \text{otherwise.} \end{cases}$$

for $i = 0, 1$. □

Theorem 3.3.34. [7, Theorem 7.8] *Every quasi-unipotent object in* $\mathrm{MC}(\mathcal{R}_K/K)$ *is special.*

Sketch of the proof. Let (M, ∇) be a quasi-unipotent object in $\mathrm{MC}(\mathcal{R}_K/K)$. As in the proof of Lemma 3.3.31, we can assume that M is indecomposable. The strategy here is to use induction on the rank of M. By Proposition 3.3.30, any irreducible subobject Q of M is étale. By the induction hypothesis and the argument in Lemma 3.3.31, if $N = M/Q$, then $N \cong Q \otimes U_1$ for some indecomposable unipotent object U_1. Moreover, we can choose

K'/K a finite étale Galois extension such that

$$Q' = Q \otimes K' \bigoplus_{i \in I} P'_i$$

with geometrically irreducible étale objects P'_i in $\mathrm{MC}(\mathcal{R}_{K'}/K)$ also by Lemma 3.3.31. This decomposition can be rewritten as

$$Q' = \bigoplus_{\sigma \in S} \sigma(P')$$

where S is a subset of $\mathrm{Gal}(K'/K)$. Let $M' = M \otimes_K K'$ and $U'_1 = U_1 \otimes_K K'$. Treating with indecomposability and ranks of these $\sigma(P')$ as in the proof of [7, Theorem 7.8], we can prove that

$$M' \cong \bigoplus_{\sigma \in S} \sigma(P') \otimes U' \cong Q' \otimes U'.$$

Using Galois descent, we have our decomposition for M. □

Now we can establish Katz correspondence for overconvergent isocrystals, as follows.

Theorem 3.3.35. [7, Theorem 7.15] *The inverse image functor*

$$\mathrm{MC}^{sp}(A_K^{\dagger}/K) \to \mathrm{MC}^{qu}(\mathcal{R}_K/K).$$

is an equivalence of categories.

Proof. The following result is necessary:

Lemma 3.3.36. [7, Proposition 7.11] *For étale objects Q and Q', unipotent objects U and U' in $\mathrm{MC}(\mathcal{R}_K/K)$. Then we have*

$$\mathrm{Hom}_{\nabla}(Q, Q') \otimes \mathrm{Hom}_{\nabla}(U, U') \cong \mathrm{Hom}_{\nabla}(Q \otimes U, Q' \otimes U').$$

The similar argument holds for special étale objects and unipotent objects in $\mathrm{MC}(A_K^{\dagger}/K)$.

This lemma can be proven by choosing a suitable extension K' of K; thus we are reduced to the case that Q and Q' are geometrically irreducible. This result follows directly from the calculation in the proof of Lemma 3.3.33.

Back to the main theorem, the following inverse image functor

$$\mathrm{MC}^{\mathrm{sp}}(A_K^{\dagger}/K) \to \mathrm{MC}^{\mathrm{sp}}(\mathcal{R}_K/K).$$

is an equivalence of categories by Corollary 3.3.3, Corollary 3.3.26 and the above lemma. Then the assertion follows from Theorem 3.3.34. □

Remark 3.3.37. Although Matsuda [7] constructed such a p-adic analogue of Katz correspondence, it is remarkable that this correspondence only deals with a small class of overconvergent isocrystals compared to those in classical Katz correspondence. Specifically, we recall that in the decomposition of Katz's special objects [5, Section 2], he considered rank one, regular-singular objects instead of étale objects in Matsuda's version. In the next section, we will consider some directions to extend this correspondence.

3.4 Further results

In this section, firstly we will prove that objects in both categories of Matsuda's version of Katz correspondence (Theorem 3.3.35) satisfy regular-singular conditions. Our results are based on [7, Corollary 7.13].

We denote by $\widetilde{A}^{\dagger}_K := \widetilde{A}^{\dagger} \otimes K$ the subalgebra of $\mathcal{R}_K$, where

$$\widetilde{A}^{\dagger} := \left\{ \sum_{n=0}^{\infty} a_n t^n \middle| a_n \in \mathcal{O}_K, \forall \rho < 1, |a_n|\rho^n \to 0 |a_n|\rho^n \to 0, n \to +\infty \right\}.$$

Definition 3.4.1. An object (M, ∇) in $\mathrm{MC}(\mathcal{R}_K/K)$ is called **regular** if there exists a submodule of $\widetilde{M}$ over $\widetilde{A}^{\dagger}_K$ such that

$$M \cong \widetilde{M} \otimes_{\widetilde{A}^{\dagger}_K} \mathcal{R}_K$$

and the restriction of ∇ to $\widetilde{M}$ is a logarithmic connection (cf. Theorem 1.5.7)

$$\nabla^{\log}_{\mathcal{R}} : \widetilde{M} \to \widetilde{M} \otimes \Omega_{\widetilde{A}^{\dagger}_K}$$

with the differential module with logarithmic pole at $t = 0$:

$$\Omega_{\widetilde{A}^{\dagger}_K} := \widetilde{A}^{\dagger}_K \frac{dt}{t}$$

Proposition 3.4.2. *If M is a quasi-unipotent module with connection over $\mathcal{R}_K$, M is regular.*

Proof. By Theorem 3.3.34, there is a decomposition

$$M = \bigoplus_{i \in I} Q_i \otimes U_i$$

with étale objects Q_i and unipotent objects U_i. By Theorem 3.3.2, there exists a basis $\mathbf{e} = (e_1, \ldots, e_{n_i})$ of U_i such that

$$\nabla \mathbf{e} = \mathbf{e} C \otimes \frac{dt}{t}$$

with $C \in M_{n_i}(K)$, n_i the rank of U_i. Thus we can easily choose sub-$\widetilde{A}_K^\dagger$-modules $\widetilde{U}_i$ of U_i such that

$$\nabla(\widetilde{U}_i) \subset \widetilde{U}_i \otimes \Omega_{\widetilde{A}_K^\dagger}.$$

We can equip each Q_i with a φ^n-structure $\varphi_{n,i}$ for some positive integer n.

By the quasi-unipotence of M, there exists a finite Galois extension F of the residue field $E = k((t))$ of $\mathcal{R}_K$ such that every $(Q_i, \nabla, \varphi_{n,i}) \otimes \mathcal{R}_K(F)$ is trivial. Using the discussion of finite étale Galois extensions of the Robba ring after Lemma 3.2.2, there exists a finite Galois extension $\widetilde{B}^\dagger$ of $\widetilde{A}^\dagger$ with the Galois group $\mathrm{Gal}(F/E)$. Let $\widetilde{B}_K^\dagger = \widetilde{B}^\dagger \otimes_{\mathcal{O}_K} K$. We define the sub $\widetilde{A}_K^\dagger$-module $\widetilde{Q}_i$ of Q_i by

$$\widetilde{Q}_i = (V_{\mathcal{R}_K,n}(Q_i) \otimes_{K_n} \widetilde{B}_K^\dagger)^G,$$

with K_n the subfield of K fixed by the nth power of Frobenius endomorphism φ of K and $V_{\mathcal{R}_K,n}$ the functor defined in Lemma 3.3.22. It is remarkable that the condition that $\mathbb{F}_{p^n} \subset k$ in Lemma 3.3.22 can be satisfied by Galois descent.

Let $\Omega_{\widetilde{B}_K^\dagger} = \Omega_{\widetilde{A}_K^\dagger} \otimes_{\widetilde{A}^\dagger} \widetilde{B}_K^\dagger$. Obviously the derivation $d : \widetilde{A}_K^\dagger \to \Omega_{\widetilde{A}_K^\dagger}$ extends uniquely to $d : \widetilde{B}_K^\dagger \to \Omega_{\widetilde{B}_K^\dagger}$ and hence $\widetilde{Q}_i$ has a connection

$$\nabla : \widetilde{Q}_i \to \widetilde{Q}_i \otimes \Omega_{\widetilde{A}_K^\dagger}$$

which is compatible with that of Q_i. Then $\widetilde{M} = \bigoplus \widetilde{Q}_i \otimes \widetilde{U}_i$ satisfies the condition.

$\square$

Consider two subalgebras of $A^\dagger = \mathcal{O}_K[t, t^{-1}]^\dagger$:

$$A_0^\dagger := \mathcal{O}_K[t]^\dagger \text{ and } A_\infty^\dagger := \mathcal{O}_K[t^{-1}]^\dagger.$$

Let $A_{0,K}^\dagger := A_0^\dagger \otimes K$ and $A_{\infty,K}^\dagger := A_\infty^\dagger \otimes K$

Definition 3.4.3. An object (M, ∇) in $\mathrm{MC}(A_K^\dagger/K)$ is called regular singular at 0 if there exists a submodule of M_0 over $A_{0,K}^\dagger$ such that

$$M \cong M_0 \otimes_{A_{0,K}^\dagger} A_K^\dagger$$

and the restriction of ∇ to M_0 is a logarithmic connection (cf. Theorem 1.5.7)

$$\nabla_0^{\log} : M_0 \to M_0 \otimes \Omega_{A^\dagger_{0,K}}$$

with the differential module with logarithmic pole at $t = 0$:

$$\Omega_{A^\dagger_{0,K}} := A^\dagger_{0,K}\frac{dt}{t}.$$

Similarly, an object (M, ∇) in $\mathrm{MC}(A^\dagger_K/K)$ is called regular singular at ∞ if there exists a submodule of M_∞ over $A^\dagger_{\infty,K}$ such that

$$M \cong M_\infty \otimes_{A^\dagger_{\infty,K}} A^\dagger_K$$

and the restriction of ∇ to M_∞ is a logarithmic connection (cf. 1.5.7)

$$\nabla_\infty^{\log} : M_\infty \to M_\infty \otimes \Omega_{A^\dagger_{\infty,K}}$$

with the differential module with logarithmic pole at $t = \infty$:

$$\Omega_{A^\dagger_{\infty,K}} := A^\dagger_{\infty,K}\frac{dt}{t}.$$

With the similar argument as in the proof of Proposition 3.4.2 and [7, Corollary 7.13], we obtain the following result:

Proposition 3.4.4. *If M be a special module with connection over $A^\dagger_K$, M is regular-singular at both 0 and ∞.*

Proof. We will prove that M is regular-singular at 0. The other assertion at ∞ is proved similarly.

By Definition 3.3.27(i), a special object (M, ∇) of $\mathrm{MC}(A^\dagger_K/K)$ admits a decomposition

$$M = \bigoplus_{i \in I} Q_i \otimes U_i$$

with special étale objects Q_i and unipotent objects U_i. By Theorem 3.3.2, there is a basis $\mathbf{e} = (e_1, \dots, e_{n_i})$ of U_i such that

$$\nabla \mathbf{e} = \mathbf{e}C \otimes \frac{dt}{t}$$

with $C \in M_{n_i}(K)$, n_i the rank of U_i. Thus we can easily choose a sub-$A^\dagger_{0,K}$-module $\widetilde{U}_{i,0}$ of U_i such that

$$\nabla(\widetilde{U}_{i,0}) \subset \widetilde{U}_{i,0} \otimes \Omega_{A^\dagger_{0,K}}.$$

By Galois descent, we can assume that $\mathbb{F}_{p^n} \subset k$. By Definition 3.3.16, each Q_i is equipped with a φ^n-structure $\varphi_{n,i}$ for some positive integer n such that $(Q_i, \nabla, \varphi_{n,i})$ is special unit-root, i.e. there exists a finite-dimensional continuous representation of $\pi_1^{\text{ét}}(X, x)$ over K_n (the subfield of K fixed by the nth power of Frobenius endomorphism of K), denoted by V_i, such that

$$Q_i = (V_i \otimes_{K_n} B_K^\dagger)^{\pi_1^{\text{ét}}(X,x)}$$

for $B_K^\dagger = B^\dagger \otimes_{\mathcal{O}_K} K$, a finite étale Galois extension $B^\dagger$ of $A^\dagger$ corresponding to a special étale cover at 0, denoted by $U \to X = \operatorname{Spec} k[t, t^{-1}]$ (cf. Definition 3.3.11). Recall that the étale fundamental group acts on V via the Galois group $\operatorname{Gal}(B^\dagger/A^\dagger)$.

By the equivalence (i)$\Leftrightarrow$(iii) in Proposition 3.3.12, there exists an integer $N \geqslant 1$ and a finite Galois extension k' of k such that k' contains N distinct Nth roots of unity, such that there exists a finite étale Galois extension B_0 of $A'_0 := k'[t^{1/N}]$ satisfying $U = \operatorname{Spec}(B_0 \otimes_{A'_0} k'[t^{1/N}, t^{-1/N}])$. Let $\mathcal{O}_{K'}$ be the ring of integers of an unramified finite extension K' of K with residue field k' and $A_0'^\dagger$ a weak completion of $\mathcal{O}_{K'}[t^{1/N}]$. Since $(A_0'^\dagger, \pi)$ is a Henselian pair (by Lemma 3.2.2), there exists a finite étale covering $B_0^\dagger$ of $A_0'^\dagger$ unique up to an isomorphism such that $B_0^\dagger \otimes_{\mathcal{O}_K} k \cong B_0$. Since $B_0^\dagger$ is integrally closed in $B^\dagger = B_0^\dagger \otimes_{A_0^\dagger} A^\dagger$ and $B^\dagger$ is also finite Galois over $A^\dagger$ with Galois group $G = \operatorname{Gal}(B^\dagger/A^\dagger)$, $B_0^\dagger$ is also finite Galois over $A_0^\dagger$ with the same Galois group.

Let $B_{0,K}^\dagger := B_0^\dagger \otimes_{\mathcal{O}_K} K$. Thus we can define

$$Q_{i,0} := (V_i \otimes_{K_n} B_{0,K}^\dagger)^G.$$

Let $\Omega_{B_{0,K}^\dagger} := \Omega_{A_{0,K}^\dagger} \otimes_{A_0^\dagger} B_0^\dagger$. It is clearly seen that $d : A_{0,K}^\dagger \to \Omega_{A_{0,K}^\dagger}$ naturally extends to $d : B_{0,K}^\dagger \to \Omega_{B_{0,K}^\dagger}$ and hence $Q_{i,0}$ has a connection

$$\nabla : Q_{i,0} \to Q_{i,0} \otimes \Omega_{A_{0,K}^\dagger}$$

which is compatible with that of Q_i. Then $M_0 = \bigoplus Q_{i,0} \otimes U_{i,0}$ satisfies the condition. $\square$

Combining Theorem 3.3.35, Proposition 3.4.2 and Proposition 3.4.4, it is possible to extend Matsuda's version of Katz correspondence, as follows.

Question 3.4.5 (p-adic Deligne correspondence)**.** *There is an equivalence between the following subcategories of regular-singular objects in* $\mathrm{MC}(A_K^\dagger/K)$ *and* $\mathrm{MC}(\mathcal{R}_K/K)$*, respectively:*

$$\mathrm{MC}_{rs\ at\ 0,\ \infty}(A_K^\dagger/K) \to \mathrm{MC}_{rs}(\mathcal{R}_K/K).$$

In the next step, we state without proofs the following result, which is considered as the p-adic analogue of Katz correspondence for rank-one objects (Proposition 1.6.3).

Theorem 3.4.6. [30, Theorem 3.3.2] *The inverse image functor*

$$\mathrm{MC}(A_K^\dagger/K) \to \mathrm{MC}(\mathcal{R}_\mathcal{K}/\mathcal{K})$$

induces an equivalence between the full subcategory of $\mathrm{MC}(A_K^\dagger/K)$ *consisting of rank-one objects which are regular-singular at infinity and the full subcategory of* $\mathrm{MC}(\mathcal{R}_\mathcal{K}/\mathcal{K})$ *consisting of rank-one objects.*

Remark 3.4.7.

(i) Garnier proved Theorem 3.4.6 in an analytic approach. Specifically, he used the notions of "radius of convergence" of differential modules at the "points" $\rho \in (0,1)$ i.e. the radius of convergence of solutions of associated p-adic differential equations with respect to the semi-norms $|\cdot|_\rho$, $\rho \in (0,1)$. This approach was firstly used by Robba and completely developed by Christol-Mebkhout. For an introduction of this approach, see [31] or [32].

(ii) By the construction of Katz correspondence in characteristic zero (Section 1.6), this result allows us to expect for the Katz correspondence of larger categories compared to Matsuda's version (Theorem 3.3.35). However, this requires a p-adic analogue of Turrittin-Levelt-Jordan decomposition (Theorem 1.5.9) for differential modules over the Robba ring, which seems to be extremely difficult without restricting to differential modules with some additional structure. One of the difficulties is the existence of irreducible differential modules over $\mathcal{R}_K$ of rank $\geqslant 2$.

(iii) The well-known p-adic local monodromy theorem, which was conjectured by Crew and proved independently in different approaches by André [33], Mebkhout [34] and Kedlaya [35], is considered the biggest effort to construct a Katz correspondence for isocrystals. Specifically, this theorem states that any differential module over $\mathcal{R}_K$ endowed with a Frobenius structure is quasi-unipotent.

Conclusion

In this thesis, we have presented the following:

1. The concepts of differential modules and modules with connection. We also introduce Turrittin-Levelt-Jordan decomposition for differential modules in characteristic zero and propose the construction of Deligne-Katz correspondence in characteristic zero.

2. An introduction to rigid geometry in terms of Tate's viewpoint and Raynaud's generic fiber. Specifically, we have proposed the rigid analytification functor and the relation of rigid geometry with formal geometry via Raynaud's generic fiber.

3. The concepts of overconvergent isocrystals and Matsuda's version of Deligne-Katz correspondence for them. We have also suggested some directions to extend this result to an equivalence between larger categories.

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