BỘ GIÁO DỤC VÀ ĐÀO TẠO

VIỆN HÀN LÂM KHOA HỌC
VÀ CÔNG NGHỆ VIỆT NAM

HỌC VIỆN KHOA HỌC VÀ CÔNG NGHỆ

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SỰ TỒN TẠI, DUY NHẤT NGHIỆM VÀ PHƯƠNG PHÁP LẶP GIẢI MỘT SỐ BÀI TOÁN BIÊN CHO PHƯƠNG TRÌNH VI PHÂN PHI TUYẾN CẤP BA

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THE EXISTENCE, UNIQUENESS AND ITERATIVE METHODS FOR SOME NONLINEAR BOUNDARY VALUE PROBLEMS OF THIRD ORDER DIFFERENTIAL EQUATIONS

by

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Supervisor: Prof. Dr. NGUYEN DONG ANH

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DECLARATION OF AUTHORSHIP

I hereby declare that this thesis was carried out by myself under the guidance and supervision of Prof. Dr. Nguyen Dong Anh. The results in it are original, genuine and have not been published by any other author. The numerical experiments performed in MATLAB are honest and precise. The joint-authored publications have been granted permission to be used in this thesis by the co-authors.

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The author

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Introduction

Overview of research situation and the necessity of the research

Numerous problems in the fields of mechanics, physics, biology, environment, etc. are reduced to boundary value problems for high order nonlinear ordinary differential equations (ODE), integro-differential equations (IDE) and functional differential equations (FDE). The study of qualitative aspects of these problems such as the existence, uniqueness and properties of solutions, and the methods for finding the solutions always are of interests of mathematicians and engineers. One can find exact solutions of the problems in a very small number of special cases. In general, one needs to seek their approximations by approximate methods, mainly numerical methods. Below we review some important topics in the above field of nonlinear boundary value problems and justify why we select problems for studying in this thesis.

a) Existence of solutions and numerical methods for two-point third order nonlinear boundary value problems

High order differential equations, especially third order and fourth order differential equations describe many problems of mechanics, physics and engineering such as bending of beams, heat conduction, underground water flow, thermoelasticity, plasma physics and so on [1, 2, 3, 4]. The study of qualitative aspects and solution methods for linear problems, when the equations and boundary conditions are linear, is basically resolved. In recent years, ones draw a great attention to nonlinear differential equations. There are numerous researches on the existence and solution methods for fourth order nonlinear boundary value problems. It is worthy to mention some typical works concerning the existence of solutions and positive solutions, the multiplicity of solutions, and analytical and numerical methods for finding solutions [5, 6, 7, 8, 9, 10]. Among the contributions to the study of fourth order nonlinear boundary value problems, there are some results of Vietnamese authors (see, e.g., [11, 12, 13, 14]).

Concerning the not fully or fully third order differential equations

$$u'''(t) = f(t, u(t), u'(t), u''(t)), \quad 0 < t < 1 \tag{0.0.1}$$

there are also many researches. A number of works are devoted to the existence, uniqueness and positivity of solutions of the problems with different boundary conditions. The methods for investigating qualitative aspects of the problems are diverse, including the method of lower and upper solutions and monotone technique [7, 15, 16, 17, 18, 19], Leray-Schauder continuation principle [20], fixed point theory on cones [21], etc. It should be emphasized that in the above works there is an essential assumption that the function $f(t, x, y, z) : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}$ satisfies a Nagumo-type condition on the last two variables [22], or linear growth in x, y, z at infinity [20], or some complicated conditions including monotone increase in each of x and y [23], or one-sided Lipschitz condition in x for $f = f(t,x)$ [19] and in x, y for $f = f(t,x,y)$ [17]. Sun et al. in [24] studied the existence of monotone positive solution of the BVP for the case $f = f(u(t))$ under conditions which are difficult to be verified.

Differently from the above approaches to the third order boundary value problems, very recently Kelevedjiev and Todorov [25] using barrier strips type conditions gave sufficient conditions guaranteeing positive or non-negative, monotone, convex or concave solutions.

It should be said that in the mentioned works, no examples of solutions are shown although the sufficient conditions are satisfied and the verification of them is difficult. Therefore, it is desired to overcome the above shortcoming, namely, to construct easily verified sufficient conditions and show examples when these conditions are satisfied and solutions in these examples.

For solving third order linear and nonlinear boundary value problems for the equation (0.0.1) having in mind that the problems under consideration have solutions, there is a great number of methods including analytical and numerical methods. Below we briefly review these methods via some typical works. First we mention some works where analytical methods are used. Specifically, in [26] the authors proposed an iterative method based on embedding Green's functions into well-known fixed point iterations, including Picard's and Krasnoselskii–Mann's schemes. The uniform convergence is proved but the method is very difficult to realize because it requires to calculate integrals of the product the Green function of the problem with the function $f(t, u_n(t), u'_n(t), u''_n(t))$ at each iteration. In [27, 28] the Adomian decomposition method and its modification are applied. Recently, in 2020, He [29] suggests a simple but effective way to the third-order ordinary differential equations by the Taylor series technique. In general, for solving the BVPs for nonlinear third order equations numerical methods are widely used. Namely, Al Said et al. [30] have solved a third order two point BVP using cubic splines. Noor et al. [31] generated second order method based on quartic splines. Other authors [32, 33] generated finite difference schemes using fourth degree B-spline and quintic polynomial spline for this problem subject to other boundary conditions. El-Danaf [34] constructed a new spline method based on quartic nonpolynomial spline functions that has a polynomial part and a trigonometric part to develop numerical methods for a linear differential equation. Recently, in 2016 Pandey [35] solved the problem for the case $f = f(t,u)$ by the use of quartic polynomial splines. The convergence of the method of at least $O(h^2)$ for the linear case $f = f(t)$ was proved. In the following year, this author in [36] proposed two difference schemes for the general case $f = f(t, u(t), u'(t), u''(t))$ and also established the second order accuracy for the linear case. In 2019, Chaurasia et al. [37] used exponential amalgamation of cubic spline functions to form a novel numerical method of second-order accuracy. *It should be emphasized that all of above mentioned authors only drew attention to the construction of the discrete analogue of the equation* (0.0.1) *associated with some boundary conditions and estimated the error of the obtained solution assuming that the nonlinear system of algebraic equations can be solved by known iterative methods. Thus, they did not take into account the errors arising in the last iterative methods.*

Motivated by the above facts we wish to construct iterative numerical methods of competitive accuracy or more accurate compared with some existing methods, and importantly, to obtain the total error combining the error of iterative process and the error of discretization of continuous problems at each iteration.

b) Boundary value problems with integral boundary conditions

Recently, boundary value problems for nonlinear differential equations with integral boundary conditions have attracted attention from many researchers. They constitute a very interesting and important class of problems because they arise in many applied fields such as heat conduction, chemical engineering, underground water flow, thermoelasticity and plasma physics. It is worth mentioning some works concerning the problems with integral boundary conditions for second order equations such as [38, 39, 40, 41, 42, 43]. There are also many papers devoted to the third order and fourth order equations with integral boundary conditions.

Below we mention some works concerning the third order nonlinear equations. The first work we would mention, is of Boucherif et al. [44] in 2009. It is about the problem

$$\begin{aligned}
&u'''(t) = f(t, u(t), u'(t), u''(t)), \quad 0 < t < 1,\\
&u(0) = 0,\\
&u'(0) - au''(0) = \int_0^1 h_1(u(s), u'(s))ds,\\
&u'(1) + bu''(1) = \int_0^1 h_2(u(s), u'(s))ds,
\end{aligned}$$

where a, b are positive real numbers, f, h_1, h_2 are continuous functions. Based on a priori bounds and a fixed point theorem for a sum of two operators, one a compact operator and the other a contraction, the authors established the existence of solutions to the problem under complicated conditions on the functions f, h_1, h_2. Independently from the above work, in 2010 Sun and Li [24] considered the problem

$$\begin{aligned}
&u'''(t) + f(t, u(t), u'(t)) = 0, \quad 0 < t < 1,\\
&u(0) = u'(0) = 0, \ u'(1) = \int_0^1 g(t)u'(t)dt.
\end{aligned}$$

By using the Krasnoselskii's fixed point theorem, some sufficient conditions are obtained for the existence and nonexistence of monotone positive solutions to the above problem.

Next, in 2012 Guo, Liu and Liang [45] studied the boundary value problem with second derivative

$$\begin{aligned}
&u'''(t) + f(t, u(t), u''(t)) = 0, \quad 0 < t < 1,\\
&u(0) = u''(0) = 0, \ u(1) = \int_0^1 g(t)u(t)dt.
\end{aligned}$$

The authors obtained sufficient conditions for the existence of positive solutions by using the fixed point index theory in a cone and spectral radius of a linear operator. No examples of the functions f and g satisfying the conditions of existence were shown.

In another paper, in 2013 Guo and Yang [46] considered a problem with other boundary conditions, namely, the problem

$$\begin{aligned}
&u'''(t) = f(t, u(t), u'(t)), \quad 0 < t < 1,\\
&u(0) = u''(0) = 0, \ u(1) = \int_0^1 g(t)u(t)dt.
\end{aligned}$$

Based on the Krasnoselskii fixed-point theorem on cone, the authors established the existence of positive solutions of the problem under very complicated and artificial growth conditions posed on the nonlinearity $f(t, x, y)$.

Very recently, in [47] Guendouz et al. studied the problem

$$\begin{aligned} u'''(t) + f(u(t)) &= 0, \quad 0 < t < 1, \\ u(0) = u'(0) = 0,\ u(1) &= \int_0^1 g(t)u(t)dt. \end{aligned}$$

By applying the Krasnoselskii's fixed point theorem on cones they established the existence results of positive solutions of the problem. This technique was used also by Benaicha and Haddouchi in [48] for an integral boundary problem for a fourth order nonlinear equation.

Many authors also studied fourth order differential equations with integral boundary conditions (see, e.g., [48,49,50,51,52,53,54,55,56,57,58]). Below we mention only some typical works. First it is worthy to mention the work of Zhang and Ge [58], where they studied the problem

$$\begin{aligned} u''''(t) &= w(t)f(t, u(t), u''(t)), \quad 0 < t < 1, \\ u(0) &= \int_0^1 g(s)u(s)ds,\ u(1) = 0, \\ u''(0) &= \int_0^1 h(s)u''(s)ds,\ u''(1) = 0, \end{aligned}$$

where w may be singular at $t = 0$ and/or $t = 1$, $f : [0, 1] \times \mathbb{R}^+ \times \mathbb{R}^- \to \mathbb{R}^+$ is continuous, and $g, h \in L^1[0, 1]$ are nonnegative. Using the fixed point theorem of cone expansion and compression of norm type, the authors established the existence and nonexistence of positive solutions.

In 2013, Li et al. [54] studied the fully nonlinear fourth-order boundary value problem

$$\begin{aligned} u''''(t) &= f(t, u(t), u'(t), u''(t), u'''(t)), \quad t \in [0, 1], \\ u(0) = u'(1) = u'''(1) = 0,\ u''(0) &= \int_0^1 h(s, u(s), u'(s), u''(s))ds, \end{aligned}$$

where $f : [0, 1] \times \mathbb{R}^4 \to \mathbb{R},\ h : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}$ are continuous functions. Based on a fixed point theorem for a sum of two operators, one is completely continuous and the other is a nonlinear contraction, the authors established the existence of solutions and monotone positive solutions for the problem.

Later, in 2015, Lv et al. [55] considered a simplified form of the above problem

$$\begin{aligned} u''''(t) &= f(t, u(t), u'(t), u''(t)), \quad t \in [0, 1], \\ u(0) = u'(1) = u'''(1) = 0,\ u''(0) &= \int_0^1 g(s)u''(s)ds, \end{aligned}$$

where $f : [0, 1] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^- \to \mathbb{R}^+,\ g : [0, 1] \to \mathbb{R}^+$ are continuous functions. Using the fixed point theorem of cone expansion and compression of norm type, they obtained the existence and nonexistence of concave monotone positive solutions.

It should be emphasized that in all mentioned above works of integral boundary value problems the authors could only show examples of the nonlinear terms satisfying required sufficient conditions, but no exact solutions are shown. Moreover, the known results are of purely theoretical character concerning the existence of solutions but not methods for finding solutions.

Therefore, it is needed to give conditions for existence of solutions, to show examples with solutions, and importantly, to construct methods for finding the solutions for integral boundary value problems.

c) Boundary value problems for integro-differential equations

Integro-differential equations are the mathematical models of many phenomena in physics, biology, hydromechanics, chemistry, etc. In general, it is impossible to find the exact solutions of the problems involving these equations, especially when they are nonlinear. Therefore, many analytical approximation methods and numerical methods have been developed for these equations (see, e.g. [59, 61, 62, 63, 64, 65, 66, 67, 68, 69]).

Below, we mention some works concerning the solution methods for integro-differential equations. First, it is worthy to mention the recent work of Tahernezhad and Jalilian in 2020 [65]. In this work, the authors consider the second order linear problem

$$u''(x) + p(x)u'(x) + q(x)u(x) = f(x) + \int_a^b k(x,t)u(t)dt, \quad a < x < b,$$
$$u(a) = \alpha, \ u(b) = \beta,$$

where $p(x), q(x), k(x,t)$ are sufficiently smooth functions.

Using non-polynomial spline functions, namely, the exponential spline functions, the authors constructed the numerical solution of the problem and proved that the error of the approximate solution is $O(h^2)$, where h is the grid size on $[a, b]$. Before [65] there are interesting works of Chen et al. [60, 69], where the authors used a multiscale Galerkin method for constructing an approximate solution of the above second order problem, for which the computed convergence rate is two.

Besides the researches evolving the second order integro-differential equations, recently many authors have been interested in fourth order integro-differential equations due to their wide applications. We first mention the work of Singh and Wazwaz [63]. In this work, the authors developed a technique based on the Adomian decomposition method with the Green's function for constructing a series solution of the nonlinear Voltera equation associated with the Dirichlet boundary conditions

$$y^{(4)}(x) = g(x) + \int_0^x k(x,t)f(y(t))dt, \quad 0 < x < b,$$
$$y(0) = \alpha_1, \ y'(0) = \alpha_2, \ y(b) = \alpha_3, \ y'(b) = \alpha_4.$$

Under some conditions it was proved that the series solution converges as a geometric progression.

For the linear Fredholm IDE [59]

$$y^{(4)}(x) + \alpha y''(x) + \beta y(x) - \int_a^b K(x,t)y(t)dt = f(x), \quad a < x < b,$$

with the above Dirichlet boundary conditions, the difference method and the trapezoidal rule are used to design the corresponding linear system of algebraic equations. A new variant called the Modified Arithmetic Mean iterative method is proposed for solving the latter system, but the error estimate of the method is not obtained.

The boundary value problem for the nonlinear IDE

$$y^{(4)}(x) - \varepsilon y''(x) - \frac{2}{\pi}\Big(\int_0^\pi |y'(t)|^2 dt\Big) y''(x) = p(x), \quad 0 < x < \pi,$$
$$y(0) = 0, \ y(\pi) = 0, \ y''(0) = 0, \ y''(\pi) = 0$$

was considered in [12,68], where the authors constructed approximate solutions by the iterative and spectral methods, respectively. Recently, Dang and Nguyen [11] studied the existence and uniqueness of solution and constructed iterative method for finding the solution for the IDE

$$\begin{aligned} &u^{(4)}(x) - M\Big(\int_0^L |u'(t)|^2 dt\Big)u''(x) = f(x,u,u',u'',u'''), \quad 0 < x < L, \\ &u(0) = 0,\ u(L) = 0,\ u''(0) = 0,\ u''(L) = 0, \end{aligned}$$

where M is a continuous non-negative function.

Very recently, Wang [66] considered the problem

$$\begin{aligned} y^{(4)}(x) &= f(x, y(x), \int_0^1 k(x,t)y(t)dt),\ 0 < x < 1, \\ y(0) &= 0,\ y(1) = 0,\ y''(0) = 0,\ y''(1) = 0. \end{aligned} \tag{0.0.2}$$

This problem can be seen as a generalization of the linear fourth order problem

$$\begin{aligned} &u^{(4)}(x) + Mu(x) - N\int_0^1 k(x,t)u(t)dt) = p(x),\ 0 < x < 1, \\ &u(0) = 0,\ u(1) = 0,\ u''(0) = 0,\ u''(1) = 0, \end{aligned}$$

where M, N are constants, $p \in C[0,1]$. The latter problem arises from the models for suspension bridges [70,71], quantum theory [72].

Using the monotone method and a maximum principle, Wang constructed the sequences of functions, which converge to the extremal solutions of the problem (0.0.2).

From the above reviewed works we see that some integro-differential equations, linear and nonlinear, are studied by different methods. The development of a unified method for investigating both the qualitative and quantitative aspects of extended integro-differential equations is necessary and is of great interest.

d) Boundary value problems for functional differential equations

Functional differential equations have numerous applications in engineering and sciences [73]. Therefore, for the last decades they have been studied by many authors. There are many works concerning the numerical solution of both initial and boundary value problems for them. The methods used are diverse including collocation method [74], iterative methods [75,76], neural networks [77,78], and so on. Below we mention some typical results.

First it is worthy to mention the work of Reutskiy in 2015 [74]. In this work, the author considered the linear pantograph functional differential equation with proportional delay

$$u^{(n)} = \sum_{j=0}^{J}\sum_{k=0}^{n-1} p^{jk}(x)u^{(k)}(\alpha_j x) + f(x), \quad x \in [0,T]$$

associated with initial or boundary conditions. Here α_j are constants $(0 < \alpha_j < 1)$. The author proposed a method, where the initial equation is replaced by an approximate equation which has an exact analytic solution with a set of free parameters. These free parameters are determined by the use of the collocation procedure. Many examples show the efficiency of the method but no error estimates are obtained.

In 2016 Bica et al. [75] considered the boundary value problem

$$\begin{aligned} x^{(2p)}(t) &= f(t, x(t), x(\varphi(t))), \quad t \in [a, b], \\ x^{(i)}(a) &= a_i, \ x^{(i)}(b) = b_i, \quad i = \overline{0, p-1} \end{aligned} \tag{0.0.3}$$

where $\varphi : [a, b] \to \mathbb{R}, \ a \le \varphi(t) \le b, \forall t \in [a, b]$. For solving the problem, the authors constructed successive approximations for the equivalent integral equation with the use of cubic spline interpolation at each iterative step. The error estimate was obtained for the approximate solution under very strong conditions including $(\alpha + 13\beta)(b - a)M_G < 1$, where α and β are the Lipschitz coefficients of the function $f(s, u, v)$ in the variables u and v in the domain $[a, b] \times \mathbb{R} \times \mathbb{R}$, respectively; M_G is a number such that $|G(t, s)| \le M_G \ \forall t, s \in [a, b]$, $G(t, s)$ being the Green function for the above problem. Some numerical experiments demonstrate the convergence of the proposed iterative method. But it is a regret that in the proof of the error estimate for fourth order nonlinear BVP there is a mistake when the authors by default considered that the partial derivatives $\frac{\partial^3 G}{\partial s^3}, \frac{\partial^4 G}{\partial s^4}$ are continuous in $[a, b] \times [a, b]$. Indeed, it is invalid because $\frac{\partial^3 G}{\partial s^3}$ has discontinuity on the line $s = t$. Due to this mistake the authors obtained that the error of the method for fourth order BVP is $O(h^4)$. This mistake and a similar mistake in the proof of $O(h^2)$ convergence for the second order problem are corrected in the recent corrigendum [79]. Although in [75] the method was constructed for the general function $\varphi(t)$ but in all numerical examples only the particular case $\varphi(t) = \alpha t$ was considered and the conditions of convergence were not verified. It is a regret that in all examples the Lipschitz conditions for the function $f(s, u, v)$ are not satisfied in unbounded domains as required in the conditions (ii) and (iv) [75, page 131].

Recently, in 2018 Khuri and Sayfy [76] proposed a Green function based iterative method for functional differential equations of arbitrary orders. But the scope of application of the method is very limited due to the difficulty in calculation of integrals at each iteration.

For solving functional differential equations, beside analytical and numerical methods, recently computational intelligence algorithms also are used (see, e.g., [77, 78]), where feed-forward artificial neural networks of different architecture are applied. These algorithms are heuristic, so no errors estimates are obtained and they require large computational efforts.

The further investigation of the existence of solutions for functional differential equations and effective solution methods for them has a great significance. It is why in this thesis we shall study this topic.

Objectives and contents of the research

The aim of the thesis is to study the existence, uniqueness of solutions and solution methods for some BVPs for high order nonlinear differential, integro-differential and functional differential equations. Specifically, the thesis intends to study the following contents:

Content 1 The existence, uniqueness of solutions and iterative methods for some BVPs for third order nonlinear differential equations.

Content 2 The existence, uniqueness of solutions and iterative methods for some problems for third and fourth order nonlinear differential equations with integral boundary conditions.

Content 3 The existence, uniqueness of solutions and iterative methods for some BVPs for integro-differential and functional differential equations.

Approach and the research method

We shall approach to the above contents from both theoretical and practical points of view, which are the study of qualitative aspects of the existence solutions and construction of numerical methods for finding the solutions. The methodology throughout the thesis is the reduction of BVPs to operator equations in appropriate spaces, the use of fixed point theorems for establishing the existence and uniqueness of solutions and for proving the convergence of iterative methods.

The achievements of the thesis

The thesis achieves the following results:
Result 1 The establishment of theorems on the existence, uniqueness of solutions and positive solutions for third order nonlinear BVPs and the construction of numerical methods for finding the solutions.
These results are published in the two papers [AL1] and [AL2]. Specifically,
- in [AL1] we propose a unified approach to investigate boundary value problems (BVPs) for fully third order differential equations. It is based on the reduction of BVPs to operator equations for the nonlinear terms but not for the functions to be sought as some authors did. By this approach we have established the existence, uniqueness, positivity and monotony of solutions and the convergence of the iterative method for approximating the solutions under some easily verified conditions in bounded domains. These conditions are much simpler and weaker than those of other authors for studying solvability of the problems before by using different methods. Many examples illustrate the obtained theoretical results.
- in [AL2] we establish the existence and uniqueness of solution and propose simple iterative methods on both continuous and discrete levels for a fully third order BVP. We prove that the discrete methods are of second order and third order of accuracy due to the use of appropriate formulas for numerical integration and obtain estimate for total error.

Result 2 The establishment of the existence, uniqueness of solutions and construction of iterative methods for finding the solutions for nonlinear third and fourth order differential equations with integral boundary conditions. These results are published in the two papers [AL3] and [AL5]. Specifically,
- The work [AL3] is devoted to third order differential equations.
- The work [AL6] concerns fourth order differential equations.

Result 3 The establishment of the existence, uniqueness of solutions and construction of numerical methods for finding the solutions of nonlinear integro-differential equations. The results are published in [AL6].
Result 4 The establishment of the existence, uniqueness of solutions and construction of numerical methods for finding the solutions of nonlinear functional differential equations. The results are published in [AL4].

The obtained results of the thesis are published in the six papers [AL1]-[AL6] (see "List of the works of the author related to the thesis").

Structure of the thesis

Except for "Introduction", "Conclusions" and "References", the thesis contains 4 chapters. In Chapter 1 we recall some auxiliary knowledges. The results of the thesis are presented in Chapters 2, 3 and 4. Namely,

1. Chapter 2 presents the results on the existence, uniqueness of solutions and positive solutions for third order nonlinear BVPs and the construction of numerical methods for finding the solutions.

2. Chapter 3 is devoted to the study of the existence, uniqueness of solutions and construction of iterative methods for finding the solutions for nonlinear third and fourth order differential equations with integral boundary conditions.

3. Chapter 4 presents the results on the existence, uniqueness of solutions and construction of numerical methods for finding the solutions of nonlinear integro-differential equations and functional differential equations.

Chapter 1

Preliminaries

In this chapter we recall some preliminaries on fixed point theorems, Green's functions and quadrature formulas which will be used in the next chapters.

1.1. Some fixed point theorems

1.1.1. Schauder Fixed-Point Theorem

The material of this subsection is taken from [80].

Theorem 1.1.1 (Brouwer Fixed-Point Theorem (1912))**.** Suppose that U is a nonempty, convex, compact subset of $\mathbb{R}^N$, where $N \geq 1$, and that $f : U \to U$ is a continuous mapping. Then f has a fixed point.

A typical example of the Brouwer Fixed-Point Theorem is proof of the existence of solutions of system of nonlinear algebraic equations.

Remark that Brouwer Fixed-Point Theorem is applicable only to continuous mappings in finite dimensional spaces. A generalization of the theorem to infinite dimensional spaces is the Schauder fixed-point theorem.

Definition 1.1.1. Let X and Y be B-spaces, and $T : D(T) \subseteq X \to Y$ an operator. T is called compact iff:
(i) T is continuous;
(ii) T maps bounded sets into relatively compact sets.

Compact operators play a central role in nonlinear functional analysis. Their importance stems from the fact that many results on continuous operators on $\mathbb{R}^N$ carry over to B-spaces when "continuous" is replaced by "compact".

Typical examples of compact operators on infinite-dimensional B-spaces are integral operators with sufficiently regular integrands. Set

$$(Tx)(t) = \int_a^b K(t, s, x(s))ds,$$

$$(Sx)(t) = \int_a^t K(t, s, x(s))ds, \quad \forall t \in [a, b].$$

Suppose

$$K : [a, b] \times [a, b] \times [-R, R] \to \mathbb{K},$$

where $-\infty < a < b < +\infty,\ 0 < R < \infty$ and $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Denote

$$U = \{x \in C([a,b], \mathbb{K}) : \|x\| \leq R\},$$

where $\|x\| = \max_{a \leq s \leq b}$ and $C([a,b], \mathbb{K})$ is the space of continuous maps $x : [a,b] \to \mathbb{K}$. Then the integral operators T and S map U into $C([a,b], \mathbb{K})$ and are compact.

Theorem 1.1.2 (Schauder Fixed-Point Theorem (1930))**.** Let U be a nonempty, closed, bounded, convex subset of a B-space X, and suppose $T : U \to U$ is a compact operator. Then T has a fixed point.

Corollary 1.1.3 (Alternate Version of the Schauder Fixed-Point Theorem)**.** Let U be a nonempty, compact, convex subset of a B-space X, and suppose $T : U \to U$ is a continuous operator. Then T has a fixed point.

The corollary is the direct translation of the Brouwer fixed-point theorem to B-spaces. The first verison (Theorem 1.1.2) is more frequently used in applications, in which case U is often chosen to be a ball.

1.1.2. Krasnoselskii Fixed-Point Theorem

Theorem 1.1.4 (Krasnoselskii Fixed-Point Theorem [81,82])**.** Let X be Banach space, $P \subset X$ be a cone, Ω_1, Ω_2 be open sets in X with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. Suppose $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ are compact operators satisfying the conditions:
(i) $\|T(x)\| \leq \|x\|$, $x \in P \cap \partial\Omega_1$ and $\|T(x)\| \geq \|x\|$, $x \in P \cap \partial\Omega_2$
or
(ii) $\|T(x)\| \geq \|x\|$, $x \in P \cap \partial\Omega_1$ and $\|T(x)\| \leq \|x\|$, $x \in P \cap \partial\Omega_2$.
Then T has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

This theorem usually is used for studying the existence of positive solutions of operator equations to which nonlinear boundary value problems are reduced. An improvement of the above theorem is the following theorem.

Theorem 1.1.5. [81] Let X be a Banach space, and $P \subset X$ be a closed convex cone. Assume that Ω_1, Ω_2 are bounded open subsets of X with $\theta \in \omega_1, \overline{\Omega}_1 \subset \Omega_2$. Let $A : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be a completely continuous mapping. If A satisfies the following conditions:
(1) $\lambda Au \neq u$ for $u \in P \cap \delta\Omega_1, 0 < \lambda \leq 1$;
(2) there exists $e \in P \setminus \{\theta\}$ such that $u - Au \neq \tau e$ for $u \in P \cap \partial\Omega_2, \tau \geq 0$;
or the following conditions:
(3) there exists $e \in P \setminus \{\theta\}$ such that $u - Au \neq \tau e$ for $u \in P \cap \partial\Omega_1, \tau \geq 0$;
(4) $\lambda Au \neq u$ for $u \in P \cap \partial\Omega_2, 0 < \lambda \leq 1$;
then A has a fixed point in $P \cap (\Omega_2 \setminus \overline{\Omega_1})$.

1.1.3. Banach Fixed-Point Theorem

Theorem 1.1.6 (Banach Fixed-Point Theorem (1922) [80])**.** Suppose that
(i) we are given an operator $T : M \subset X \to M$, i.e., M is mapped into itself by T;
(ii) M is a closed nonempty set in a complete metric space (X, d);
(iii T is q-contractive, i.e.,

$$d(Tx, Ty) < qd(x, y) \tag{1.1.1}$$

for all $x, y \in M$ and for a fixed $q, 0 < q < 1$.
Then we may conclude the following:
(a) Existence and uniqueness: Equation (1.1.1) has exactly one solution, i.e., T has exactly one fixed point on M;
(b) Convergence of the iteration: The sequence $x_{n+1} = Tx_n$ of successive approximations converges to the solution, x,for an arbitrary choice of initial point x_0 in M;
(c) Error estimates: For all $n = 0, 1, 2, ...$ we have the a priori error estimate

$$d(x_n, x) \leq \frac{q^n}{1-q} d(x_0, x_1).$$

and the a posteriori error estimate

$$d(x_{n+1}, x) \leq \frac{q}{1-q} d(x_n, x_{n+1}).$$

(d) Rate of convergence: For all $n = 0, 1, 2, ...$ we have

$$d(x_{n+1}, x) \leq q d(x_n, x).$$

Banach Fixed-Point Theorem has many important applications in qualitative study as well as in approximate solution of nonlinear equations, system of linear or nonlinear equations, integral equations, differential equations,...

1.2. Green's functions

Green's functions play an important role in the study of existence and uniqueness of boundary value problems for ordinary differential equations.

Consider the linear homogeneous boundary-value problem

$$L[y(x)] \equiv p_0(x)\frac{d^n y}{dx^n} + p_1(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + p_n(x)y = 0, \qquad (1.2.1)$$

$$M_i(y(a), y(b)) \equiv \sum_{k=0}^{n-1}\left(\alpha_k^i \frac{d^k y(a)}{dx^k} + \beta_k^i \frac{d^k y(b)}{dx^k}\right) = 0, \quad i = 1, ..., n, \qquad (1.2.2)$$

where $p_i(x), i = 0, ..., n$ are continuous functions on (a, b), $p_0(x) \neq 0$ in all points in (a, b).

Definition 1.2.1. *(see [83])* The function $G(x, t)$ is said to be the Green's function for the boundary value problem (1.2.1)-(1.2.2) if, as a function of its first variable x, it meets the following defining criteria, for any $t \in (a, b)$:
(i) On both intervals $[a, t)$ and $(t, b]$, $G(x, t)$ is a continuous function having continuous derivatives up to n-th order and satisfies the governing equation in (1.2.1) on (a, t) and (t, b), that is:

$$L[G(x, t)] = 0, x \in (a, t); \quad L[G(x, t)] = 0, x \in (t, b).$$

(ii) $G(x, t)$ satisfies the boundary conditions in (1.2.2), that is

$$M_i(G(a, t), G(b, t)) = 0, \quad i = 1, ..., n.$$

(iii) For $x = t$, $G(x,t)$ and all its derivatives up to $(n-2)$ are continuous

$$\lim_{x\to t^+} \frac{\partial^k G(x,t)}{\partial x^k} - \lim_{x\to t^-} \frac{\partial^k G(x,t)}{\partial x^k} = 0, \quad k = 0, ..., n-2.$$

(iv) The $(n-1)$th derivative of $G(x,t)$ is discontinuous when $x = t$, providing

$$\lim_{x\to t^+} \frac{\partial^{n-1} G(x,t)}{\partial x^{n-1}} - \lim_{x\to t^-} \frac{\partial^{n-1} G(x,t)}{\partial x^{n-1}} = -\frac{1}{p_0(t)}.$$

The following theorem specifies the conditions for existence and uniqueness of the Green's function.

Theorem 1.2.1. *(see [83])* If the homogeneous boundary-value problem in (1.2.1)-(1.2.2) has only a trivial solution, then there exists an unique Green's function associated with the problem.

Consider the linear nonhomogeneous equation

$$L[y(x)] \equiv p_0(x)\frac{d^n y}{dx^n} + p_1(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + p_n(x)y = -f(x), \tag{1.2.3}$$

subject to the homogeneous boundary conditions

$$M_i(y(a), y(b)) \equiv \sum_{k=0}^{n-1} \left(\alpha_k^i \frac{d^k y(a)}{dx^k} + \beta_k^i \frac{d^k y(b)}{dx^k} \right) = 0, \quad i = 1, ...n, \tag{1.2.4}$$

where $p_j(x)$ and the right-hand side term $f(x)$ in (1.2.3) are continous functions, with $p_0(x) \neq 0$ on (a,b) and M_i represent linearly independent forms with constant coefficients.

The following theorem establishes the link between the uniqueness of the solution of (1.2.3)-(1.2.4) and the corresponding homogeneous problem.

Theorem 1.2.2. *(see [83])* If the homogeneous boundary-value problem corresponding to (1.2.3)-(1.2.4) has only the trivial solution, then the problem in (1.2.3)-(1.2.4) has a unique solution in the form

$$y(x) = \int_a^b G(x,t) f(t) dt,$$

where $G(x,t)$ is the Green's function of the là hàm Green of the corresponding homogeneous problem.

Let us consider some Green's functions that will later be used in the thesis.

Example 1.2.1. Consider the problem

$$\begin{cases} u''(x) = \varphi(x), \quad 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases} \tag{1.2.5}$$

The corresponding Green's function is of the form

$$G(x,t) = \begin{cases} A_1 + A_2 x, & 0 \le x \le t \le 1 \\ B_1 + B_2(1-x), & 0 \le t \le x \le 1, \end{cases} \tag{1.2.6}$$

where A_1, A_2 and B_1, B_2 are the functions of t. $G(x,t)$ satisfies the condition (i). Because $G(x,t)$ must satisfy the homogeneous boundary conditions in (ii), it follows that $A_1 = B_1 = 0$. Therefore

$$G(x,t) = \begin{cases} A_2 x, & 0 \le x \le t \le 1 \\ B_2(1-x), & 0 \le t \le x \le 1. \end{cases} \tag{1.2.7}$$

The condition (iii) leads to

$$B_2(1-t) - A_2 t = 0. \tag{1.2.8}$$

From the condition (iv) we have

$$B_2 + A_2 = 1. \tag{1.2.9}$$

We can find A_2, B_2 by solving (1.2.8) and (1.2.9). It follows that $A_2 = 1-t$, $B_2 = t$. Substitute into (1.2.7) we obtain the Green's function

$$G(x,t) = \begin{cases} x(1-t), & 0 \le x \le t \le 1, \\ t(1-x), & 0 \le t \le x \le 1. \end{cases} \tag{1.2.10}$$

The solution of the problem (1.2.5) can be represented in the form

$$u(x) = \int_0^1 G(x,t)\varphi(t)dt.$$

Example 1.2.2. Consider the problem

$$\begin{cases} u'''(x) = \varphi(x), & 0 < x < 1, \\ u(0) = u'(0) = u'(1) = 0. \end{cases} \tag{1.2.11}$$

The corresponding Green's function is of the form

$$G(x,t) = \begin{cases} A_1 + A_2 x + A_3 x^2, & 0 \le x \le t \le 1 \\ B_1 + B_2(1-x) + B_3(1-x)^2, & 0 \le t \le x \le 1, \end{cases} \tag{1.2.12}$$

where A_1, A_2, A_3 and B_1, B_2, B_3 are the functions of t. $G(x,t)$ satisfies the condition (i). Because $G(x,t)$ must satisfy the homogeneous boundary conditions in (ii), it follows that

$$A_1 = A_2 = B_2 = 0.$$

Therefore

$$G(x,t) = \begin{cases} A_3 x^2, & 0 \le x \le t \le 1 \\ B_1 + B_3(1-x)^2, & 0 \le t \le x \le 1. \end{cases} \tag{1.2.13}$$

The condition (iii) leads to

$$\begin{cases} B_1 + B_3(1-t)^2 = A_3 t^2 \\ -B_3(1-t) = A_3 t. \end{cases} \tag{1.2.14}$$

From the condition (iv) we have

$$B_3 - A_3 = -1/2. \tag{1.2.15}$$

We can find A_3, B_1, B_3 by solving (1.2.14) and (1.2.15). It follows that

$$A_3 = -\frac{t}{2} + \frac{1}{2}, \quad B_1 = -\frac{t^2}{2} + \frac{t}{2}, \quad B_3 = -\frac{t}{2}.$$

Substitute into (1.2.13) we obtain the Green's function

$$G(x,t) = \begin{cases} x^2(t-1)/2, & 0 \le x \le t \le 1, \\ t(x^2 - 2x + t)/2, & 0 \le t \le x \le 1. \end{cases} \tag{1.2.16}$$

The solution of the problem (1.2.11) can be represented in the form

$$u(x) = \int_0^1 G(x,t)\varphi(t)dt.$$

Example 1.2.3. Consider the problem

$$\begin{cases} u^{(4)}(x) = \varphi(x), \quad 0 < x < 1, \\ u(0) = u''(0) = u(1) = u''(1) = 0. \end{cases} \tag{1.2.17}$$

The corresponding Green's function is of the form

$$G(x,t) = \begin{cases} A_1 + A_2 x + A_3 x^2 + A_4 x^3, & 0 \le x \le t \le 1 \\ B_1 + B_2(1-x) + B_3(1-x)^2 + B_4(1-x)^3, & 0 \le t \le x \le 1, \end{cases} \tag{1.2.18}$$

where A_1, A_2, A_3, A_4 and B_1, B_2, B_3, B_4 are the functions of t. $G(x,t)$ satisfies the condition (i). Because $G(x,t)$ must satisfy the homogeneous boundary conditions in (ii), it follows that

$$A_1 = A_3 = B_1 = B_3 = 0.$$

Therefore

$$G(x,t) = \begin{cases} A_2 x + A_4 x^3, & 0 \le x \le t \le 1 \\ B_2(1-x) + B_4(1-x)^3, & 0 \le t \le x \le 1. \end{cases} \tag{1.2.19}$$

The condition (iii) leads to

$$\begin{cases} B_2(1-t) + B_4(1-t)^3 = A_2 t + A_4 t^3 \\ -B_2 - 3B_4(1-t)^2 = A_2 + 3A_4 t^2 \\ 6B_4(1-t) = 6A_4 t. \end{cases} \tag{1.2.20}$$

From the condition (iv) we have

$$B_4 + A_4 = -1/6. \tag{1.2.21}$$

We can find A_4, B_1, B_2, B_4 by solving (1.2.20) and (1.2.21). It follows that

$$A_2 = \frac{t^3}{6} - \frac{t^2}{2} + \frac{t}{3}, \quad A_4 = \frac{t}{6} - \frac{1}{6},$$

$$B_2 = -\frac{t^3}{6} + \frac{t}{6}, \quad B_4 = -\frac{t}{6}.$$

Substitute into (1.2.19) we obtain the Green's function

$$G(x,t) = \begin{cases} t(x-1)(t^2 - 2x + x^2)/6, & 0 \le t \le x \le 1, \\ x(t-1)(t^2 - 2t + x^2)/6, & 0 \le x \le t \le 1. \end{cases} \tag{1.2.22}$$

The solution of the problem (1.2.17) can be represented in the form

$$u(x) = \int_0^1 G(x,t)\varphi(t)dt.$$

1.3. Some quadrature formulas

The material of this section is taken from [84]).

Trapezoidal rule:

Let $f \in C^2[a,b], h = b-a$. Then there exists a point $\xi \in (a,b)$ such that

$$\int_a^b f(x)dx = \frac{h}{2}[f(a)+f(b)] - \frac{1}{12}h^3 f''(\xi).$$

Theorem 1.3.1. Let $f \in C^2[a,b], h = (b-a)/n$, and $x_j = a + jh$, for each $j = 0, 1, ..., n$. There exists a $\mu \in (a,b)$ for which the Composite Trapezoidal rule for n subintervals can be written with its error term as

$$\int_a^b f(x)dx = \frac{h}{2}\Big[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\Big] - \frac{b-a}{12}h^2 f''(\mu).$$

Briefly,

$$\int_a^b f(x)dx = \frac{h}{2}\Big[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\Big] + O(h^2).$$

Simpson's rule:

Let $f \in C^4[a,b], x_j = a + jh$ for $j = 0, 1, 2$. Then there exists a $\xi \in (a,b)$ such that

$$\int_a^b f(x)dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] - \frac{h^5}{90}f^{(4)}(\xi).$$

Theorem 1.3.2. Let $f \in C^4[a,b]$, n be even, $n = 2m, h = (b-a)/n$, and $x_j = a+jh$, for each $j = 0, 1, ..., n$. There exists a $\mu \in (a,b)$ for which the Composite Simpson's rule for n subintervals can be written with its error term as

$$\int_a^b f(x)dx = \frac{h}{3}\Big[f(a) + 2\sum_{j=1}^{m-1} f(x_{2j}) + 4\sum_{j=1}^{m} f(x_{2j-1}) + f(b)\Big] - \frac{b-a}{180}h^4 f^{(4)}(\mu).$$

Briefly,

$$\int_a^b f(x)dx = \frac{h}{3}\Big[f(a) + 2\sum_{j=1}^{m-1} f(x_{2j}) + 4\sum_{j=1}^{m} f(x_{2j-1}) + f(b)\Big] + O(h^4).$$

Chapter 2

Existence results and iterative method for two-point third order nonlinear BVPs

2.1. Existence results and continuous iterative method for third order nonlinear BVPs

2.1.1. Introduction

In this section we propose a unified efficient method to investigate the solvability and approximation of BVPs for the fully third order equation

$$u'''(t) = f(t, u(t), u'(t), u''(t)), \quad 0 < t < 1 \tag{2.1.1}$$

with general boundary conditions

$$\begin{aligned} B_1[u] &= \alpha_1 u(0) + \beta_1 u'(0) + \gamma_1 u''(0) = 0, \\ B_2[u] &= \alpha_2 u(0) + \beta_2 u'(0) + \gamma_2 u''(0) = 0, \\ B_3[u] &= \alpha_3 u(1) + \beta_3 u'(1) + \gamma_3 u''(1) = 0, \end{aligned} \tag{2.1.2}$$

and

$$\begin{aligned} B_1[u] &= \alpha_1 u(0) + \beta_1 u'(0) + \gamma_1 u''(0) = 0, \\ B_2[u] &= \alpha_2 u(1) + \beta_2 u'(1) + \gamma_2 u''(1) = 0, \\ B_3[u] &= \alpha_3 u(1) + \beta_3 u'(1) + \gamma_3 u''(1) = 0, \end{aligned} \tag{2.1.3}$$

such that

$$Rank \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 & 0 & 0 & 0 \\ \alpha_2 & \beta_2 & \gamma_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix} = 3.$$

The boundary conditions (2.1.2) include as particular cases the boundary conditions considered in [16, 17, 19, 20, 23], meanwhile the boundary conditions (2.1.3) include as particular cases the boundary conditions considered in [16, 22]. Notice that the boundary conditions of the form (2.1.3) can be transformed to the boundary conditions of the form (2.1.2) by changing variable $t = 1 - s$.

To investigate the BVP (2.1.1)-(2.1.2) as the BVP (2.1.1), (2.1.3) we use a new approach based on the reduction of them to operator equations for the nonlinear terms but not for the functions to be sought. This approach was used to some boundary value problems for fourth nonlinear equations in very recent works [11, 13, 14, 85, 86]. Here, by this approach we have established the existence, uniqueness, positivity and monotony of solutions and the convergence of the iterative method for approximating the solutions of the problems (2.1.1)-(2.1.2) under some easily verified conditions in bounded domains. These conditions are much simpler and weaker than those of other authors for studying solvability of particular cases of the problems before by using different methods. Many examples illustrate the obtained theoretical results.

2.1.2. Existence results

Since the problem (2.1.1)-(2.1.2) and the problem (2.1.1), (2.1.3) are completely similar, below we consider only the first of them.

For convenience we rewrite the problem (2.1.1)-(2.1.2) in the form

$$\begin{aligned} &u'''(t) = f(t, u(t), u'(t), u''(t)), \quad 0 < t < 1 \\ &B_1[u] = B_2[u] = B_3[u] = 0, \end{aligned} \tag{2.1.4}$$

where $B_1[u], B_2[u], B_3[u]$ are defined by (2.1.2). We shall associate this problem with an operator equation as follows.

For functions $\varphi(x) \in C[0, 1]$ consider the nonlinear operator A defined by

$$(A\varphi)(t) = f(t, u(t), u'(t), u''(t)), \tag{2.1.5}$$

where $u(t)$ is the solution of the problem

$$\begin{aligned} &u'''(t) = \varphi(t), \quad 0 < t < 1 \\ &B_1[u] = B_2[u] = B_3[u] = 0 \end{aligned} \tag{2.1.6}$$

provided that it is uniquely solvable. It is easy to verify the following:

Proposition 2.1.1. If the function $\varphi(t)$ is a fixed point of the operator A, i.e., $\varphi(t)$ is a solution of the operator equation

$$A\varphi = \varphi, \tag{2.1.7}$$

then the function $u(t)$ determined from the boundary value problem (2.1.6) solves the problem (2.1.4). Conversely, if $u(t)$ is a solution of the boundary value problem (2.1.4) then the function

$$\varphi(t) = f(t, u(t), u'(t), u''(t))$$

is a fixed point of the operator A defined above by (2.1.5), (2.1.6).

Thus, the solution of the original problem (2.1.4) is reduced to the solution of the operator equation (2.1.7).

Now consider the problem (2.1.6). Suppose that the Green function of it exists and is denoted by $G(t, s)$. Then the unique solution of the problem is represented in the form

$$u(t) = \int_0^1 G(t, s)\varphi(s)ds. \tag{2.1.8}$$

By differentiation of both sides of the above formula we obtain

$$u'(t) = \int_0^1 G_1(t, s)\varphi(s)ds, \quad u''(t) = \int_0^1 G_2(t, s)\varphi(s)ds, \tag{2.1.9}$$

where $G_1(t,s) = G'_t(t,s)$ is a function continuous in the square $Q = [0,1]^2$ and $G_2(t,s) = G'''_{tt}(t,s)$ is continuous in the square Q except for the line $t = s$.
Further, let

$$\begin{aligned} &\max_{0\le t\le 1} \int_0^1 |G(t,s)|ds = M_0 \\ &\max_{0\le t\le 1} \int_0^1 |G_1(t,s)|ds = M_1, \ \max_{0\le t\le 1} \int_0^1 |G_2(t,s)|ds = M_2. \end{aligned} \tag{2.1.10}$$

Next, for each fixed real number $M > 0$ introduce the domain

$$\mathcal{D}_M = \{(t,x,y,z)|\ 0 \le t \le 1,\ |x| \le M_0M,\ |y| \le M_1M,\ |z| \le M_2M\},$$

and as usual, by $B[O,M]$ we denote the closed ball of radius M centered at 0 in the space of continuous in $[0,1]$ functions, namely,

$$B[O,M] = \{\varphi \in C[0,1]|\ \|\varphi\| \le M\},$$

where

$$\|\varphi\| = \max_{0\le t\le 1} |\varphi(t)|.$$

Theorem 2.1.2 (Existence of solutions)**.** Suppose that there exists a number $M > 0$ such that the function $f(t,x,y,z)$ is continuous and bounded by M in the domain $\mathcal{D}_M$, i.e.,

$$|f(t,x,y,z)| \le M \tag{2.1.11}$$

for any $(t,x,y,z) \in \mathcal{D}_M$.

Then, the problem (2.1.4) has a solution $u(t)$ satisfying

$$|u(t)| \le M_0M,\ |u'(t)| \le M_1M,\ |u''(t)| \le M_2M \text{ for any } 0 \le t \le 1. \tag{2.1.12}$$

Proof. Having in mind Proposition 2.1.1 the theorem will be proved if we show that the operator A associated with the problem (2.1.4) has a fixed point. For this purpose, it is not difficult to show that the operator A maps the closed ball $B[0,M]$ into itself. Next, from the compactness of integral operators (2.1.8), (2.1.9), which put each $\varphi \in C[0,1]$ in correspondence to the functions u, u', u'', respectively [87, Sec. 31] and the continuity of the function $f(t,x,y,z)$ it follows that A is a compact operator in the Banach space $C[0,1]$. By the Schauder Fixed Point Theorem [80] the operator A has a fixed point in $B[0,M]$. The estimates (2.1.12) hold due to the equalities (2.1.8), (2.1.9) and (2.1.10). □

Now suppose that the Green function $G(x,t)$ and its first derivative $G_1(x,t)$ are of constant signs in the square $Q = [0,1]^2$. Let's adopt the following convention for simplification of writing:
For a function $H(x,t)$ defined and having a constant sign in the square Q we define

$$\sigma(H) = sign(H(t,s)) = \begin{cases} 1, & \text{if } H(t,s) \ge 0, \\ -1, & \text{if } H(t,s) < 0. \end{cases}$$

In order to investigate the existence of positive solutions of the problem (2.1.1),(2.1.2) we introduce the notations

$$\begin{aligned} \mathcal{D}_M^+ = \{(t,x,y,z)|\ &0 \le t \le 1,\ 0 \le x \le M_0M, \\ &0 \le \sigma(G)\sigma(G_1)y \le M_1M,\ |z| \le M_2M\} \end{aligned}$$

and

$$S_M = \{\varphi \in C[0,1]|\ 0 \le \sigma(G)\varphi \le M\}.$$

Theorem 2.1.3 (Existence of positive solution)**.** Suppose that there exists a number $M > 0$ such that the function $f(t, x, y, z)$ is continuous and

$$0 \le \sigma(G) f(t, x, y, z) \le M \tag{2.1.13}$$

for any $(t, x, y, z) \in \mathcal{D}_M^+$. Then, the problem (2.1.1),(2.1.2) has a monotone nonnegative solution $u(t)$ satisfying

$$0 \le u(t) \le M_0 M,\ 0 \le \sigma(G)\sigma(G_1) u'(t) \le M_1 M,\ |u''(t)| \le M_2 M. \tag{2.1.14}$$

In addition, if $\sigma(G)\sigma(G_1) = 1$ then the problem has a nonnegative, increasing solution, and if $\sigma(G)\sigma(G_1) = -1$ then the problem has a nonnegative, decreasing solution.

Besides, if $f(t, 0, 0, 0) \not\equiv 0$ for $t \in (0, 1)$ then the solution is positive.

Proof. The proof of the existence of monotone nonnegative solution of the problem is similar to that of solution in Theorem 2.1.2 with the replacements of $\mathcal{D}_{\mathcal{M}}$ by $\mathcal{D}_M^+$, $B[0, M]$ by S_M and the condition (2.1.11) by the condition (2.1.13). From the estimates (2.1.14) it is obvious that if $\sigma(G)\sigma(G_1) = 1$ then $u'(t) \ge 0$, consequently, the solution is increasing function, otherwise, if $\sigma(G)\sigma(G_1) = -1$ then $u'(t) \le 0$, therefore, the solution is decreasing function. Moreover, if $f(t, 0, 0, 0) \not\equiv 0$ for $t \in (0, 1)$ then $u = 0$ cannot be the solution of the problem. Therefore, it must be positive. □

Theorem 2.1.4 (Existence and uniqueness of solution)**.** Assume that there exist numbers $M, L_0, L_1,\ L_2 \ge 0$ such that

$$|f(t, x, y, z)| \le M,$$

$$|f(t, x_2, y_2, z_2) - f(t, x_1, y_1, z_1)| \le L_0|x_2 - x_1| + L_1|y_2 - y_1| + L_2|z_2 - z_1| \tag{2.1.15}$$

for any $(t, x, y, z), (t, x_i, y_i, z_i) \in \mathcal{D}_M\ (i = 1, 2)$ and

$$q := L_0 M_0 + L_1 M_1 + L_2 M_2 < 1. \tag{2.1.16}$$

Then, the problem (2.1.1),(2.1.2) has a unique solution $u(t)$ such that $|u(t)| \le M_0 M,\ |u'(t)| \le M_1 M,\ |u''(t)| \le M_2 M$ for any $0 \le t \le 1$.

Proof. It is easy to show that under the conditions of the theorem, the operator A associated with the problem (2.1.1),(2.1.2) is a contraction mapping from the closed ball $B[0, M]$ into itself. By the contraction principle the operator A has a unique fixed point in $B[O, M]$, which corresponds to a unique solution $u(t)$ of the problem (2.1.1),(2.1.2).
The estimates for $u(t)$ and its derivatives are obtained as in Theorem 2.1.2. Thus, the theorem is proved. □

Analogously, we have the following theorem for the existence and uniqueness of positive solution of the problem (2.1.1),(2.1.2).

Theorem 2.1.5 (Existence and uniqueness of positive solution)**.** Assume that all the conditions of Theorem 2.1.3 are satisfied in the domain $\mathcal{D}_M^+$. Moreover, assume that there exist numbers $L_0, L_1, L_2 \ge 0$ such that the function $f(t, x, y, z)$ satisfies the Lipschitz conditions (2.1.15), (2.1.16). Then, the problem (2.1.1),(2.1.2) has a unique monotone nonnegative solution $u(t)$ satisfying (2.1.14). Besides, if $f(t, 0, 0, 0) \not\equiv 0$ for $t \in (0, 1)$ then the solution is positive.

Remark 2.1.1. Due to the representation (2.1.9) for $u''(t)$, based on the sign of $G(t, s)$ and $G_2(t, s)$ we can conclude of the convexity or concavity of solutions of the problem (2.1.4).

2.1.3. Iterative method

Consider the following iterative method for solving the problem (2.1.1), (2.1.2):

1. Given a starting approximation $\varphi_0 \in B[0, M]$, say

$$\varphi_0(t) = 0. \tag{2.1.17}$$

2. Knowing φ_k $(k = 0, 1, ...)$ compute

$$u_k(t) = \int_0^1 G(t, s)\varphi_k(s)\ ds, \tag{2.1.18}$$

$$y_k(t) = u_k'(t),\ z_k(t) = u_k''(t), \tag{2.1.19}$$

or equivalently,

$$\begin{aligned} y_k(t) &= \int_0^1 G_1(t, s)\varphi_k(s)\ ds, \\ z_k(t) &= \int_0^1 G_2(t, s)\varphi_k(s)\ ds. \end{aligned} \tag{2.1.20}$$

3. Update the new approximation

$$\varphi_{k+1}(t) = f(t, u_k(t), y_k(t), z_k(t)). \tag{2.1.21}$$

Set

$$p_k = \frac{q^k}{1-q}\|\varphi_1 - \varphi_0\|.$$

Theorem 2.1.6 (Convergence)**.** Under the assumptions of Theorem 2.1.4 the above iterative method converges and there hold the estimates

$$\|u_k - u\| \le M_0 p_k, \quad \|u_k' - u'\| \le M_1 p_k, \quad \|u_k'' - u''\| \le M_2 p_k, \tag{2.1.22}$$

where u is the exact solution of the problem (2.1.1), (2.1.2), and M_0, M_1, M_2 are given by (2.1.10).

Proof. Indeed, the above iterative process is the successive approximation of the fixed point of the operator A associated with the problem (2.1.1)-(2.1.2). Therefore, it converges with the rate of geometric progression and there is the estimate

$$\|\varphi_k - \varphi\| \le p_k, \tag{2.1.23}$$

where φ is the fixed point of A. Taking into account the representations (2.1.8), (2.1.9), (2.1.18), (2.1.20) and the formulas (2.1.10), from the above estimate we obtain the estimates (2.1.22). Thus, the theorem is proved. □

In many problems when the Green function and its derivatives have constant sign and the right-hand side function $f(t, x, y, z)$ is monotone in variables x, y, z we can establish the monotony of the sequence of approximations $u_k(t)$. Below we consider a particular case, which will be met in some examples in the next section.

Theorem 2.1.7 (Monotony)**.** Consider the problem (2.1.1)-(2.1.2), where the Green function $G(t,s)$ and its derivative $G_1(t,s)$ are nonpositive in the square $Q=[0,1]^2$, the function $f=f(t,x,y)\le 0$ is decreasing in x,y for $x,y\ge 0$. Then the sequence of approximations $u_k(t)$ generated by the above iterative process is increasing, i.e.

$$0=u_0(t)\le u_1(t)\le \ldots\le u_k(t)\le \ldots,\quad t\in[0,1]. \tag{2.1.24}$$

Proof. Indeed, starting from $\varphi_0=0$ by the iterative process (2.1.17)-(2.1.21) we obtain $u_0=0,\ y_0=0$. Since $f=f(t,x,y)\le 0$ we have $\varphi_1=f(t,0,0)\le 0$. Therefore, $u_1(t)=\int_0^1 G(t,s)\varphi_1(s)ds\ge 0$ due to $G(t,s)\le 0$. Analogously, $y_1(t)\ge 0$. Thus, we have $u_1\ge u_0,\ y_1\ge y_0$. Due to the decrease of $f(t,x,y)$ in x,y we have $\varphi_2(t)=f(t,u_1,y_1)\le f(t,u_0,y_0)=\varphi_1(t)$. Therefore, from the formulas for computing $u_2(t),y_2(t)$ it follows that $u_2\ge u_1,\ y_2\ge y_1$. Repeating the above argument we obtain (2.1.24). The theorem is proved. □

2.1.4. Some particular cases and examples

Consider some particular cases of the general BVP (2.1.1)-(2.1.2) and BVP (2.1.1), (2.1.3), which cover the problems studied by other authors using different methods. For each case, the theoretical results obtained in the previous section will be illustrated on examples, including some examples considered before by other authors. In numerical realization of the proposed iterative method, for computing definite integrals the trapezium formula with second order accuracy is used. In all examples, numerical computations are performed on the uniform grid on the interval $[0,1]$ with the gridsize $h=0.01$ until $\|\varphi_k-\varphi_{k-1}\|\le 10^{-6}$. The number of iterations for reaching the above accuracy will be indicated.

From the particular cases together with examples it will be clear of the efficiency of the proposed approach to BVPs for nonlinear third order differential equations by the reduction of them to operator equations for the nonlinear terms.

2.1.4.1. Case 1.

Consider the problem

$$\begin{aligned} u^{(3)}(t)&=f(t,u(t),u'(t),u''(t)),\quad 0<t<1,\\ u(0)&=u'(0)=u'(1)=0. \end{aligned}$$

Notice that for the case $f=f(t,u(t))$ in [19], using the lower and upper solutions method and the fixed point theorem on cones Yao and Feng established several results of solution and positive solution. For the case $f=f(t,u(t),u'(t))$ in [17] Feng and Liu also obtained existence results by the use of the upper and lower solutions method and a new maximum principle. It should be emphasized that the results of these two works are pure existence but not uniqueness.

The Green function associated with the considered problem has the form

$$G(t,s)=\begin{cases} \dfrac{s}{2}(t^2-2t+s), & 0\le s\le t\le 1,\\ \dfrac{t^2}{2}(s-1), & 0\le t\le s\le 1.\end{cases}$$

After differentiation of $G(t,s)$ we obtain

$$G_1(t,s)=\begin{cases} s(t-1), & 0\le s\le t\le 1,\\ t(s-1), & 0\le t\le s\le 1,\end{cases}$$

$$G_2(t,s) = \begin{cases} s, & 0 \le s \le t \le 1, \\ s-1, & 0 \le t \le s \le 1. \end{cases}$$

It is obvious that

$$G(t,s) \le 0, \; G_1(t,s) \le 0, \;\; 0 \le t, s \le 1$$

and we have

$$M_0 = \max_{0\le t\le 1} \int_0^1 |G(t,s)|\, ds = \frac{1}{12}, \quad M_1 = \max_{0\le t\le 1} \int_0^1 |G_1(t,s)|\, ds = \frac{1}{8},$$
$$M_2 = \max_{0\le t\le 1} \int_0^1 G_2(t,s)\, ds = \frac{1}{2}.$$

Example 2.1.1 (Example 7 in [19]). Consider the problem

$$\begin{aligned} u^{(3)}(t) &= -e^{u(t)}, \quad 0 < t < 1, \\ u(0) &= u'(0) = u'(1) = 0. \end{aligned} \tag{2.1.25}$$

Yao and Feng [19] using the lower and upper solutions method and the fixed point theorem on cones proved that the above problem has a solution $u(t)$ such that $\|u\| \le 1$, $u(t) > 0$ for $t \in (0,1)$ and $u(t)$ is an increasing function. Here, using the theoretical results obtained in the previous section we establish the results which are more strong than the above results.

Indeed, for the problem (2.1.25) $f = f(t,x) = -e^x$. In the domain

$$\mathcal{D}_M^+ = \Big\{(t,x) |\; 0 \le t \le 1, \; 0 \le x \le \frac{M}{12}\Big\}$$

there hold $-e^{M/12} \le f(t,x) \le 0$. So, with the choice $M = 1.1$ we have $-M \le f(t,x) \le 0$. Further, in $\mathcal{D}_M^+$ the function $f(t,x)$ satisfies the Lipschitz condition with $L_0 = e^{M/12} = 1.096$. Therefore, $q = L_0/12 = 0.0913$. By Theorem 2.1.5 the problem has a *unique* monotone positive solution $u(t)$ satisfying the estimates

$$0 \le u(t) \le \frac{M}{12} = \frac{1.1}{12} = 0.0917, \; 0 \le u'(t) \le \frac{M}{8} = \frac{1.1}{8} = 0.1357,$$
$$|u''(t)| \le \frac{M}{2} = \frac{1.1}{2} = 0.55.$$

Clearly, these results are better than those in [19].

The numerical solution of the problem obtained by the iterative method (2.1.17)-(2.1.21) after 5 iterations is depicted in Figure 2.1. From this figure it is clear that the solution is monotone, positive and is bounded by 0.0917 as shown above by the theory.

Example 2.1.2 (Example 8 in [19]). Consider the problem

$$\begin{aligned} u^{(3)}(t) &= -\frac{5u^3(t) + 4u(t) + 3}{u^2(t) + 1}, \quad 0 < t < 1, \\ u(0) &= u'(0) = u'(1) = 0. \end{aligned} \tag{2.1.26}$$

Yao and Feng in [19] showed that the above problem has a solution $u(t)$ such that $u(t) > 0$ for $t \in (0,1)$ and $u(t)$ is an increasing function. Similarly as in Example 4.1.1 we established that the problem (2.1.26) has a *unique* monotone positive solution $u(t)$ satisfying

$$0 \le u(t) \le 0.3417, \; 0 \le u'(t) \le 0.5125, \; |u''(t)| \le 2.05.$$

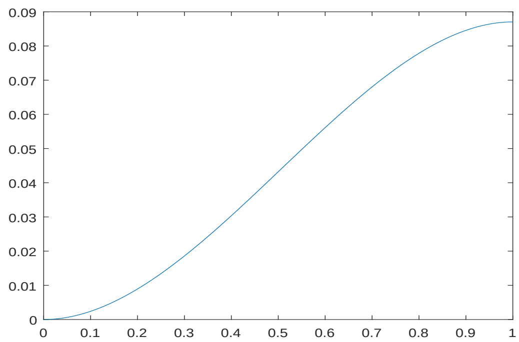

Figure 2.1: The graph of the approximate solution in Example 2.1.1

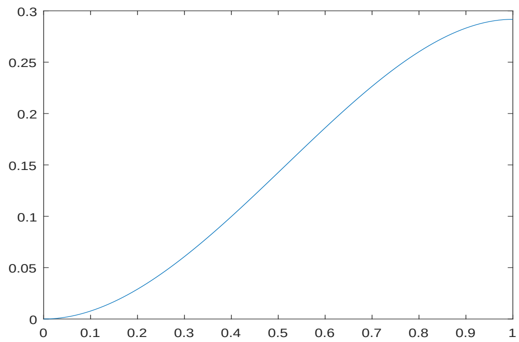

Figure 2.2: The graph of the approximate solution in Example 2.1.2

The numerical solution of the problem obtained by the iterative method (2.1.17)-(2.1.21) after 8 iterations is depicted in Figure 2.2. From this figure it is clear that the solution is monotone, positive and is bounded by 0.3417 as shown above by the theory.

Example 2.1.3 (Example 4.2 in [17])**.** Consider the problem

$$\begin{aligned} u^{(3)}(t) &= -e^{u(t)} - e^{u'(t)}, \quad 0 < t < 1, \\ u(0) &= u'(0) = u'(1) = 0. \end{aligned}$$

Using the lower and upper solutions method and a new maximum principle, Feng and Liu in [17] established that the above problem has a solution $u(t)$ such that $\|u\| \leq 1$, $u(t) > 0$ for $t \in (0,1)$ and $u(t)$ is an increasing function. Here, using Theorem 2.1.5 with the choice $M = 2.7$ we conclude that the problem has a *unique* monotone positive solution $u(t)$ satisfying the estimates

$$0 \leq u(t) \leq 0.2250, \; 0 \leq u'(t) \leq 0.3375, \; |u''(t)| \leq 1.350.$$

The numerical solution of the problem obtained by the iterative method (2.1.17)-(2.1.21) after 9 iterations is depicted in Figure 2.3. From this figure it is clear that the solution is monotone, positive and is bounded by 0.2250 as shown above by the theory.

Remark 2.1.2. In the above examples, it is easy to see that all the conditions of Theorem 2.1.7 are satisfied. Therefore, the sequences of approximations are increasing. This fact is also confirmed by the numerical experiments.

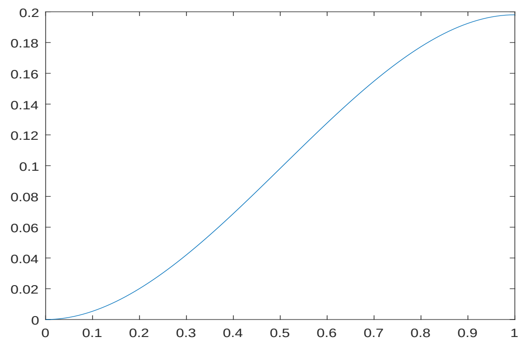

Figure 2.3: The graph of the approximate solution in Example 2.1.3

Remark 2.1.3. It should be emphasized that in [17] and [19] the authors used one very important assumption, which means that the nonlinear functions $f(t,x)$ or $f(t,x,y)$ satisfy one-side Lipschitz condition in x or x,y in the whole space $\mathbb{R}$ or $\mathbb{R}^2$, respectively. If now change the sign of the right-hand sides then this condition is not satisfied. Therefore, it is impossible to say anything about the solution of the problem. But Theorem 2.1.4 ensures the existence and uniqueness of a solution. Moreover, in a similar way as in Theorem 2.1.4 it is possible conclude that this solution is nonpositive.

2.1.4.2. Case 2.

Consider the problem

$$\begin{aligned} u^{(3)}(t) &= f(t,u(t),u'(t),u''(t)), \quad 0<t<1,\\ u(0) &= u'(0)=u''(1)=0. \end{aligned} \tag{2.1.27}$$

In [20] under the assumptions that the function $f(t,x,y,z)$ defined on $[0,1]\times\mathbb{R}^3\to\mathbb{R}$ is L_p-Caratheodory, and there exist functions $\alpha,\beta,\gamma,\delta\in L_p[0,1]$, $p\geq 1$, such that

$$|f(t,x,y,z)\leq \alpha(t)x+\beta(t)y+\gamma(t)z+\delta(t)|, \quad t\in(0,1)$$

and

$$A_0\|\alpha\|_p+A_1\|\beta\|_p+\|\gamma\|_p<1,$$

where A_0, A_1 are some constants depending on p, the problem has at least one solution. The tool used is the Leray-Schauder continuation principle. No examples are given for illustrating the theoretical results.

Here, assuming that the function $f(t,x,y,z)$ is continuous, we establish the existence of unique solution by Theorem 2.1.5. For the problem (2.1.27) the Green function is

$$G(t,s)=\begin{cases} -st+\dfrac{s^2}{2}, & 0\leq s\leq t\leq 1,\\ -\dfrac{t^2}{2}, & 0\leq t\leq s\leq 1. \end{cases}$$

The first and the second derivatives of this function are

$$G_1(t,s)=\begin{cases} -s, & 0\leq s\leq t\leq 1,\\ -t, & 0\leq t\leq s\leq 1, \end{cases}$$

$$G_2(t,s)=\begin{cases} 0, & 0\leq s\leq t\leq 1,\\ -1, & 0\leq t\leq s\leq 1. \end{cases}$$

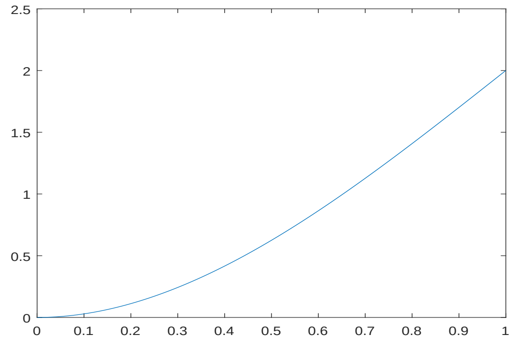

Figure 2.4: The graph of the approximate solution in Example 2.1.4

It is easy to see that

$$G(t,s) \le 0,\ G_1(t,s) \le 0,\ 0 \le t,s \le 1$$

and

$$M_0 = \max_{0\le t\le 1} \int_0^1 |G(t,s)|\, ds = \frac{1}{3}, \quad M_1 = \max_{0\le t\le 1} \int_0^1 |G_1(t,s)|\, ds = \frac{1}{2},$$
$$M_2 = \max_{0\le t\le 1} \int_0^1 |G_2(t,s)|\, ds = 1.$$

Example 2.1.4. Consider the following problem

$$\begin{aligned} &u'''(t) = -\frac{1}{36}\big(u'(t)\big)^2 + \frac{1}{24}u(t)u''(t) + \frac{1}{4}t^2 - 6, \quad 0 \le t \le 1, \\ &u(0) = u'(0) = u''(1) = 0. \end{aligned} \tag{2.1.28}$$

In this example

$$f(t,x,y,z) = -\frac{1}{36}y^2 + \frac{1}{24}xz + \frac{1}{4}t^2 - 6.$$

It is possible to verify that with $M = 7.5,\ L_1 = 0.3125,\ L_2 = 0.2083,\ L_3 = 0.1042.$ So, all the conditions of Theorem 2.1.5 are met, and the problem (2.1.28) has a unique positive solution satisfying the estimates $0 \le u(t) \le 2.5,\ 0 \le u'(t) \le 3.75,\ |u''(t)| \le 7.5.$

The numerical solution of the problem obtained by the iterative method (2.1.17)-(2.1.21) after 5 iterations is depicted in Figure 2.4. From this figure it is clear that the solution is bounded by 2.5 as shown above by the theory.

It is interesting that the problem (2.1.28) *has the exact solution* $u(t) = -t^3 + 3t^2$.

This solution satisfies the exact estimates $0 \le u(t) \le 2,\ 0 \le u'(t) \le 3,\ 0 \le u''(t) \le 6$ for $0 \le t \le 1$, which are better than the theoretical estimates above. On the grid with the gridsize $h = 0.01$ the maximal deviation of the obtained approximate solution and the exact solution is $3.7665e-04$.

2.1.4.3. Case 3.

Consider the problem

$$\begin{aligned} u^{(3)}(t) &= f(t,u(t),u'(t),u''(t)), \quad 0 < t < 1, \\ u(0) &= u'(1) = u''(1) = 0. \end{aligned} \tag{2.1.29}$$

Under the conditions similar to those in the previous case, Hopkins and Kosmatove in [20] established the existence of a solution of the problem without illustrative examples. Very recently, in [22] Li Yongxiang and Li Yanhong studied the existence of positive solutions of the problem (2.1.29) under conditions on the growth of the function $f(t,x,y,z)$ as $|x|+|y|+|z|$ tends to zero and infinity, including a Nagumo-type condition on y and z. The tool used is the fixed point index theory on cones.

Here, assuming that the function $f(t,x,y,z)$ is continuous, we can establish the existence results by the above theorems. For the problem (2.1.29) the Green function is

$$G(t,s)=\begin{cases}\dfrac{s^2}{2}, & 0\le s\le t\le 1,\\ st-\dfrac{t^2}{2}, & 0\le t\le s\le 1.\end{cases}$$

The first and the second derivatives of this function are

$$G_1(t,s)=\begin{cases}0, & 0\le s\le t\le 1,\\ s-t, & 0\le t\le s\le 1,\end{cases}$$

$$G_2(t,s)=\begin{cases}0, & 0\le s\le t\le 1,\\ -1, & 0\le t\le s\le 1.\end{cases}$$

It is easy to see that

$$G(t,s)\ge 0,\ G_1(t,s)\ge 0,\ 0\le t,s\le 1$$

and we obtain

$$M_0=\frac{1}{6},\ M_1=\frac{1}{2},\ M_2=1.$$

Example 2.1.5. Consider the following problem

$$\begin{aligned} u'''(t)&=\frac{1}{18}\big(u'(t)\big)^2-\frac{1}{12}u(t)u''(t)+\frac{1}{2}t+\frac{11}{2}, \quad 0\le t\le 1,\\ u(0)&=u'(1)=u''(1)=0. \end{aligned} \tag{2.1.30}$$

In this example

$$f(t,x,y,z)=\frac{1}{18}y^2-\frac{1}{12}xz+\frac{1}{2}t+\frac{11}{2}.$$

It is possible to verify that with $M=8,\ L_1=\frac{2}{3},\ L_2=\frac{4}{9},\ L_3=\frac{1}{9}$, and all the conditions of Theorem 2.1.5 are met. Therefore, the problem (2.1.30) has a unique positive solution, which is increasing and satisfies the estimates $0\le u(t)\le\frac{4}{3},\ 0\le u'(t)\le 4,\ -8\le u''(t)\le 0$.

The numerical solution of the problem obtained by the iterative method (2.1.17)-(2.1.21) after 6 iterations is depicted in Figure 2.5. From this figure it is clear that the solution is monotone, positive and is bounded by 4/3 as shown above by the theory.

It is possible to verify that the function $u(t)=t^3-3t^2+3t$ is the exact solution of the problem (2.1.30). This solution is positive, increasing and satisfies the exact estimates $0\le u(t)\le 1,\ 0\le u'(t)\le 3,\ -6\le u''(t)\le 0$ for $0\le t\le 1$, which are better than the theoretical estimates above. On the grid with the gridsize $h=0.01$ the maximal deviation of the obtained approximate solution and the exact solution is $3.6256e-04$.

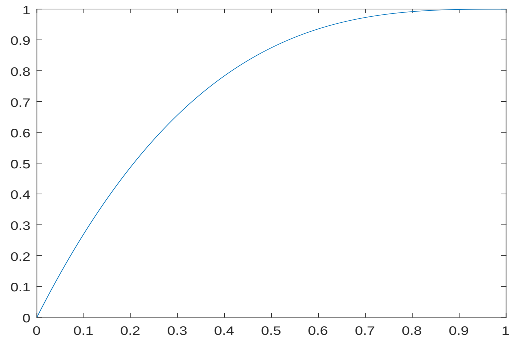

Figure 2.5: The graph of the approximate solution in Example 2.1.5

2.1.4.4. Case 4.

Consider the problem

$$\begin{aligned} &u^{(3)}(t) = f(t, u(t), u'(t), u''(t)), \quad 0 < t < 1, \\ &u(0) = u''(0) = u'(1) = 0. \end{aligned} \tag{2.1.31}$$

Using the lower and upper solutions method and Schauder fixed theorem on cones, Bai [23] established the existence of a solution under complicated conditions on the right-hand side function.

For the problem (2.1.31) the Green function is

$$G(t,s) = \begin{cases} \dfrac{t^2}{2} - t + \dfrac{s^2}{2}, & 0 \le s \le t \le 1, \\ t(s-1), & 0 \le t \le s \le 1. \end{cases}$$

The first and the second derivatives of this function are

$$G_1(t,s) = \begin{cases} t-1, & 0 \le s \le t \le 1, \\ s-1, & 0 \le t \le s \le 1, \end{cases}$$

$$G_2(t,s) = \begin{cases} 1, & 0 \le s \le t \le 1, \\ 0, & 0 \le t \le s \le 1. \end{cases}$$

Obviously,

$$G(t,s) \le 0, \; G_1(t,s) \le 0, \; 0 \le t, s \le 1$$

and it is easy to obtain

$$M_0 = \frac{1}{3}, \; M_1 = \frac{1}{2}, \; M_2 = 1.$$

In view of the above facts concerning the Green function, using theorems in the previous section we can establish the results on the existence of solution of the problem (2.1.31).

Example 2.1.6 (Example 3.5 in [23]).

$$\begin{aligned} u^{(3)}(t) &= -\frac{1}{4}\big[t + e^{u(t)} + (u'(t))^2 + u''(t)\big], \quad 0 < t < 1, \\ u(0) &= u''(0) = u'(1) = 0. \end{aligned} \tag{2.1.32}$$

Defining

$$\mathcal{D}_M^+ = \{(t,x,y,z) |\; 0 \le t \le 1, \; 0 \le x \le \frac{M}{3}, \; 0 \le y \le \frac{M}{2}, \; |z| \le M\},$$

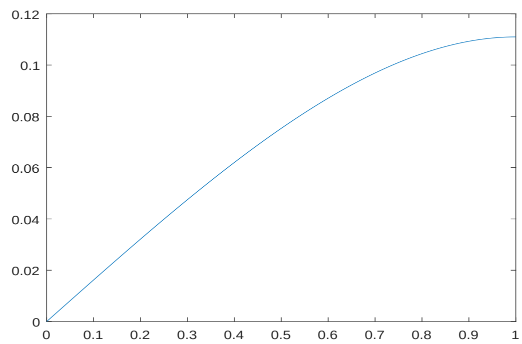

Figure 2.6: The graph of the approximate solution in Example 2.1.6

for $M = 0.835$ we have

$$-M \le f(t,x,y,z) = -\frac{1}{4}\big[t + e^x + y^2 + z\big] \le 0.$$

Further, it is easy to calculate the Lipschitz coefficients of $f(t,x,y,z)$:

$$L_0 = \frac{1}{4}e^{M/3} = 0.3302,\ L_1 = \frac{M}{4} = 0.2087,\ L_2 = 1.$$

Therefore, $q = L_0/3 + L_1/2 + L_2 = 0.4851 < 1$. By Theorem 2.1.5 the problem has a *unique monotone positive* solution $u(t)$ such that

$$0 \le u(t) \le M/3 = 0.2783,\ 0 \le u'(t) \le M/2 = 0.5,\ |u''(t)| \le 1.$$

Notice that in [23] Bai could only conclude that the problem has a positive solution.
The numerical solution obtained by the iterative method (2.1.17)-(2.1.21) after 5 iterations is depicted in Figure 2.6. From this figure it is clear that the solution is monotone, positive and is bounded by 0.2783 as shown above by the theory.

2.1.4.5. Case 5.

Consider the problem

$$\begin{aligned} u^{(3)}(t) &= f(t,u(t),u'(t),u''(t)), \quad 0<t<1,\\ u(0) &= u'(1) = u''(1) = 0. \end{aligned} \tag{2.1.33}$$

In [22], based on the fixed point index theory in cones authors established the existence of positive solution under complicated conditions posed on the growth of the function f including a Nagumo-type condition.

For the problem (2.1.33) the Green function is

$$G(t,s) = \begin{cases} \dfrac{s^2}{2}, & 0 \le s \le t \le 1,\\ st - \dfrac{t^2}{2}, & 0 \le t \le s \le 1. \end{cases}$$

The first and the second derivatives of this function is

$$G_1(t,s) = \begin{cases} 0, & 0 \le s \le t \le 1,\\ s - t, & 0 \le t \le s \le 1, \end{cases}$$

$$G_2(t,s) = \begin{cases} 0, & 0 \le s \le t \le 1, \\ -1, & 0 \le t \le s \le 1. \end{cases}$$

It is easy to see that

$$G(t,s) \ge 0,\ G_1(t,s) \ge 0,\ G_2(t,s) \le 0,\ 0 \le t,s \le 1$$

and

$$M_0 = \frac{1}{6},\ M_1 = \frac{1}{2},\ M_2 = 1.$$

In view of the above facts concerning the Green function, using theorems in the previous section we can establish the results on the existence of solution of the problem (2.1.33).

Example 2.1.7. Consider the following problem

$$\begin{aligned} u'''(t) &= \frac{1}{18}\big(u'(t)\big)^2 - \frac{1}{12}u(t)u''(t) + \frac{1}{2}t + \frac{11}{2}, \quad 0 \le t \le 1, \\ u(0) &= u'(1) = u''(1) = 0. \end{aligned} \tag{2.1.34}$$

In this example

$$\begin{aligned} f(t,x,y,z) &= \frac{1}{18}y^2 - \frac{1}{12}xz + \frac{1}{2}t + \frac{11}{2}, \\ f(t,0,0,0) &= \frac{1}{2}t + \frac{11}{2} > 0 \quad \forall t \in [0,1]. \end{aligned}$$

It is easy to verify that with $M = 8$ all the conditions of Theorem 2.1.4 are satisfied. Due to this the problem has a unique positive increasing solution satisfying the estimates $0 \le u(t) \le \frac{4}{3},\ 0 \le u'(t) \le 4,\ |u''(t)| \le 8.$

Notice that the function $u(t) = t^3 - 3t^2 + 3t$ is the exact solution of the problem (2.1.34). This solution is positive, increasing and satisfies the exact estimates $0 \le u(t) \le 1,\ 0 \le u'(t) \le 3,\ -6 \le u''(t) \le 0$ for $0 \le t \le 1$, which are better than the theoretical estimates above.

Example 2.1.8. Consider the following problem

$$\begin{aligned} u'''(t) &= u^3(t) + u(t)(u'(t))^2 + u(t)(u''(t))^2,\ 0 \le t \le 1, \\ u(0) &= u'(1) = u''(1) = 0. \end{aligned} \tag{2.1.35}$$

In this example

$$f(t,x,y,z) = x^3 + xy^2 + xz^2.$$

It is easy to verify that with $0 < M \le \sqrt{\frac{108}{23}}$ Theorem 2.1.5 guarantees that the problem (2.1.35) has a unique nonnegative monotone solution. Because $u(t) \equiv 0$ is a nonnegative solution of the problem, we conclude that the problem cannot have positive solution. This conclusion is contrary to the conclusion in [22]. Therefore, we think that there may be some inaccuracy in their results.

2.1.5. Conclusion

In this section, we have proposed a novel efficient approach to study fully third order differential equation with general two-point linear boundary conditions. The approach is based on the reduction of boundary value problems to fixed point problems for nonlinear operators for the right-hand sides of the equation but not for the function to be sought. The results are that we have established the existence, uniqueness, positivity and monotony of solution under the conditions, which are simpler and easier to verify than those of other authors. The applicability and advantages of the proposed approach are illustrated on some examples taken from the papers of other authors, where our approach gives better results.

The proposed approach can be applicable to other boundary value problems for the third order and higher orders nonlinear differential equations. This is the subject of our researches in the future.

2.2. Numerical methods for third order nonlinear BVPs

2.2.1. Introduction

In the previous section we have established the existence,uniqueness of solutions and the convergence of an iterative method on continuous level for the fully third order differential equations with general two-point linear boundary conditions. We also have shown some particular cases and examples for illustrating the obtained theoretical results. In this section we will discuss numerical realization of the proposed iterative method on continuous level. The investigation will be done for a case, namely, for Case 1 in the previous section. So, we consider the BVP

$$\begin{aligned} &u^{(3)}(t) = f(t, u(t), u'(t), u''(t)), \quad 0 < t < 1, \\ &u(0) = 0, u'(0) = 0, u'(1) = 0. \end{aligned} \tag{2.2.1}$$

In order to be easily tracked we recall some facts concerning the existence of solutions of the above problem. The Green function of the problem, and its first and second derivatives are

$$G_0(t,s) = \begin{cases} \dfrac{s}{2}(t^2 - 2t + s), & 0 \le s \le t \le 1, \\ \dfrac{t^2}{2}(s-1), & 0 \le t \le s \le 1. \end{cases}$$

$$G_1(t,s) = G'_t(t,s) = \begin{cases} s(t-1), & 0 \le s \le t \le 1, \\ t(s-1), & 0 \le t \le s \le 1, \end{cases}$$

$$G_2(t,s) = G''_{tt}(t,s) \begin{cases} s, & 0 \le s \le t \le 1, \\ s-1, & 0 \le t \le s \le 1. \end{cases} \tag{2.2.2}$$

We have $G_0(t,s) \le 0,\ G_1(t,s) \le 0$ in $Q = [0,1]^2$ and

$$\begin{aligned} &M_0 = \max_{0\le t\le 1} \int_0^1 |G(t,s)|\, ds = \frac{1}{12}, \quad M_1 = \max_{0\le t\le 1} \int_0^1 |G_1(t,s)|\, ds = \frac{1}{8}, \\ &M_2 = \max_{0\le t\le 1} \int_0^1 |G_2(t,s)|\, ds = \frac{1}{2}. \end{aligned} \tag{2.2.3}$$

Next, for each fixed real number $M > 0$ introduce the domain

$$\mathcal{D}_M = \{(t,x,y,z) |\ 0 \le t \le 1,\ |x| \le M_0 M,\ |y| \le M_1 M,\ |z| \le M_2 M\},$$

Theorem 2.2.1 (Existence and uniqueness of solution)**.** Assume that there exist numbers $M, L_0, L_1,\ L_2 \ge 0$ such that

$$|f(t,x,y,z)| \le M,$$

$$|f(t,x_2,y_2,z_2) - f(t,x_1,y_1,z_1)| \le L_0|x_2-x_1| + L_1|y_2-y_1| + L_2|z_2-z_1| \quad (2.2.4)$$

for any $(t,x,y,z),(t,x_i,y_i,z_i) \in \mathcal{D}_M \ (i=1,2)$ and

$$q := L_0M_0 + L_1M_1 + L_2M_2 < 1.$$

Then, the problem (2.2.1) has a unique solution $u(t)$ such that $|u(t)| \le M_0M$, $|u'(t)| \le M_1M, \ |u''(t)| \le M_2M$ for any $0 \le t \le 1$.

Below we recall the iterative method on continuous level for the problem:

1. Given
$$\varphi_0(t) = f(t,0,0,0). \quad (2.2.5)$$

2. Knowing $\varphi_k(t)$ $(k=0,1,...)$ compute
$$\begin{aligned} u_k(t) &= \int_0^1 G_0(t,s)\varphi_k(s)ds, \\ y_k(t) &= \int_0^1 G_1(t,s)\varphi_k(s)ds, \\ z_k(t) &= \int_0^1 G_2(t,s)\varphi_k(s)ds. \end{aligned} \quad (2.2.6)$$

3. Update
$$\varphi_{k+1}(t) = f(t,u_k(t),y_k(t),z_k(t)). \quad (2.2.7)$$

Set

$$p_k = \frac{q^k}{1-q}, \ d = \|\varphi_1 - \varphi_0\|. \quad (2.2.8)$$

Theorem 2.2.2 (Convergence)**.** Under the assumptions of Theorem 2.2.1 the above iterative method converges and there hold the estimates

$$\|u_k - u\| \le M_0p_kd, \quad \|u_k' - u'\| \le M_1p_kd, \quad \|u_k'' - u''\| \le M_2p_kd,$$

where u is the exact solution of the problem (2.2.1) and M_0, M_1, M_2 are given by (2.2.3).

2.2.2. Discrete iterative method 1

To numerically realize the above iterative method we construct the corresponding discrete iterative methods. For this purpose cover the interval $[0,1]$ by the uniform grid $\bar{\omega}_h = \{t_i = ih, \ h = 1/N, i = 0,1,...,N\}$ and denote by $\Phi_k(t), U_k(t), Y_k(t), Z_k(t)$ the grid functions, which are defined on the grid $\bar{\omega}_h$ and approximate the functions $\varphi_k(t), u_k(t), y_k(t), z_k(t)$ on this grid, respectively.

First, consider the following discrete iterative method, named **Method 1**:

1. Given
$$\Phi_0(t_i) = f(t_i,0,0,0), \ i = 0,...,N. \quad (2.2.9)$$

2. Knowing $\Phi_k(t_i),\ k = 0, 1, ...;\ i = 0, ..., N$, compute approximately the definite integrals (2.2.6) by the trapezoidal rule

$$\begin{aligned} U_k(t_i) &= \sum_{j=0}^{N} h\rho_j G_0(t_i, t_j)\Phi_k(t_j), \\ Y_k(t_i) &= \sum_{j=0}^{N} h\rho_j G_1(t_i, t_j)\Phi_k(t_j), \\ Z_k(t_i) &= \sum_{j=0}^{N} h\rho_j G_2^*(t_i, t_j)\Phi_k(t_j),\ i = 0, ..., N, \end{aligned} \tag{2.2.10}$$

where ρ_j are the weights

$$\rho_j = \begin{cases} 1/2,\ j = 0, N \\ 1,\ j = 1, 2, ..., N-1 \end{cases}$$

and

$$G_2^*(t, s) = \begin{cases} s, & 0 \le s < t \le 1, \\ s - 1/2, & s = t, \\ s - 1, & 0 \le t < s \le 1. \end{cases} \tag{2.2.11}$$

3. Update

$$\Phi_{k+1}(t_i) = f(t_i, U_k(t_i), Y_k(t_i), Z_k(t_i)). \tag{2.2.12}$$

In order to get the error estimates for the numerical approximate solution for $u(t)$ and its derivatives on the grid we need some following auxiliary results.

Proposition 2.2.3. Assume that the function $f(t, x, y, z)$ has all continuous partial derivatives up to second order in the domain $\mathcal{D}_M$. Then for the functions $u_k(t), y_k(t), z_k(t),\ k = 0, 1, ...,$ constructed by the iterative method (2.2.5)-(2.2.7), we have $z_k(t) \in C^3[0, 1],\ y_k(t) \in C^4[0, 1],\ u_k(t) \in C^5[0, 1]$.

Proof. We prove the proposition by induction. For $k = 0$, by the assumption on the function f we have $\varphi_0(t) \in C^2[0, 1]$ since $\varphi_0(t) = f(t, 0, 0, 0)$. Taking into account the expression (2.2.2) of the function $G_2(t, s)$ we have

$$z_0(t) = \int_0^1 G_2(t, s)\varphi_0(s)ds = \int_0^t s\varphi_0(s)ds - \int_t^1 (s-1)\varphi_0(s)ds.$$

It is easy to see that $z_0'(t) = \varphi_0(t)$. Therefore, $z_0(t) \in C^3[0, 1]$. This implies $y_0(t) \in C^4[0, 1],\ u_0(t) \in C^5[0, 1]$.

Now suppose $z_k(t) \in C^3[0, 1],\ y_k(t) \in C^4[0, 1],\ u_k(t) \in C^5[0, 1]$. Then, because $\varphi_{k+1}(t) = f(t, u_k(t), y_k(t), z_k(t))$ and the function f by the assumption has continuous derivative in all variables up to order 2, it follows that $\varphi_{k+1}(t) \in C^2[0, 1]$. Repeating the same argument as for $\varphi_0(t)$ above we obtain that $z_{k+1}(t) \in C^3[0, 1],\ y_{k+1}(t) \in C^4[0, 1],\ u_{k+1}(t) \in C^5[0, 1]$. Thus, the proposition is proved. □

Proposition 2.2.4. For any function $\varphi(t) \in C^2[0, 1]$ we have

$$\int_0^1 G_n(t_i, s)\varphi(s)ds = \sum_{j=0}^{N} h\rho_j G_n(t_i, t_j)\varphi(t_j) + O(h^2), \quad (n = 0, 1) \tag{2.2.13}$$

$$\int_0^1 G_2(t_i, s)\varphi(s)ds = \sum_{j=0}^{N} h\rho_j G_2^*(t_i, t_j)\varphi(t_j) + O(h^2). \tag{2.2.14}$$

Proof. In the case $n = 0, 1$, since the functions $G_n(t_i, s)$ are continuous at $s = t_i$ and are polynomials in s in the intervals $[0, t_i]$ and $[t_i, 1]$ we have

$$\int_0^1 G_n(t_i,s)\varphi(s)ds = \int_0^{t_i} G_n(t_i,s)\varphi(s)ds + \int_{t_i}^1 G_n(t_i,s)\varphi(s)ds$$
$$= h\big(\tfrac{1}{2}G_n(t_i,t_0)\varphi(t_0) + G_n(t_i,t_1)\varphi(t_1) + ... + G_n(t_i,t_{i-1})\varphi(t_{i-1}) + \tfrac{1}{2}G_2(t_i,t_i)\varphi(t_i)\big)$$
$$+ h\big(\tfrac{1}{2}G_n(t_i,t_i)\varphi(t_i) + G_n(t_i,t_{i+1})\varphi(t_{i+1}) + ... + G_n(t_i,t_{N-1})\varphi(t_{N-1})$$
$$+ \tfrac{1}{2}G_n(t_i,t_N)\varphi(t_N)\big) + O(h^2)$$
$$= \sum_{j=0}^{N} h\rho_j G_n(t_i,t_j)\varphi(t_j) + O(h^2) \quad (n = 0,1).$$

Thus, the estimate (2.2.13) is established. The estimate (2.2.14) is obtained using the following result, which is easily proved.

Lemma 2.2.1. Let $p(t)$ be a function having continuous derivatives up to second order in the interval $[0, 1]$ except for the point t_i, $0 < t_i < 1$, where it has a jump. Denote $\lim_{t\to t_i-0} p(t) = p_i^-$, $\lim_{t\to t_i+0} p(t) = p_i^+$, $p_i = \frac{1}{2}(p_i^- + p_i^+)$. Then

$$\int_0^1 p(t)dt = \sum_{j=0}^{N} h\rho_j p(j) + O(h^2), \tag{2.2.15}$$

where $p_j = p(t_j), j \neq i$.

□

Proposition 2.2.5. Under the assumption of Proposition 2.2.3 for any $k = 0, 1, ...$ we have

$$\|\Phi_k - \varphi_k\| = O(h^2), \tag{2.2.16}$$

$$\|U_k - u_k\| = O(h^2),\ \|Y_k - y_k\| = O(h^2),\ \|Z_k - z_k\| = O(h^2), \tag{2.2.17}$$

where $\|.\|_{C(\bar{\omega}_h)}$ is the max-norm of function on the grid $\bar{\omega}_h$.

Proof. We prove the proposition by induction. For $k = 0$ we have $\|\Phi_0 - \varphi_0\| = 0$. Next, by the first equation in (2.2.6) and Proposition 2.2.4 we have

$$u_0(t_i) = \int_0^1 G_0(t_i,s)\varphi_0(s)ds = \sum_{j=0}^{N} h\rho_j G_0(t_i,t_j)\varphi_0(t_j) + O(h^2) \tag{2.2.18}$$

for any $i = 0, ..., N$. On the other hand, in view of the first equation in (2.2.10) we have

$$U_0(t_i) = \sum_{j=0}^{N} h\rho_j G_0(t_i,t_j)\varphi_0(t_j). \tag{2.2.19}$$

Therefore, $|U_0(t_i) - u_0(t_i)| = O(h^2)$. Consequently, $\|U_0 - u_0\| = O(h^2)$.
Similarly, we have

$$\|Y_0 - y_0\| = O(h^2),\ \|Z_0 - z_0\| = O(h^2). \tag{2.2.20}$$

Now suppose that (2.2.16) and (2.2.17) are valid for $k \geq 0$. We shall show that these estimates are valid for $k + 1$.

Indeed, by the Lipschitz condition of the function f and the estimates (2.2.17) it is easy to obtain the estimate

$$\|\Phi_{k+1} - \varphi_{k+1}\| = O(h^2) \tag{2.2.21}$$

Now from the first equation in (2.2.6) by Proposition 2.2.4 we have

$$u_{k+1}(t_i) = \int_0^1 G_0(t_i, s)\varphi_{k+1}(s)ds = \sum_{j=0}^{N} h\rho_j G_0(t_i, t_j)\varphi_{k+1}(t_j) + O(h^2)$$

On the other hand by the first formula in (2.2.10) we have

$$U_{k+1}(t_i) = \sum_{j=0}^{N} h\rho_j G_0(t_i, t_j)\Phi_{k+1}(t_j).$$

From the above equalities, having in mind the estimate (2.2.21) we obtain the estimate

$$\|U_{k+1} - u_{k+1}\| = O(h^2).$$

Similarly, we obtain

$$\|Y_{k+1} - y_{k+1}\| = O(h^2),\ \|Z_{k+1} - z_{k+1}\| = O(h^2).$$

Thus, by induction we have proved the proposition. □

Combining Proposition 2.2.5 and Theorem 2.2.2 results in the following theorem.

Theorem 2.2.6. For the approximate solution of the problem (2.2.1) obtained by the discrete iterative method (2.2.9)-(2.2.12) on the uniform grid with gridsize h we have the estimates

$$\|U_k - u\| \le M_0 p_k d + O(h^2),\ \|Y_k - u'\| \le M_1 p_k d + O(h^2),\ \|Z_k - u''\| \le M_2 p_k d + O(h^2),$$

where M_0, M_1, M_2 are defined by (2.2.3) and p_k, d are defined by (2.2.8).

Remark 1. We perform the discrete iterative process (2.2.9)-(2.2.12) until $\|\Phi_{k+1} - \Phi_k\| \le TOL$, where TOL is a given tolerance. From Theorem 2.2.6 it is seen that the accuracy of the discrete approximate solution depends on both the number q defined in Theorem 2.2.1, which determines the number of iterations of the continuous iterative method and the gridsize h. The number q describes the nature of the BVP, therefore, it is necessary to choose an appropriate h consistent with q as the choice of very small h does not increase the accuracy of the approximate discrete solution.

2.2.3. Discrete iterative method 2

Consider another discrete iterative method, named **Method 2**. The steps of this method are the same as of Method 1 with an essential difference in Step 2 and now the number of grid points is even, $N = 2n$. Namely,

2'. Knowing $\Phi_k(t_i),\ k = 0, 1, ...;\ i = 0, ..., N$, compute approximately the definite integrals (2.2.6) by the modified Simpson rule

$$U_k(t_i) = F(G_0(t_i, .)\Phi_k(.)),$$
$$Y_k(t_i) = F(G_1(t_i, .)\Phi_k(.)),$$
$$Z_k(t_i) = F(G_2^*(t_i, .)\Phi_k(.)),$$

where

$$F(G_l(t_i,.)\Phi_k(.)) = \begin{cases} \sum_{j=0}^{N} h\rho_j G_l(t_i,t_j)\Phi_k(t_j) & \text{if } i \text{ is even} \\ \sum_{j=0}^{N} h\rho_j G_l(t_i,t_j)\Phi_k(t_j) + \dfrac{h}{6}\Big(G_l(t_i,t_{i-1})\Phi_k(t_{i-1}) - 2G_l(t_i,t_i)\Phi_k(t_i) \\ \quad +G_l(t_i,t_{i+1})\Phi_k(t_{i+1})\Big) & \text{if } i \text{ is odd,} \\ l=0,1;\ i=0,1,2,...,N. \end{cases}$$

ρ_j are the weights of the Simpson rule

$$\rho_j = \begin{cases} 1/3, & j=0,N \\ 4/3, & j=1,3,...,N-1 \\ 2/3, & j=2,4,...,N-2, \end{cases}$$

$F(G_2^*(t_i,.)\Phi_k(.))$ is calculated in the same way as $F(G_l(t_i,.)\Phi_k(.))$ above, where G_l is replaced by G_2^* defined by the formula (2.2.11).

Proposition 2.2.7. Assume that the function $f(t,x,y,z)$ has all continuous partial derivatives up to fourth order in the domain $\mathcal{D}_M$. Then for the functions $u_k(t), y_k(t), z_k(t), \varphi_{k+1}(t)$, $k=0,1,...,$ constructed by the iterative method (2.2.5)-(2.2.7) we have $z_k(t) \in C^5[0,1],\ y_k(t) \in C^6[0,1],\ u_k(t) \in C^7[0,1], \varphi_{k+1}(t) \in C^4[0,1]$.

Proposition 2.2.8. For any function $\varphi(t) \in C^4[0,1]$ we have

$$\int_0^1 G_l(t_i,s)\varphi(s)ds = F(G_l(t_i,.)\varphi(.)) + O(h^3), \quad (l=0,1) \tag{2.2.22}$$

$$\int_0^1 G_2(t_i,s)\varphi(s)ds = F(G_2^*(t_i,.)\varphi(.)) + O(h^3). \tag{2.2.23}$$

Proof. Recall that the interval $[0,1]$ is divided into $N=2n$ subintervals by the points $t_i = ih, h=1/N$. In each subinterval $[0,t_i]$ and $[t_i,1]$ the functions $G_l(t_i,s)$ are continuous as polynomials. Therefore, if i is an even number, $i=2m$ then we represent

$$\int_0^1 G_l(t_i,s)\varphi(s)ds = \int_0^{t_{2m}} G_l(t_i,s)\varphi(s)ds + \int_{t_{2m}}^1 G_l(t_i,s)\varphi(s)ds.$$

Applying the Simpson rule to the integrals in the right-hand side we obtain

$$\int_0^1 G_l(t_i,s)\varphi(s)ds = F(G_l(t_i,.)\varphi(.)) + O(h^4)$$

because by assumption $\varphi(t) \in C^4[0,1]$.
Now consider the case when i is an odd number, $i=2m+1$. In this case we represent

$$\begin{aligned} I = \int_0^1 G_l(t_i,s)\varphi(s)ds &= \int_0^{t_{2m}} G_l(t_i,s)\varphi(s)ds + \int_{t_{2m}}^{t_{2m+1}} G_l(t_i,s)\varphi(s)ds \\ &+ \int_{t_{2m+1}}^{t_{2m+2}} G_l(t_i,s)\varphi(s)ds + \int_{t_{2m+2}}^1 G_l(t_i,s)\varphi(s)ds. \end{aligned} \tag{2.2.24}$$

For simplicity we denote

$$f_j = G_l(t_i,s_j)\varphi(s_j)$$

Applying the Simpson rule to the first and the fourth integrals in the right-hand side (2.2.24) and the trapezoidal rule to the second and the third integrals, we obtain

$$\begin{aligned}
I = &\frac{h}{3}[f_0 + f_{2m} + 4(f_1 + f_3 + ... + f_{2m-1}) + 2(f_2 + f_4 + ... + f_{2m-2})] + O(h^4)\\
&+ \frac{h}{2}(f_{2m} + f_{2m+1}) + O(h^3) + \frac{h}{2}(f_{2m+1} + f_{2m+2}) + O(h^3)\\
&+ \frac{h}{3}[f_{2m+2} + f_{2n} + 4(f_{2m+3} + f_{2m+5} + ... + f_{2n-1}) + 2(f_{2m+4} + f_{2m+6} + ... + f_{2n-2})] + O(h^4)\\
&= \frac{h}{3}[f_0 + f_{2n} + 4(f_1 + f_3 + ... + f_{2n-1}) + 2(f_2 + f_4 + ... + f_{2n-2})]\\
&+ \frac{h}{6}(f_{2m} - 2f_{2m+1} + f_{2m+2}) + O(h^3)\\
&= F(G_l(t_i, .)\varphi(.)) + O(h^3).
\end{aligned}$$

Thus, in both cases of i, even or odd, we have the estimate (2.2.22).
The estimate (2.2.23) is obtained analogously as (2.2.22) if taking into account that

$$2G_2^*(t_i, t_i) = G_2^-(t_i, t_i) + G_2^+(t_i, t_i),$$

where $G_2^{\pm}(t_i, t_i) = \lim_{s\to t_i\pm 0} G_2(t_i, s)$. □

Theorem 2.2.9. Under the assumptions of Proposition 2.2.7, for the approximate solution of the problem (2.2.1) obtained by Discrete iterative method 2 on the uniform grid with gridsize h we have the estimates

$$\begin{aligned}
&\|U_k - u\| \le M_0 p_k d + O(h^3),\ \|Y_k - u'\| \le M_1 p_k d + O(h^3),\\
&\|Z_k - u''\| \le M_2 p_k d + O(h^3).
\end{aligned}$$

2.2.4. Examples

Consider some examples for confirming the validity of the obtained theoretical results and the efficiency of the proposed iterative method. For the first two examples the exact solutions are known, and for the third example the exact solution is not known.

Example 2.2.1 (Problem 2 in [35]). Consider the problem

$$\begin{aligned}
u'''(x) &= x^4 u(x) - u^2(x) + g(x),\ 0 < x < 1,\\
u(0) &= 0,\ u'(0) = -1,\ u'(1) = \sin(1),
\end{aligned} \tag{2.2.25}$$

where $g(x) = -3\sin(x) - (x-1)\cos(x) - x^4(x-1)\sin(x) + (x-1)^2\sin^2(x)$. It is possible to verify that the function $u^*(x) = (x-1)\sin(x)$ is the exact solution of the problem.

By setting $u(x) = v(x) + P(x)$, where $P(x) = \frac{1}{2}(1+\sin(1))x^2 - x$ is the polynomial of the second degree satisfying the boundary conditions in (2.2.25), the problem for $u(x)$ is reduced to the following problem for $v(x)$:

$$\begin{aligned}
v'''(x) &= x^4 v(x) - v^2(x) - 2P(x)v(x) + x^4 P(x) - P^2(x) + g(x),\ 0 < x < 1,\\
v(0) &= 0,\ v'(0) = 0,\ v'(1) = 0,
\end{aligned} \tag{2.2.26}$$

In order to apply Theorem 2.2.1, we need to determine the number M. For the right-hand side function

$$f(x, v) = -v^2(x) + x^4 v(x) - 2P(x)v(x) + x^4 P(x) - P^2(x) + g(x)$$

in the domain $\mathcal{D}_M = \{(x,v)|\ 0 \le t \le 1,\ |v| \le M_0 M\}$, where $M_0 = \frac{1}{12}$ we have

$$\begin{aligned}
|f| &\le |v|^2 + |v| + 2|P(x)||v| + |x^4 P(x)| + |P(x)|^2 + |g(x)| \\
&\le (\frac{M}{12})^2 + (1 + 2 * 0.2715)\frac{M}{12} + 0.1 + 0.2715^2 + 4.12 \\
&< \frac{M^2}{144} + \frac{1.55M}{12} + 4.3.
\end{aligned}$$

Here we use the estimates

$$|P(x)| \le 0.2715,\ |x^4 P(x)| \le 0.1,\ x \in [0,1],$$

that are easily obtained. Besides, for estimating $|g(x)|$ we use the estimates

$$|(x-1)\sin(x)| \le 0.2401,\ |x^4(x-1)\sin(x)| \le 0.0596,\ x \in [0,1].$$

It is easy to verify that with $M = 6$ then $\frac{M^2}{144} + \frac{1.55M}{12} + 4.3 < M$. Hence, for this chosen M we have $|f(x,v)| \le M$ in $\mathcal{D}_M$. Furthermore, in this domain the function $f(x,v)$ satisfies the Lipschitz condition in the variable v with the coefficient $L_0 = 2.543$. Therefore, $q = 0.2119$. Hence, all conditions of Theorem 2.2.1 are satisfied, so the problem has a unique solution and the iterative method converges. The results of the numerical experiments with two different tolerances are given in Tables 2.1- 2.3.

Table 2.1: The convergence in Example 2.2.1 for $TOL = 10^{-4}$

N	K	$Error_{trap}$	$Order$	$Error_{Simp}$	$Order$
8	3	9.9153e-04		9.7143e-04	
16	3	2.4646e-04	2.0083	1.3101e-04	2.8905
32	3	6.0906e-05	2.0167	1.6020e-05	3.0317
64	3	1.4563e-05	2.0643	1.2587e-06	3.6696
128	3	2.9796e-06	2.2891	8.8553e-07	0.5073
256	3	4.3187e-07	2.7865	8.8165e-07	0.0063

Table 2.2: The convergence in Example 2.2.1 for $TOL = 10^{-6}$

N	K	$Error_{trap}$	$Order$	$Error_{Simp}$	$Order$
8	4	9.99237e-04		9.7223e-04	
16	4	2.4734e-04	2.0044	1.3189e-04	2.8820
32	4	6.1802e-05	2.0008	1.6915e-05	2.9629
64	4	1.5462e-05	1.9989	2.1492e-06	2.9765
128	4	3.8797e-06	1.9947	2.8688e-07	2.9053
256	4	9.8437e-07	1.9787	5.2749e-08	2.4439
512	4	2.6054e-07	1.9177	2.3446e-08	1.1698
1024	4	7.9583e-08	1.7110	1.9786e-08	0.2448

In the above tables N is the number of grid points, K is the number of iterations, $Error_{trap}$, $Error_{Simp}$ are errors $\|U_K - u^*\|$ in the cases of using Method 1 and Method 2, respectively, $Order$ is the order of convergence calculated by

$$Order = \log_2 \frac{\|U_K^{N/2} - u^*\|}{\|U_K^N - u^*\|}.$$

Table 2.3: The convergence in Example 2.2.1 for $TOL = 10^{-10}$

N	K	$Error_{trap}$	$Order$	$Error_{Simp}$	$Order$
8	7	9.9235e-04		9.7222e-04	
16	7	2.4732e-04	2.0045	1.3187e-04	2.8822
32	7	6.1782e-05	2.0011	1.6896e-05	2.9643
64	7	1.5443e-05	2.0003	2.1301e-06	2.9877
128	7	3.8605e-06	2.0001	2.6774e-07	2.9923
256	7	9.6511e-07	2.0000	3.3544e-08	2.9965
512	7	2.4128e-07	2.0000	4.1977e-09	2.9984
1024	7	6.0319e-08	2.0000	5.2483e-10	2.9997

In the above formula the superscripts $N/2$ and N of U_K mean that U_K is computed on the grid with the corresponding number of grid points.

From the tables we observe that for each tolerance the number of iterations is constant and the errors of the approximate solution decrease with the rate (or order) close to 2 for Method 1 and close to 3 for Method 2 until they cannot improved. This can be explained as follows. Since the total error of the actual approximate solution consists of two terms: the error of the iterative method on continuous level and the error of numerical integration at each iteration, when these errors are balanced, the further increase of number of grid points N(or equivalently, the decrease of grid size h) cannot in general improve the accuracy of approximate solution.

Notice that in [35] the author used Newton-Raphson iteration method to solve nonlinear system of equations arising after discretization of the differential problem. The iteration process is continued until the maximum difference between two successive iterations , i.e., $\|U_{k+1} - U_k\|$ is less than 10^{-10}. The number of iterations for achieving this tolerance is not reported. The accuracy for some different N is given in Table 2.4 (see [35, Table 2]).

Table 2.4: The results in [35] for the problem in Example 2.2.1

N	8	16	32	64
Error	0.11921225e-01	0.33391170e-02	0.87742222e-03	0.23732412e-03

From the tables of our results and of Pandey it is clear that our method gives much better accuracy.

Example 2.2.2 (Problem 2 in [36]). Consider the problem

$$\begin{aligned} u'''(x) &= -xu''(x) - 6x^2 + 3x - 6,\ 0 < x < 1, \\ u(0) &= 0,\ u'(0) = 0,\ u'(1) = 0. \end{aligned}$$

It is easy to verify that with $M = 18, L_0 = L_1 = 0, L_2 = 1, q = 0.5$ all conditions of Theorem 2.2.1 are satisfied, so the problem has a unique solution. This solution is $u(x) = x^2(\frac{3}{2} - x)$. The results of the numerical experiments with different tolerances are given in Tables 2.5, 2.6 and 2.7.

Notice that in [36] the author used Gauss-Seidel iteration method to solve linear system of equations arisen after discretization of the differential problem. The iteration process is continued until the maximum difference between two

Table 2.5: The convergence in Example 2.2.2 for $TOL = 10^{-4}$

N	K	$Error_{trap}$	$Order$	$Error_{Simp}$	$Order$
8	6	0.0078		9.7662e-04	
16	6	0.0020	2.0000	1.2215e-04	2.9991
32	6	4.8837e-04	1.9998	1.5345e-05	2.9929
64	6	1.2216e-04	1.9992	1.9936e-06	2.9443
128	6	3.0604e-05	1.9969	3.2471e-07	2.6181
256	6	7.7157e-06	1.9878	1.1612e-07	1.4835

Table 2.6: The convergence in Example 2.2.2 for $TOL = 10^{-6}$

N	K	$Error_{trap}$	$Order$	$Error_{Simp}$	$Order$
8	8	0.0078		9.7662e-04	
16	6	0.0020	2.0000	1.2215e-04	2.9991
32	6	4.8837e-04	1.9998	1.5345e-05	2.9929
64	6	1.2216e-04	1.9992	1.9936e-06	2.9443
128	6	3.0604e-05	1.9969	3.2471e-07	2.6181
256	6	7.7157e-06	1.9878	1.1612e-07	1.4835
512	6	1.9937e-06	1.9524	9.0051e-08	0.3868
1024	6	5.6316e-07	1.8238	8.6794e-08	0.0532

Table 2.7: The convergence in Example 2.2.2 for $TOL = 10^{-10}$

N	K	$Error_{trap}$	$Error_{Simp}$	N	K	$Error_{trap}$	$Error_{Simp}$
8	11	0.0078	2.0650e-13	64	11	1.2207e-04	2.5890e-13
16	11	0.0020	2.6790e-13	128	11	3.0518e-05	2.5790e-13
32	11	4.8828e-04	2.6279e-13	256	11	7.6294e-06	2.5802e-13

Table 2.8: The results in [36] for the problem in Example 2.2.2

N	128	256	512	1024
Error	0.30696392e-4	0.61094761(-5)	0.14379621e-5	0.41723251e-6
Iter	53	5	3	4

successive iterations, i.e., $\|U_{k+1} - U_k\|$ is less than 10^{-10}. The results for some different N are given in Table 2.8.
From Tables 2.5-2.7 of our results and Table 2.8 of Pandey's results, it is clear that our method gives better accuracy and requires less computational work.

Example 2.2.3. Consider the problem for fully third order differential equation

$$\begin{aligned} &u'''(x) = -e^{u(x)} - e^{u'(x)} - \frac{1}{10}(u''(x))^2, \ 0 < x < 1, \\ &u(0) = 0, \ u'(0) = 0, \ u'(1) = 0. \end{aligned} \tag{2.2.27}$$

For the above problem the exact solution is not known. It is easy to verify that all conditions of Theorem 2.2.1 are satisfied with $M = 3, L_0 = 1.284, L_1 = 1.455, L_2 = 0.3$ and $q = 0.4389$. So, the problem has a unique solution and the iterative method for it converges.

The numerical solution of the problem is depicted in Figure 2.7.

Table 2.9: The convergence in Example 2.2.3 for $TOL = 10^{-10}$

N	8	16	32	64	128	256
K	15	15	15	15	15	15

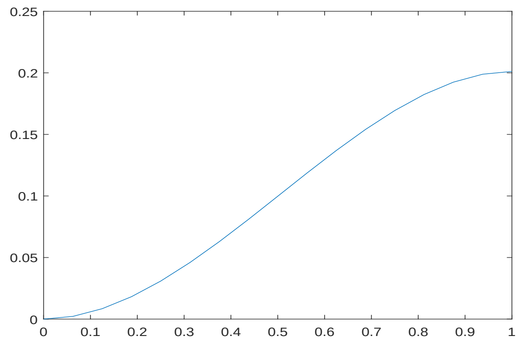

Figure 2.7: The graph of the approximate solution in Example 2.2.3.

Notice that in [17] the authors could only establish the existence but not the uniqueness of a solution to the equation $u'''(x) = -e^{u(x)}$ associated with the boundary conditions as in (2.2.27), and later, in [19] Yao and Feng also could only obtain the similar result for the equation $u'''(x) = -e^{u(x)} - e^{u'(x)}$.

Remark 2.2.1 (Convergence of the iterative method)**.** It should be remarked that Theorem 2.2.1 provides only sufficient conditions for the existence and uniqueness of a solution to the problem (2.2.1) and Theorem 2.2.2 gives the convergence rate of the iterative method for finding the solution. When these conditions are not satisfied the iterative method may converge or not converge. Below we give some examples for illustrating this statement.

First, consider the problem

$$u'''(x) = -e^{u(x)} - e^{u'(x)} - (u''(x))^2, \; 0 < x < 1,$$
$$u(0) = 0, \; u'(0) = 0, \; u'(1) = 0.$$

For this problem the right-hand side function is $f(x, u, y, z) = -e^u - e^y - z^2$. In the domain

$$\mathcal{D}_M = \left\{(x, u, y, z) | \; 0 \le x \le 1, \; |u| \le \frac{M}{12}, \; |y| \le \frac{M}{8}, \; |z| \le \frac{M}{2}\right\}$$

we have

$$g(M) := \max_{(x,u,y,z)\in\mathcal{D}_M} |f(x, u, y, z)| = e^{M/12} + e^{M/8} + \big(\frac{M}{2}\big)^2.$$

It is easy to verify that $g(M) \ge M + 1.4019 > M$ for any $M > 0$. Hence, there does not exist $M >$ such that $|f(x, u, y, z)| \le M \; \forall (x, u, y, z) \in \mathcal{D}_M$. Therefore, Theorem 2.2.1 cannot guarantee the existence and uniqueness of a solution and the convergence of the iterative method. Nevertheless, for $TOL = 10^{-10}$ the iterative method converges after 23 iterations.

Next, an example when the conditions of Theorem 2.2.1 are not satisfied and the iterative method does not converge is for the equation

$$u'''(x) = -e^{u(x)} - e^{u'(x)} - (u''(x))^2 + 5u''(x) + 10, \; 0 < x < 1.$$

2.2.5. On some extensions of the problem

2.2.5.1. The problem on large intervals

First consider the problem (2.2.1) on the interval $[0, T]$, i.e., the problem

$$\begin{aligned} u^{(3)}(t) &= f(t, u(t), u'(t), u''(t)), \quad 0 < t < T, \\ u(0) &= 0, u'(0) = 0, u'(T) = 0. \end{aligned} \tag{2.2.28}$$

For this problem, it is easy to verify that the Green function is

$$G_0(t,s) = \begin{cases} \frac{s}{2}(\frac{t^2}{T} - 2t + s), & 0 \le s \le t \le T, \\ \frac{t^2}{2}(\frac{s}{T} - 1), & 0 \le t \le s \le T. \end{cases}$$

The first and the second derivatives of this function with respect to t are

$$G_1(t,s) = \begin{cases} s(\frac{t}{T} - 1), & 0 \le s \le t \le T, \\ t(\frac{s}{T} - 1), & 0 \le t \le s \le T, \end{cases}$$

$$G_2(t,s) = \begin{cases} \frac{s}{T}, & 0 \le s \le t \le T, \\ \frac{s}{T} - 1, & 0 \le t \le s \le T. \end{cases}$$

It is easy to see that $G_0(t,s) \le 0$, $G_1(t,s) \le 0$ in $Q = [0,1]^2$ and

$$\begin{aligned} M_0 &= \max_{0 \le t \le T} \int_0^T |G(t,s)|\, ds = \frac{T^3}{12}, \quad M_1 = \max_{0 \le t \le T} \int_0^T |G_1(t,s)|\, ds = \frac{T^2}{8}, \\ M_2 &= \max_{0 \le t \le T} \int_0^T |G_2(t,s)|\, ds = \frac{T}{2}. \end{aligned} \tag{2.2.29}$$

Clearly, the numbers M_i $(i = 0, 1, 2)$ increase with the increase of T. Therefore, the domain $\mathcal{D}_M$ becomes more extended. This implies that the Lipschitz coefficients L_0, L_1, L_2 of the function $f(t, x, y, z)$ with respect to x, y, z do not decrease, and accordingly, the number $q = L_0M_0 + L_1M_1 + L_2M_2$ increases. This leads to narrowing the scope of applicability of Theorem 2.2.1 on the existence and uniqueness of solution and Theorem 2.2.2 on the convergence of the iterative method.

For demonstrating the above remark we consider some examples.

Example 2.2.4. Consider the problem on $[0, T]$ for the equation of Example 2.2.2, namely, the problem

$$\begin{aligned} u'''(x) &= -xu''(x) - 6x^2 + 3x - 6, \; 0 < x < T, \\ u(0) &= 0, \; u'(0) = 0, \; u'(T) = 0. \end{aligned}$$

Below are the results of convergence for Discrete iterative method 2 with $n = 256$ for some T:

Table 2.10: The convergence in Example 2.2.4 for $TOL = 10^{-6}$

T	1	2	3	4	5
K	8	18	82	2009	no convergence

Here K is the number of iterations for achieving the given tolerance TOL. Notice that from $T = 2$ the conditions of Theorem 2.2.1 are not satisfied but only from $T = 5$ the iterative method diverges. From Table 2.10 clearly that the convergence of the iterative method depends on the width of the interval, where the problem is considered.

Example 2.2.5. Consider the problem

$$u'''(x) = -\frac{1}{6}e^{-u^2} + e^{-(u'')^2}, \ 0 < x < T,$$
$$u(0) = 0, \ u'(0) = 0, \ u'(T) = 0.$$

For this example the right-hand side function is $f = f(x,u,y,z) = -\frac{1}{6}e^{-u^2} + e^{-(z)^2}$. In any domain

$$\mathcal{D}_M = \left\{ (x,u,y,z) |\ 0 \le x \le T, \ |u| \le \frac{T^3}{12}M, \ |y| \le \frac{T^2}{8}M, \ |z| \le \frac{T}{2}M \right\},$$

we always have $|f| \le \frac{7}{6}$. Therefore, in Theorem 2.2.1 we take $M = \frac{7}{6}$. The Lipschitz coefficients of the function f are $L_0 = 0.1430, L_1 = 0, L_2 = 0.8579$. So, $q = 0.1430\ \frac{T^3}{12} + 0.8579\ \frac{T}{2} = 0.0119\ T^3 + 0.4289\ T$. Clearly, for large values of T not all conditions of Theorem 2.2.1 are satisfied, and it is expected that the iterative method will diverge for large T. But it is interesting that this does not occur. Below are the results of the convergence of the iterative method for $n = 200$.

Table 2.11: The convergence in Example 2.2.5 for $TOL = 10^{-6}$

T	1	3	5	10	15	20	40	100
K	6	12	13	16	18	20	27	37

The approximate solution for $T = 100$ is depicted in Figure 2.8.

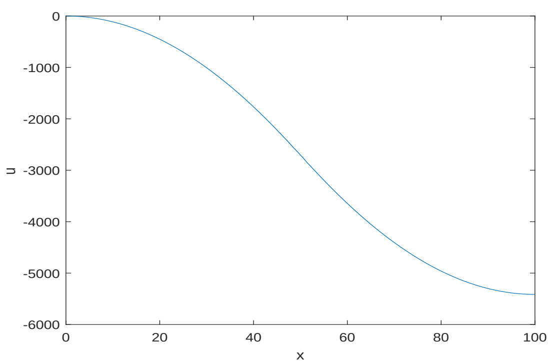

Figure 2.8: The graph of the approximate solution in Example 2.2.5.

2.2.5.2. The problem for unbounded nonlinear terms

For the problem with unbounded nonlinear terms (right-hand sides) $f(t,u,y,z)$ caused by singular points, of course, Theorem 2.2.1 cannot work, and Theorem 2.2.2 cannot ensure the convergence of the iterative method. But it is interesting that in some special cases the discrete iterative methods still converge. Below we report some nonlinear terms $f(t,u,y,z)$ for which the iterative method converge:

$$(i)\ \frac{u^2}{\sqrt{|t-\frac{\pi}{4}|}}+e^y+1,\quad (ii)\ \frac{u^2}{|t-\frac{\pi}{4}|}+e^y+z^2+1,\quad (iii)\ \frac{u^2}{|t-\frac{\sqrt{2}}{4}|}+e^y+z^2+1.$$

Notice that in the above three functions the singular points are irrational points, therefore, when using the discrete methods on the grids with rational points then the denominators always are not zero. For this reason the computations can be performed.

When we use the uniform grids with the number of grid points $n=2^k, k=3,4,5,...,$ the iterative methods also converge for $f=\dfrac{u^2}{\sqrt{|t-\frac{1}{3}|}}+e^y+1$. This is due to the fact that $i/2^k \neq 1/3$ for any i and k.

Above we only made some remarks on the problem (2.2.1) when the nonlinear term is unbounded. In the future we will study this issue deeply.

2.2.6. Conclusion

In this section, we established the existence and uniqueness of solution for a boundary value problem for fully third order differential equations. Next, for finding this solution we proposed iterative methods at both continuous and discrete levels. The numerical realization of the discrete iterative methods are very simple. It is based on the popular trapezoidal rule and a modified Simpson rule for numerical integration. One of the important results is that we obtained an estimate for the total error of the approximate solution which is actually obtained. This total error depends on the number of iterations performed and the discretization parameter. The validity of the theoretical results and the efficiency of the iterative methods are illustrated in examples. In addition, we made some remarks on the iterative method for two extensions of the problem for large intervals and when the nonlinear terms are unbounded due to interior singular points. In the future we will deeply study these issues.

The method for investigating the existence and uniqueness of solution and the iterative schemes for finding solution in this section can be applied to other third order nonlinear boundary value problems, and in general, for higher order nonlinear boundary value problems.

Chapter 3

Existence results and iterative method for some nonlinear ODEs with integral boundary conditions

3.1. Existence results and iterative method for fully third order nonlinear integral boundary value problems

3.1.1. Introduction

In this section, we consider the boundary value problem

$$u'''(t) = f(t, u(t), u'(t), u''(t)), \quad 0 < t < 1, \tag{3.1.1}$$

$$u(0) = u'(0) = 0, \quad u(1) = \int_0^1 g(s)u(s)ds, \tag{3.1.2}$$

where $f : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}^+, g : [0, 1] \to \mathbb{R}^+$.

This problem is a natural generalization of the problem

$$\begin{aligned} u'''(t) + f(u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = 0, \ u(1) = \int_0^1 g(t)u(t)dt. \end{aligned} \tag{3.1.3}$$

studied recently by Guendouz et al. in [47]. There, by applying the Krasnoselskii's fixed point theorem on cones they established the existence results of positive solutions of the problem. This technique was used also by Benaicha and Haddouchi in [48] for an integral boundary problem for a fourth order nonlinear equation.

It should be emphasized that in all of the above-mentioned works the authors only could (even could not) show examples of the nonlinear terms satisfying required sufficient conditions, but *no exact solutions are shown. Moreover, the known results are of purely theoretical characteristics concerning the existence of solutions but not methods for finding solutions.*

Here, by the method of reducing BVPs to operator equation for right-hand sides developed in [13, 14, 85, 88] we establish the existence, uniqueness and positivity of solution and propose an iterative method for finding the solution. Some examples demonstrate the validity of the obtained theoretical results and the efficiency of the iterative method. Especially, one example of exact solution of the problem is constructed so that the functions f and g satisfy the required conditions.

3.1.2. Existence results

To investigate the problem (3.1.1)-(3.1.2) we associate it with an operator equation as follows.

First, we denote the space of pairs $w = (\varphi, \alpha)^T$, where $\varphi \in C[0,1], \alpha \in \mathbb{R}$, by $\mathcal{B}$, i.e., set $\mathcal{B} = C[0,1] \times \mathbb{R}$, and equip it with the norm

$$\|w\|_{\mathcal{B}} = \max(\|\varphi\|, k|\alpha|), \tag{3.1.4}$$

where $\|\varphi\| = \max_{0\le t\le 1} |\varphi(t)|$, k is a number, $k \ge 1$. The constant k will have a significance in the conditions for the existence and uniqueness of solution. Later, in examples the selection of it will depend on particular cases.

Further, define the operator $A : \mathcal{B} \to \mathcal{B}$ by the formula

$$Aw = \begin{pmatrix} f(t, u(t), u'(t), u''(t)) \\ \int_0^1 g(s)u(s)ds \end{pmatrix}, \tag{3.1.5}$$

where $u(t)$ is the solution of the problem

$$u'''(t) = \varphi(t), \quad 0 < t < 1, \tag{3.1.6}$$

$$u(0) = u'(0) = 0, \; u(1) = \alpha. \tag{3.1.7}$$

It is easy to verify the following lemma.

Lemma 3.1.1. If $w = (\varphi, \alpha)^T$ is a fixed point of the operator A in the space $\mathcal{B}$, i.e., is a solution of the operator equation

$$Aw = w \tag{3.1.8}$$

in $\mathcal{B}$, then the function $u(t)$ defined from the problem (3.1.6)-(3.1.7) is a solution of the original problem (3.1.1)-(3.1.2).
Conversely, if $u(t)$ is a solution of (3.1.1)-(3.1.2), then the pair $(\varphi, \alpha)^T$, where

$$\varphi(t) = f(t, u(t), u'(t), u''(t)), \tag{3.1.9}$$

$$\alpha = \int_0^1 g(s)u(s)ds, \tag{3.1.10}$$

is a solution of the operator equation (3.1.8).

Thus, by this lemma, the problem (3.1.1)-(3.1.2) is reduced to the fixed point problem for A.

Remark that the above operator A, which is defined on pairs of functions $\varphi(t)$, $t \in [0,1]$ and boundary values α of $u(t)$ at $t = 1$, is similar to the mixed boundary-domain operator introduced in [89] for studying biharmonic type equation.

Now, we study the properties of A. For this purpose, notice that the problem (3.1.6)-(3.1.7) has a unique solution representable in the form

$$u(t) = \int_0^1 G_0(t,s)\varphi(s)ds + \alpha t^2, \quad 0 < t < 1, \tag{3.1.11}$$

where

$$G_0(t,s) = \begin{cases} -\frac{1}{2}s(1-t)(2t - ts - s), & 0 \le s \le t \le 1 \\ -\frac{1}{2}(1-s)^2 t^2, & 0 \le t \le s \le 1 \end{cases}$$

is the Green function of the operator $u'''(t)$ associated with the homogeneous boundary conditions $u(0) = u'(0) = u(1) = 0$.

Differentiating both sides of (3.1.11) gives

$$u'(t) = \int_0^1 G_1(t,s)\varphi(s)ds + 2\alpha t, \tag{3.1.12}$$

$$u''(t) = \int_0^1 G_2(t,s)\varphi(s)ds + 2\alpha, \tag{3.1.13}$$

where

$$G_1(t,s) = \begin{cases} -s(st-2t+1), & 0 \le s \le t \le 1, \\ -(1-s)^2 t, & 0 \le t \le s \le 1, \end{cases}$$

$$G_2(t,s) = \begin{cases} -s(s-2), & 0 \le s \le t \le 1, \\ -(1-s)^2, & 0 \le t \le s \le 1. \end{cases}$$

It is easily seen that $G_0(t,s) \le 0$ in $Q = [0,1]^2$, and

$$\begin{aligned} M_0 &= \max_{0\le t\le 1} \int_0^1 |G_0(t,s)|ds = \frac{2}{81}, \\ M_1 &= \max_{0\le t\le 1} \int_0^1 |G_1(t,s)|ds = \frac{1}{18}, \\ M_2 &= \max_{0\le t\le 1} \int_0^1 |G_2(t,s)|ds = \frac{2}{3}. \end{aligned} \tag{3.1.14}$$

Therefore, from (3.1.11), (3.1.12), (3.1.13) and (3.1.14) we obtain

$$\begin{aligned} \|u\| &\le M_0\|\varphi\| + |\alpha|, \\ \|u'\| &\le M_1\|\varphi\| + 2|\alpha|, \\ \|u''\| &\le M_2\|\varphi\| + 2|\alpha|. \end{aligned} \tag{3.1.15}$$

Now for any number $M > 0$ define the domain

$$\begin{aligned} \mathcal{D}_M = \{(t,x,y,z) \mid 0 \le t \le 1, \ |x| \le (M_0 + \frac{1}{k})M, \\ |y| \le (M_1 + \frac{2}{k})M, \ |z| \le (M_2 + \frac{2}{k})M\}. \end{aligned} \tag{3.1.16}$$

Next, denote

$$C_0 = \int_0^1 g(s)ds, \ C_2 = \int_0^1 s^2 g(s)ds. \tag{3.1.17}$$

Lemma 3.1.2. Suppose that the function $f(t,x,y,z)$ is continuous and bounded by M in $\mathcal{D}_M$, i.e.,

$$|f(t,x,y,z)| \le M \quad \text{in } \mathcal{D}_M \tag{3.1.18}$$

and

$$q_1 := kC_0M_0 + C_2 \le 1. \tag{3.1.19}$$

Then the operator A defined by (3.1.5) maps the closed ball $B[0,M]$ in $\mathcal{B}$ into itself.

Proof. Take any $w = (\varphi, \alpha)^T \in B[0, M]$. Then $\|\varphi\| \le M$ and $k|\alpha| \le M$. Let $u(t)$ be the solution of the problem (3.1.6)-(3.1.7). Then from the estimates (3.1.15) for the solution $u(t)$ and its derivatives we obtain

$$\|u\| \le \left(M_0 + \frac{1}{k}\right) M, \quad \|u'\| \le \left(M_1 + \frac{2}{k}\right) M, \quad \|u''\| \le \left(M_2 + \frac{2}{k}\right) M.$$

Therefore, $(t, u(t), u'(t), u''(t)) \in \mathcal{D}_M$. Hence, by the assumption (3.1.18) we have

$$|f(t, u(t), u'(t), u''(t))| \le M.$$

Now estimate $I := k|\int_0^1 g(s)u(s)ds|$. In view of the representation (3.1.11) we obtain

$$\begin{aligned} I \le & k \int_0^1 g(s) \left| \int_0^1 G_0(s, y)\varphi(y)dy \right| ds + k|\alpha| \int_0^1 g(s)s^2 ds \\ & \le kC_0M_0M + C_2M = (kC_0M_0 + C_2)M \le M. \end{aligned} \tag{3.1.20}$$

The inequalities on the above line occur due to (3.1.14), (3.1.17) and the assumption (3.1.19).

Therefore, by the definition of the norm in the space $\mathcal{B}$ we have

$$\|Aw\|_{\mathcal{B}} \le M,$$

which means that the operator A maps the closed ball $B[0, M]$ in $\mathcal{B}$ into itself. The lemma is proved. □

Lemma 3.1.3. The operator A is a compact operator in $B[0, M]$.

Proof. The compactness of A follows from the compactness of the integral operators (3.1.11), (3.1.12), (3.1.13), the continuity of the function $f(t, x, y, z)$ and the compactness of the integral operator $\int_0^1 g(s)u(s)ds$. □

Theorem 3.1.1 (Existence of solution)**.** Suppose the conditions of Lemma 3.1.2 are satisfied. Then the problem (3.1.1)-(3.1.2) has a solution.

Proof. By Lemma 3.1.2 and Lemma 3.1.3, the operator A is a compact operator mapping the closed ball $B[0, M]$ in the Banach space $\mathcal{B}$ into itself. Therefore, according to the Schauder fixed point theorem, the operator A has a fixed point in $B[0, M]$. This fixed point corresponds to a solution of the problem (3.1.1)-(3.1.2). □

In order to establish the existence of positive solutions of (3.1.1)-(3.1.2), let us introduce the domain

$$\begin{aligned} \mathcal{D}_M^+ = \{(t, x, y, z) \mid 0 \le t \le 1,\ 0 \le x \le (M_0 + \frac{1}{k})M, \\ |y| \le (M_1 + \frac{2}{k})M,\ |z| \le (M_2 + \frac{2}{k})M\}, \end{aligned} \tag{3.1.21}$$

and the strip

$$S_M = \{w = (\varphi, \alpha)^T \mid -M \le \varphi \le 0,\ 0 \le k\alpha \le M\} \tag{3.1.22}$$

in the space $\mathcal{B}$.

Theorem 3.1.2 (Positivity of solution)**.** Suppose the function $f(t,x,y,z)$ is continuous and

$$-M \le f(t,x,y,z) \le 0 \text{ in } \mathcal{D}_M^+, \tag{3.1.23}$$

and the condition (3.1.19) is satisfied. Then the problem (3.1.1)-(3.1.2) has a nonnegative solution. Moreover, if $f(t,0,0,0) \not\equiv 0$ then this solution is positive.

Proof. It is easy to verify that under the conditions of the theorem, the operator A maps S_M into itself. Indeed, for any $w \in S_M$, $w = (\varphi, \alpha)^T$, $-M \le \varphi \le 0, 0 \le k\alpha \le M$. Since $G_0(t,s) \le 0$, from (3.1.11), (3.1.12), (3.1.13) we have

$$0 \le u(t) \le (M_0 + \frac{1}{k})M, \ |u'(t)| \le (M_1 + \frac{2}{k})M, \ |u''(t)| \le (M_2 + \frac{2}{k})M$$

for $0 \le t \le 1$. Therefore, for the solution $u(t)$ of (3.1.6)-(3.1.7) we have $(t, u(t), u'(t), u''(t)) \in \mathcal{D}_M^+$, and by the condition (3.1.23) we obtain

$$-M \le f(t, u(t), u'(t), u''(t)) \le 0.$$

As in the proof of Theorem 3.1.1 we also have the estimate

$$0 \le k \int_0^1 g(s)u(s)ds \le M.$$

Hence, $\big(f(t,u(t),u'(t),u''(t)), \int_0^1 g(s)u(s)ds\big)^T \in S_M$, i.e. $A : S_M \to S_M$.
As was shown above, A is a compact operator in S. Therefore, A has a fixed point in S_M, which generates a solution of the problem (3.1.1)-(3.1.2). This solution is nonnegative. Moreover, if $f(t,0,0,0) \not\equiv 0$ then $u(t) \equiv 0$ cannot be the solution. Therefore, the solution is positive. □

Theorem 3.1.3 (Existence and uniqueness)**.** Suppose that there exist numbers $M > 0, L_0, L_1, L_2 \ge 0$ such that

(H1) $|f(t,x,y,z)| \le M, \ \forall (t,x,y,z) \in \mathcal{D}_M$.

(H2) $|f(t,x_2,y_2,z_2) - f(t,x_1,y_1,z_1)| \le L_0|x_2 - x_1| + L_1|y_2 - y_1| + L_2|z_2 - z_1|, \ \forall (t, x_i, y_i, z_i) \in \mathcal{D}_M, \ i = 1,2$.

(H3) $q := \max\{q_1, q_2\} < 1$, where $q_1 = kC_0M_0 + C_2$ as was defined by (3.1.19) and

$$q_2 = L_0(M_0 + \frac{1}{k}) + L_1(M_1 + \frac{2}{k}) + L_2(M_2 + \frac{2}{k}). \tag{3.1.24}$$

Then the problem (3.1.1)-(3.1.2) has a unique solution $u \in C^3[0,1]$.

Proof. To prove the theorem, it suffices to show that the operator A defined by (3.1.5) is a contractive mapping from the closed ball $B[0,M]$ in $\mathcal{B}$ into itself. Indeed, under the assumption $(H1)$ and the condition $q_1 < 1$ in the assumption $(H2)$, by Lemma 3.1.2 the operator A maps $B[0,M]$ into itself.

Now, we show that A is a contraction map.

Let $w_i = (\varphi_i, \alpha_i) \in B[0,M]$. We have

$$Aw_2 - Aw_1 = \begin{pmatrix} f(t, u_2(t), u_2'(t), u_2''(t) - f(t, u_1(t), u_1'(t), u_1''(t) \\ \int_0^1 g(s)(u_2(s) - u_1(s))ds \end{pmatrix},$$

where $u_i(t)$ $(i = 1, 2)$ is the solution of the problem

$$\begin{cases} u_i'''(t) = \varphi_i(t), & 0 < t < 1 \\ u_i(0) = u_i'(0) = 0, \ u_i(1) = \alpha_i. \end{cases}$$

From the proof of Lemma 3.1.2 it is known that $(t, u_i(t), u_i'(t), u_i''(t)) \in \mathcal{D}_M$. Therefore, by the Lipschitz condition $(H2)$ for f we have

$$\begin{aligned} D_1 : &= |f(t, u_2(t), u_2'(t), u_2''(t) - f(t, u_1(t), u_1'(t), u_1''(t)| \\ &\le L_0|u_2(t) - u_1(t)| + L_1|u_2'(t) - u_1'(t)| + L_2|u_2''(t) - u_1''(t)|. \end{aligned} \tag{3.1.25}$$

Since $u_2(t) - u_1(t)$ is the solution of the problem (3.1.6)-(3.1.7) with the right-hand sides $\varphi_2(t) - \varphi_1(t)$ and $\alpha_2 - \alpha_1$, we have

$$\begin{aligned} \|u_2 - u_1\| &\le M_0\|\varphi_2 - \varphi_1\| + |\alpha_2 - \alpha_1|, \\ \|u_2' - u_1'\| &\le M_1\|\varphi_2 - \varphi_1\| + 2|\alpha_2 - \alpha_1|, \\ \|u_2'' - u_1''\| &\le M_2\|\varphi_2 - \varphi_1\| + 2|\alpha_2 - \alpha_1|. \end{aligned} \tag{3.1.26}$$

As for the element $w = (\varphi, \alpha)^T \in \mathcal{B}$ we use the norm

$$\|w\|_{\mathcal{B}} = \max(\|\varphi\|, k|\alpha|) \quad (k \ge 1),$$

from (3.1.25), (3.1.26) we obtain

$$\begin{aligned} D_1 &\le L_0\left(M_0 + \frac{1}{k}\right)\|w_2 - w_1\|_{\mathcal{B}} + L_1\left(M_1 + \frac{2}{k}\right)\|w_2 - w_1\|_{\mathcal{B}} \\ &+ L_2\left(M_2 + \frac{2}{k}\right)\|w_2 - w_1\|_{\mathcal{B}} \\ &\le \left(L_0\left(M_0 + \frac{1}{k}\right) + L_1\left(M_1 + \frac{2}{k}\right) + L_2\left(M_2 + \frac{2}{k}\right)\right)\|w_2 - w_1\|_{\mathcal{B}} \\ &= q_2\|w_2 - w_1\|_{\mathcal{B}}, \end{aligned} \tag{3.1.27}$$

where q_2 is defined by (3.1.24).
Now consider

$$D_2 := k\Big|\int_0^1 g(s)(u_2(s) - u_1(s))ds\Big|.$$

By analogy with the estimate (3.1.20) it is easy to have

$$D_2 \le (kC_0M_0 + C_2)\|w_2 - w_1\|_{\mathcal{B}} = q_1\|w_2 - w_1\|_{\mathcal{B}}. \tag{3.1.28}$$

From (3.1.27) and (3.1.28) we obtain

$$\|Aw_2 - Aw_1\|_{\mathcal{B}} \le \max\{q_1, q_2\}\|w_2 - w_1\|_{\mathcal{B}}.$$

In view of condition $(H3)$ the operator A is a contraction operator in $B[0, M]$. The theorem is proved. □

Theorem 3.1.4 (Existence and uniqueness of positive solution)**.** If in Theorem 3.1.3 replace $\mathcal{D}_M$ by $\mathcal{D}_M^+$ and the condition $(H1)$ by the condition (3.1.23) then the problem (3.1.1)-(3.1.2) has a unique nonnegative solution $u(t) \in C^3[0, 1]$. Besides, if $f(t, 0, 0, 0) \not\equiv 0$ then this solution is positive.

3.1.3. Iterative method

Suppose all the conditions of Theorem 3.1.3 are met. Then the problem (3.1.1)-(3.1.2) has a unique solution. To find it, consider the following iterative method:

1. Given $w_0 = (\varphi_0, \alpha_0)^T \in B[0, M]$, for example,

$$\varphi_0(t) = f(t, 0, 0, 0),\ \ \alpha_0 = 0. \tag{3.1.29}$$

2. Knowing $\varphi_n(t)$ and $\alpha_n(t)$ $(n = 0, 1, ...)$, compute

$$u_n(t) = \int_0^1 G(t,s)\varphi_n(s)ds + \alpha_n t^2, \tag{3.1.30}$$

$$y_n(t) = \int_0^1 G_1(t,s)\varphi_n(s)ds + 2\alpha_n t, \tag{3.1.31}$$

$$z_n(t) = \int_0^1 G_2(t,s)\varphi_n(s)ds + 2\alpha_n. \tag{3.1.32}$$

3. Update

$$\varphi_{n+1}(t) = f(t, u_n(t), y_n(t), z_n(t)), \tag{3.1.33}$$

$$\alpha_{n+1} = \int_0^1 g(s)u_n(s)ds. \tag{3.1.34}$$

Theorem 3.1.5. Under the assumptions of Theorem 3.1.3 the above iterative method converges, and for the approximate solution $u_n(t)$ and its derivatives there hold the estimates

$$\|u_n - u\| \le \left(M_0 + \frac{1}{k}\right) p_n d, \tag{3.1.35}$$

$$\|u_n' - u'\| \le \left(M_1 + \frac{2}{k}\right) p_n d, \tag{3.1.36}$$

$$\|u_n'' - u''\| \le \left(M_2 + \frac{2}{k}\right) p_n d, \tag{3.1.37}$$

where $p_n = \frac{q^n}{1-q},\ d = \|w_1 - w_0\|_{\mathcal{B}},\ w_1 = (\varphi_1, \alpha_1)^T$.

Proof. In fact, the above iterative method is the successive iterative method for finding the fixed point of operator A. Therefore, it converges with the rate of geometric progression and there holds the estimate

$$\|w_n - w\|_{\mathcal{B}} \le \frac{q^n}{1-q}\|w_1 - w_0\|_{\mathcal{B}} = p_n d,$$

where $w_n - w = (\varphi_n - \varphi, \alpha_n - \alpha)^T$.
From the definition of the norm in $\mathcal{B}$ and the above estimate it follows

$$\|\varphi_n - \varphi\| \le \|w_n - w\|_{\mathcal{B}} \le p_n d,$$

$$\|\alpha_n - \alpha\| \le \frac{1}{k}\|w_n - w\|_{\mathcal{B}} \le \frac{1}{k} p_n d.$$

Now, the estimates (3.1.35)-(3.1.37) are easily obtained if taking into account the representations (3.1.11)-(3.1.13), (3.1.30)-(3.1.32), the estimates of the type (3.1.15) and the above estimates. □

To numerically realize the iterative method (3.1.29)-(3.1.34) we cover the interval $[0,1]$ by the uniform grid $\omega_h = \{t_i = ih,\ h = 1/N, i = 0, 1, ..., N\}$ and use the trapezium formula for computing definite integrals. In all examples in the next section the numerical computations will be performed on the grid with $h = 0.01$ until $\max\left\{\|\varphi_n - \varphi_{n-1}\|, k|\alpha_n - \alpha_{n-1}|\right\} \le 10^{-4}$, where k will be defined for each particular example.

3.1.4. Examples

Consider some examples for confirming the validity of the obtained theoretical results and the efficiency of the proposed iterative method.

Example 3.1.1 (Example with exact solution)**.** Consider the problem (3.1.1)-(3.1.2) with

$$f = f(t,u) = -\frac{1}{2} + \frac{1}{3}\left(\frac{1}{6}(t^2 - \frac{t^3}{2})\right)^2 - u^2,$$
$$g(s) = \frac{56}{9}s^4.$$

It is possible to verify that the positive function

$$u(t) = \frac{1}{6}\left(t^2 - \frac{t^3}{2}\right), \quad 0 \le t \le 1$$

is the exact solution of the problem.

For the given $g(s)$, simple calculations give $C_0 = \frac{56}{45}, C_2 = \frac{56}{63}$. Therefore, with $k = 2$ we obtain $q_1 = 0.9503 < 1$. For this k it is possible to choose $M = 0.6$ such that $-M \le f(t,x) \le 0$ for

$$(t,x) \in \mathcal{D}_M^+ = \{(t,x) \mid 0 \le t \le 1,\ 0 \le x \le (M_0 + \frac{1}{2})M = 0.5247M\}.$$

Indeed,

$$0 \le -f(t,x) = \frac{1}{2} + x^2 - \frac{1}{3}\left(\frac{1}{6}(t^2 - \frac{t^3}{2})\right)^2 \le \frac{1}{2} + x^2 \le \frac{1}{2} + (0.5247M)^2 \le M.$$

Thus, M must satisfy $0.2753M^2 - M + 0.5 \le 0$. The direct calculation of the left side for $M = 0.6$ gives the value $= -0.0670$. So, the choice of M is justified. Further, for $f(t,x)$ we have the Lipshitz coefficient with respect to x in $\mathcal{D}_M^+$, $L_0 = 0.3148$. Consequently, $q_2 = L_0\left(M_0 + \frac{1}{2}\right) = 0.1652$, and $q = 0.9503$. Besides, $f(t,0) \not\equiv 0$. Therefore, by Theorem 3.1.4, the problem has a unique positive solution. It is the above exact solution.

The computation shows that the iterative method (3.1.29)-(3.1.34) converges and the error of the 46th iteration compared with the exact solution is $1.1458e-04$.

Example 3.1.2 (Example 4.1 in [47])**.** Consider the boundary value problem

$$u'''(t) = -u^2e^u, \quad 0 < t < 1,$$
$$u(0) = u'(0) = 0,\ u(1) = \int_0^1 s^4u(s)ds.$$

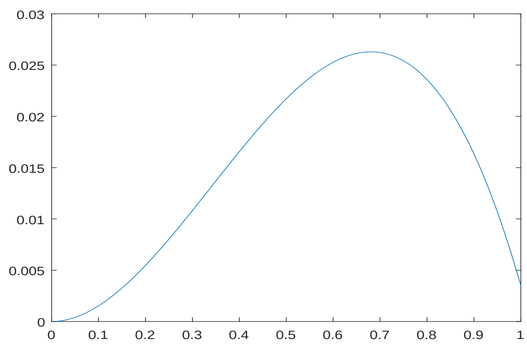

Figure 3.1: The graph of the approximate solution in Example 3.1.3.

In this example

$$f(t,x,y,z) = -x^2e^x, \quad g(s) = s^4.$$

So,

$$C_0 = \int_0^1 g(s)ds = \frac{1}{5}, \quad C_2 = \int_0^1 s^2 g(s)ds = \frac{1}{7}.$$

Choose $k = 2$ in the definition of the norm of the space $\mathcal{B}$ (3.1.4) and in the definition of $\mathcal{D}_M^+$ by (3.1.21). Then $q_1 = kC_0M_0 + C_2 = 0.1527$. For $M = 0.4$ it is possible to verify that $-M \le f(t,x) \le 0$ in $\mathcal{D}_M^+$, $\left|\frac{\partial f}{\partial x}\right| \le 0.5721$ in $\mathcal{D}_M^+$. Therefore,

$$L_0 = 0.5721, \quad q_2 = L_0\left(M_0 + \tfrac{1}{k}\right) = 0.3002.$$

Hence, by Theorem 3.1.4 the problem has a unique nonnegative solution. This solution should be $u(t) \equiv 0$ because $u(t) \equiv 0$ solves the problem. The numerical experiments by the iterative method in Section 3.1.3 confirm this conclusion.

Remark that, in [47] the authors concluded that the problem has at least one positive solution. From our result above, it is clear that their conclusion is not valid.

Example 3.1.3. Consider Example 3.1.2 with the nonlinear term $f = -(1+u^2)$. Clearly, $\frac{f(u)}{u} \to -\infty$ as $u \to +0$ and $u \to +\infty$. Thus, neither Theorem 3.1 nor Theorem 3.2 in [47] are applicable, so the existence of positive solution is not guaranteed.

Now apply our method. Choose $M = 2, k = 3$, then

$$\mathcal{D}_M^+ = \{(t,x) \mid 0 \le t \le 1,\ 0 \le x \le (M_0 + \frac{1}{k})M = 0.7160\}.$$

In $\mathcal{D}_M^+$ we have

$$-M \le f \le 0, \quad |f'_u| \le 1.4321 = L_0,$$

$$q_1 = kC_0M_0 + C_2 = 0.1577,\ q_2 = L_0\left(M_0 + \frac{1}{3}\right) = 0.5127.$$

Hence, by Theorem 3.1.4, the problem has a unique nonnegative solution. Due to $f(t,0) \neq 0$, this solution is positive. The graph of the approximate solution obtained with the given accuracy 10^{-4} after 4 iterations by the iterative method is depicted on Figure 3.1.

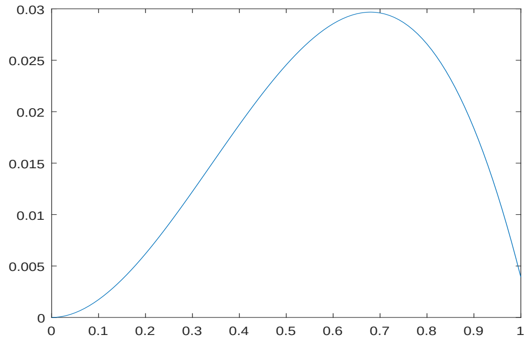

Figure 3.2: The graph of the approximate solution in Example 3.1.4.

Example 3.1.4. Consider Example 3.1.2 with the nonlinear term

$$f = -(u^2e^u + \frac{1}{5}\sin(u') + \frac{1}{8}\cos(u'') + 1).$$

In this example

$$f(t,x,y,z) = -(x^2e^x + \frac{1}{5}\sin(y) + \frac{1}{8}\cos(z) + 1).$$

Choose $M = 1.7, k = 4$. It is possible to verify that in $\mathcal{D}_M^+$ we have $-M \le f \le 0$, and the Lipschitz coefficients of f are

$$L_0 = 1.8378, \quad L_1 = \frac{1}{5}, \quad L_2 = \frac{1}{8}.$$

Therefore,

$$q_1 = 0.1626, \ q_2 = 0.7618.$$

Hence, by Theorem 3.1.4, the problem has a unique positive solution. The graph of the approximate solution obtained with the given accuracy 10^{-4} after 6 iterations by the iterative method is depicted on Figure 3.2.

3.1.5. Conclusion

In this section, we have proposed a novel method to study the fully third order differential equation with integral boundary conditions. It is based on the reduction of the boundary value problems to fixed point problem for appropriate operator defined on a space of mixed pairs of functions and numbers. By this way, we have established the existence, uniqueness and positivity of solution of the problem under easily verified conditions. Another important result is that, we have proposed an effective iterative method for finding the solution. The theoretical results have been demonstrated on some examples including an example with exact solution and other examples where the exact solutions are not known. Especially, we have shown that the conclusion on the existence of positive solutions for an example considered before by other authors, is not valid.

The proposed method can be applied to problems with other integral boundary conditions for the third and higher order differential equations. This is the subject of our researches in the future.

3.2. Existence results and iterative method for fully fourth order nonlinear integral boundary value problems

3.2.1. Introduction

In this section, we consider the boundary value problem

$$u''''(t) = f(t, u(t), u'(t), u''(t), u'''(t)), \quad 0 < t < 1, \tag{3.2.1}$$

$$u'(0) = u''(0) = u'(1) = 0, \; u(0) = \int_0^1 g(s)u(s)ds, \tag{3.2.2}$$

where $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}^+, \; g : [0,1] \to \mathbb{R}^+$ are continuous functions.

This problem is a natural generalization of the problem recently considered in [48], where instead of the fully nonlinear term it was $f(u(t))$. In the above-mentioned paper, by employing the Krasnosel'skii's fixed point theorem on cones, the authors established the existence of at least one positive solution.

In the paper [AL5], by the method developed in [11, 13, 14, 85, 86, 88, 90, 91] we establish the existence, uniqueness and positivity of solution and propose an iterative method on both continuous and discrete levels for finding the solution. We also give error analysis of the discrete approximate solution. Five examples, among them an example with exact solution and two examples taken from [48], demonstrate the validity of the obtained theoretical results and the efficiency of the iterative method.

By the way we would like to say that for numerical solution of two point nonlinear BVPs for fourth order equations there are many methods, which can be divided into three types. The first type includes methods for constructing discrete systems corresponding to BVPs, for example, [92, 93, 94, 95]. In these papers, the authors studied the convergence of the discrete systems without any analysis of errors arising in solving the discrete systems. To the second type of methods there are related the methods of construction of iterative methods on continuous level without attention to how to realize continuous problems at each iteration and error arising at each iteration, see, e.g. [1,96,97] and [11,13,14,85,86,88,90,91]. The third type includes analytical methods such as the Adomian decomposition method [98], the variational iteration method [99], the reproducing kernel method [100], when the solution is sought in series form. Spectral methods also belong to the third type since the exact solution of the problems is expressed in series representation by basis functions. For finding the coefficients of the representation it is needed to solve nonlinear systems of algebraic solutions. At present spectral methods [101] are widely used for solving BVPs for ODE, PDE, integral equations including nonlinear Volterra integral equations [102], [103].

It should be said that in all the types of methods, the total error of the actually obtained approximate numerical solution has not been addressed. In our opinion, the problem of total error in numerical solution of nonlinear BVPs must be investigated because the total error gives useful information for balancing discretization error and error of iterative process. Once this problem was considered in [12] when at each iteration it was required to solve two second order BVPs and compute an integral. In the present paper we propose an iterative method at continuous level, its discrete analog and make analysis of the total error of the approximate discrete solution for the BVP with integral boundary condition.

3.2.2. Existence results

To investigate the problem (3.2.1), (3.2.2) we associate it with an operator equation as follows.

First, we denote the space of pairs $w = (\varphi, \mu)^T$, where $\varphi \in C[0,1], \mu \in \mathbb{R}$, by $\mathcal{B}$, i.e., set $\mathcal{B} = C[0,1] \times \mathbb{R}$, and equip it with the norm

$$\|w\|_{\mathcal{B}} = \max(\|\varphi\|, r|\mu|), \tag{3.2.3}$$

where r is a number, $r \geq 1$ and $\|\varphi\| = \max_{0 \leq t \leq 1} |\varphi(t)|$.

Further, define the operator A acting on elements $w \in \mathcal{B}$ by the formula

$$Aw = \begin{pmatrix} f(t, u(t), u'(t), u''(t), u'''(t)) \\ \int_0^1 g(s)u(s)ds \end{pmatrix}, \tag{3.2.4}$$

where $u(t)$ is the solution of the problem

$$u''''(t) = \varphi(t), \quad 0 < t < 1, \tag{3.2.5}$$

$$u'(0) = u''(0) = u'(1) = 0, \; u(0) = \mu. \tag{3.2.6}$$

Obviously, due to the continuity of the functions f and g we have $Aw \in \mathcal{B}$. It is easy to verify the following

Lemma 3.2.1. If $w = (\varphi, \mu)^T$ is a fixed point of the operator A in the space $\mathcal{B}$, i.e., w is a solution of the operator equation

$$Aw = w \tag{3.2.7}$$

in $\mathcal{B}$, then the function $u(t)$ defined from the problem (3.2.5), (3.2.6) solves the original problem (3.2.1), (3.2.2).
Conversely, if $u(t)$ is a solution of (3.2.1), (3.2.2), then the pair (φ, μ), where

$$\varphi(t) = f(t, u(t), u'(t), u''(t), u'''(t)), \tag{3.2.8}$$

$$\alpha = \int_0^1 g(s)u(s)ds, \tag{3.2.9}$$

is a solution of the operator equation (3.2.7).

Thus, by this lemma, the problem (3.2.1), (3.2.2) is reduced to the fixed point problem for A.

Now, we study the properties of A. For this purpose, notice that the problem (3.2.5), (3.2.6) has a unique solution representable in the form

$$u(t) = \int_0^1 G_0(t,s)\varphi(s)ds + \mu, \quad 0 < t < 1, \tag{3.2.10}$$

where $G_0(t,s)$ is the Green function of the operator $u''''(t) = 0$ associated with the homogeneous boundary conditions

$$u'(0) = u''(0) = u'(1) = u(0) = 0,$$

$$G_0(t,s) = \frac{1}{6} \begin{cases} -t^3(1-s)^2 + (t-s)^3, & 0 \leq s \leq t \leq 1 \\ -t^3(1-s)^2, & 0 \leq t \leq s \leq 1. \end{cases} \tag{3.2.11}$$

Differentiating both sides of (3.2.10) gives

$$u'(t) = \int_0^1 G_1(t,s)\varphi(s)ds, \tag{3.2.12}$$

$$u''(t) = \int_0^1 G_2(t,s)\varphi(s)ds, \tag{3.2.13}$$

$$u'''(t) = \int_0^1 G_3(t,s)\varphi(s)ds, \tag{3.2.14}$$

where

$$G_1(t,s) = \frac{1}{2}\begin{cases} -t^2(1-s)^2 + (t-s)^2, & 0 \le s \le t \le 1, \\ -t^2(1-s)^2, & 0 \le t \le s \le 1, \end{cases} \tag{3.2.15}$$

$$G_2(t,s) = \begin{cases} -t(1-s)^2 + (t-s), & 0 \le s \le t \le 1, \\ -t(1-s)^2, & 0 \le t \le s \le 1. \end{cases} \tag{3.2.16}$$

$$G_3(t,s) = \begin{cases} -(1-s)^2 + 1, & 0 \le s < t \le 1, \\ -(1-s)^2, & 0 \le t < s \le 1. \end{cases} \tag{3.2.17}$$

It is easily seen that

$$G_0(t,s) \le 0, \quad G_1(t,s) \le 0,$$

in $Q = [0,1]^2$, and

$$\begin{aligned} M_0 &= \max_{0\le t\le 1} \int_0^1 |G_0(t,s)|ds = 0.0139, \\ M_1 &= \max_{0\le t\le 1} \int_0^1 |G_1(t,s)|ds = 0.0247, \\ M_2 &= \max_{0\le t\le 1} \int_0^1 |G_2(t,s)|ds \le 0.1883 \\ M_3 &= \max_{0\le t\le 1} \int_0^1 |G_3(t,s)|ds = 1.3333. \end{aligned} \tag{3.2.18}$$

Therefore, from (3.2.10), (3.2.12)-(3.2.14) and (3.2.18) we obtain the following estimates for the solution of the problem (3.2.5), (3.2.6):

$$\begin{aligned} \|u\| \le M_0\|\varphi\| + |\mu|, \ \|u'\| \le M_1\|\varphi\|, \\ \|u''\| \le M_2\|\varphi\|, \ \|u'''\| \le M_3\|\varphi\|. \end{aligned} \tag{3.2.19}$$

For any number $M > 0$, define the domain

$$\begin{aligned} \mathcal{D}_M = \{(t,u,y,v,z) \mid 0 \le t \le 1, \ |u| \le (M_0 + \tfrac{1}{r})M, \\ |y| \le M_1 M, \ |v| \le M_2 M, \ |z| \le M_3 M\}. \end{aligned} \tag{3.2.20}$$

From now on suppose that the function $f(t,u,y,v,z)$ is continuous in $\mathcal{D}_M$. Denote

$$C_0 = \int_0^1 g(s)ds > 0. \tag{3.2.21}$$

Lemma 3.2.2. Under the assumption that

$$|f(t,u,y,v,z)| \le M \quad \text{in } \mathcal{D}_M \tag{3.2.22}$$

and the condition

$$q_1 := C_0(rM_0 + 1) \leq 1, \tag{3.2.23}$$

where C_0 is defined by (3.2.21), holds, the operator A defined by (3.2.4) maps the closed ball $B[0, M]$ in $\mathcal{B}$ into itself.

Proof. Let $w = (\varphi, \mu)^T \in B[0, M]$. Then $\|\varphi\| \leq M$ and $|\mu| \leq \frac{M}{r}$.

Consider the problem (3.2.5), (3.2.6). From the estimates (3.2.19) for its solution $u(t)$ and the derivatives we obtain

$$\|u\| \leq \left(M_0 + \frac{1}{r}\right) M, \quad \|u'\| \leq M_1 M, \quad \|u''\| \leq M_2 M, \quad \|u'''\| \leq M_3 M.$$

Therefore, $(t, u, u', u'', u''') \in \mathcal{D}_M$. Hence, by the assumption (3.2.22) we have

$$|f(t, u(t), u'(t), u''(t), u'''(t))| \leq M.$$

Next, there hold the estimates

$$r\left|\int_0^1 g(s)u(s)ds\right| \leq r\|u\|C_0 \leq rC_0(M_0 + \frac{1}{r}) = C_0(rM_0 + 1) = q_1 M \leq M. \tag{3.2.24}$$

Therefore,

$$\|Aw\|_{\mathcal{B}} \leq M.$$

□

Lemma 3.2.3. The operator A is a compact operator in $\mathcal{B}[0, M]$.

Proof. The compactness of A follows from the compactness of the integral operators (3.2.10), (3.2.12)-(3.2.14), the continuity of the function $f(t, x, y, v, z)$ and the compactness of the integral operator $\int_0^1 g(s)u(s)ds$. □

Theorem 3.2.1. Suppose the conditions of Lemma 3.2.2 are satisfied. Then the problem (3.2.1), (3.2.2) has a solution.

Proof. By Lemma 3.2.2 and Lemma 3.2.3, the operator A is a compact operator mapping the closed ball $\mathcal{B}[0, M]$ in the Banach space $\mathcal{B}$ into itself. Therefore, according to the Schauder fixed point theorem, the operator A has a fixed point in $\mathcal{B}[0, M]$. This fixed point generates a solution of the problem (3.2.1), (3.2.2).
□

In order to establish positivity of solution of (3.2.1), (3.2.2), introduce the domain

$$\begin{aligned} \mathcal{D}_M^+ = \{(t, u, y, v, z) \mid 0 \leq t \leq 1,\ 0 \leq u \leq (M_0 + \tfrac{1}{r})M, \\ 0 \leq y \leq M_1 M,\ |v| \leq M_2 M,\ |z| \leq M_3 M\}, \end{aligned} \tag{3.2.25}$$

and the strip in $\mathcal{B}$

$$S_M = \{w = (\varphi, \mu)^T \mid -M \leq \varphi \leq 0,\ 0 \leq r\mu \leq M\}. \tag{3.2.26}$$

Theorem 3.2.2 (Positivity of solution)**.** Suppose the function $f(t, u, y, v, z)$ is continuous and

$$-M \leq f(t, u, y, v, z) \leq 0 \text{ in } \mathcal{D}_M^+, \tag{3.2.27}$$

and the condition (3.2.23) is satisfied. Then the problem (3.2.1), (3.2.2) has a nonnegative solution. Besides, if $f(t, 0, 0, 0, 0) \not\equiv 0$ in $(0, 1)$ then the solution is positive.

Proof. It is easy to verify that under the conditions of the theorem, the operator A maps S_M into itself.
Indeed, for any $w \in S_M$, $w = (\varphi, \mu)^T, -M \le \varphi \le 0, 0 \le r\mu \le M$. Since $G_i(t,s) \le 0$ for $0 \le t, s \le 1, (i = 0, 1)$ from (3.2.10), (3.2.12), (3.2.13) we have

$$0 \le u(t) \le (M_0 + \tfrac{1}{r})M,\ 0 \le u'(t) \le M_1 M,\ |u''(t)| \le M_2 M, |u'''(t)| \le M_3 M$$

for any $0 \le t \le 1$. Therefore, for the solution $u(t)$ of (3.2.5), (3.2.6) we have

$$(t, u(t), u'(t), u''(t), u'''(t)) \in \mathcal{D}_M^+,$$

and by the condition (3.2.27)

$$-M \le f(t, u(t), u'(t), u''(t), u'''(t)) \le 0.$$

Taking into account (3.2.24) we have

$$0 \le r \int_0^1 g(s)u(s)ds \le C_0(rM_0 + 1)M \le M.$$

Hence, $(f(t, u(t), u'(t), u''(t), u'''(t)), \int_0^1 g(s)u(s)ds)^T \in S_M$, i.e. $A : S_M \to S_M$.
Also, as was shown above, A is a compact operator in S, due to this A has a fixed point in S_M, which generates a solution of the problem (3.2.1), (3.2.2). This solution is nonnegative with its first derivative. Due to the condition $f(t, 0, 0, 0, 0) \not\equiv 0$ in $(0, 1)$ the function $u(t) \equiv 0$ cannot be the solution of the problem. It implies that the solution must be positive. □

Theorem 3.2.3 (Existence and uniqueness)**.** Suppose that there exist numbers $M > 0, L_0, L_1, L_2, L_3 \ge 0$ such that

1. $|f(t, u, y, v, z)| \le M,\ \forall (t, u, y, v, z) \in \mathcal{D}_M$.
2. $|f(t, u_2, y_2, v_2, z_2) - f(t, u_1, y_1, v_1, z_1)| \le L_0|u_2 - u_1| + L_1|y_2 - y_1| + L_2|v_2 - v_1| + L_3|z_2 - z_1|,\ \forall (t, u_i, y_i, v_i, z_i) \in \mathcal{D}_M,\ i = 1, 2$.
3. $q := \max\{q_1, q_2\} < 1$, where $q_1 = rC_0M_0 + C_0$ (see (3.2.23)) and
$$q_2 = L_0(M_0 + \tfrac{1}{r}) + L_1M_1 + L_2M_2 + L_3M_3.$$

Then the problem has a unique solution $u \in C^4[0, 1]$.

Proof. To prove the theorem, it suffices to show that the operator A defined by (3.2.4) is a contractive mapping from the closed ball $B[0, M]$ in $\mathcal{B}$ into itself. Indeed, under the conditions 1) and 3) by Lemma 3.2.2, the operator A maps $B[0, M]$ into itself.

Now, we show that A is a contraction map.

Let $w_i = (\varphi_i, \mu_i)^T \in B[0, M]$. We have

$$Aw_2 - Aw_1 = \begin{pmatrix} f(t, u_2(t), u_2'(t), u_2''(t), u_2'''(t)) - f(t, u_1(t), u_1'(t), u_1''(t), u_1'''(t)) \\ \int_0^1 g(s)(u_2(s) - u_1(s))ds \end{pmatrix},$$

where $u_i(t),\ (i = 1, 2)$ is the solution of the problem

$$\begin{cases} u_i''''(t) = \varphi_i(t), \quad 0 < t < 1 \\ u_i'(0) = u_i''(0) = u_i'(1) = 0,\ u_i(0) = \mu_i. \end{cases}$$

In the proof of Lemma 3.2.2 it is known that $(t, u_i(t), u_i'(t), u_i''(t), u_i'''(t)) \in \mathcal{D}_M$. Therefore, by the condition 2) for f we have

$$\begin{aligned} E_1 :&= |f(t, u_2(t), u_2'(t), u_2''(t), u_2'''(t)) - f(t, u_1(t), u_1'(t), u_1''(t), u_1'''(t))| \\ &\le L_0|u_2(t) - u_1(t)| + L_1|u_2'(t) - u_1'(t)| + L_2|u_2''(t) - u_1''(t)| \\ &+ L_3|u_2'''(t) - u_1'''(t)|. \end{aligned} \tag{3.2.28}$$

Since $u_2(t) - u_1(t)$ is the solution of the problem (3.2.5), (3.2.6) with the right-hand sides $\varphi_2(t) - \varphi_1(t)$ and $\mu_2 - \mu_1$, we have

$$\begin{aligned} \|u_2 - u_1\| &\le M_0\|\varphi_2 - \varphi_1\| + |\mu_2 - \mu_1|, \\ \|u_2' - u_1'\| &\le M_1\|\varphi_2 - \varphi_1\|, \\ \|u_2'' - u_1''\| &\le M_2\|\varphi_2 - \varphi_1\|, \\ \|u_2''' - u_1'''\| &\le M_3\|\varphi_2 - \varphi_1\|. \end{aligned} \tag{3.2.29}$$

As for the element $w = (\varphi, \mu)^T \in \mathcal{B}$ we use the norm

$$\|w\|_{\mathcal{B}} = \max(\|\varphi\|, r|\mu|) \quad (r \ge 1),$$

then in view of the above fact, from (3.2.28), (3.2.29) we obtain

$$\begin{aligned} E_1 &\le \left(L_0 \left(M_0 + \frac{1}{r} \right) + L_1M_1 + L_2M_2 + L_3M_3 \right) \|w_2 - w_1\|_{\mathcal{B}} \\ &= q_2\|w_2 - w_1\|_{\mathcal{B}}. \end{aligned} \tag{3.2.30}$$

Now consider

$$E_2 := \int_0^1 g(s)(u_2(s) - u_1(s))ds.$$

We have

$$|E_2| \le \int_0^1 g(s)|u_2(s) - u_1(s)|ds.$$

In analogy with the estimate (3.2.24) we have

$$|E_2| \le C_0(M_0 + \frac{1}{r})\|w_2 - w_1\|_{\mathcal{B}}.$$

Therefore

$$r|E_2| \le C_0(rM_0 + 1)\|w_2 - w_1\|_{\mathcal{B}} = q_1\|w_2 - w_1\|_{\mathcal{B}}. \tag{3.2.31}$$

From (3.2.30) and (3.2.31) we obtain

$$\|Aw_2 - Aw_1\|_{\mathcal{B}} \le \max\{q_1, q_2\}\|w_2 - w_1\|_{\mathcal{B}}.$$

In view of condition 3) the operator A is a contraction operator in $B[0, M]$. The theorem is proved. □

Analogously as the above theorem, it is easy to prove the following

Theorem 3.2.4 (Existence and uniqueness of positive solution)**.** If in Theorem 3.2.3 replace $\mathcal{D}_M$ by $\mathcal{D}_M^+$ and the condition 1) by the condition (3.2.27) then the problem has a unique nonnegative solution $u(t) \in C^4[0, 1]$. Besides, if $f(t, 0, 0, 0, 0) \not\equiv 0$ in $(0, 1)$ then the solution is positive.

3.2.3. Iterative method on continuous level

Consider the following iterative method

1. Given

$$\varphi_0(t) = f(t, 0, 0, 0, 0),\ \mu_0 = 0 \tag{3.2.32}$$

2. Knowing $\varphi_k(t)$ and μ_k $(k = 0, 1, ...)$ compute

$$\begin{aligned} u_k(t) &= \int_0^1 G_0(t,s)\varphi_k(s)ds + \mu_k, \\ y_k(t) &= \int_0^1 G_1(t,s)\varphi_k(s)ds, \\ v_k(t) &= \int_0^1 G_2(t,s)\varphi_k(s)ds, \\ z_k(t) &= \int_0^1 G_3(t,s)\varphi_k(s)ds, \end{aligned} \tag{3.2.33}$$

3. Update

$$\begin{aligned} \varphi_{k+1}(t) &= f(t, u_k(t), y_k(t), v_k(t), z_k(t)), \\ \mu_{k+1} &= \int_0^1 g(s)u_k(s)ds. \end{aligned} \tag{3.2.34}$$

This iterative method indeed is the successive iterative method for finding the fixed point of operator A. Therefore, it converges with the rate of geometric progression and there holds the estimate

$$\|w_k - w\|_{\mathcal{B}} \le \frac{q^k}{1-q}\|w_1 - w_0\|_{\mathcal{B}} = p_k d,$$

where $w_k - w = (\varphi_k - \varphi, \mu_k - \mu)^T$ and

$$p_k = \tfrac{q^k}{1-q},\ d = \|w_1 - w_0\|_{\mathcal{B}}. \tag{3.2.35}$$

From the definition of the norm in $\mathcal{B}$ it follows

$$\|\varphi_k - \varphi\| \le \frac{q^k}{1-q}\|w_1 - w_0\|_{\mathcal{B}} = p_k d,$$

$$\|\mu_k - \mu\| \le \frac{1}{r}\frac{q^k}{1-q}\|w_1 - w_0\|_{\mathcal{B}} = \frac{1}{r}p_k d.$$

These estimates imply the following result of the convergence of the iterative method (3.2.32)-(3.2.34).

Theorem 3.2.5. The iterative method (3.2.32)-(3.2.34) converges and for the approximate solution $u_k(t)$ there hold estimates

$$\|u_k - u\| \le \left(M_0 + \frac{1}{r}\right)p_k d,\ \|u_k' - u'\| \le M_1 p_k d,$$
$$\|u_k'' - u''\| \le M_2 p_k d,\ \|u_k''' - u'''\| \le M_3 p_k d.$$

where u is the exact solution of the problem (3.2.1)-(3.2.2), p_k and d are defined by (3.2.35), and r is the number available in (3.2.3).

3.2.4. Discrete iterative method

To numerically realize the above iterative method we construct corresponding discrete iterative method. For this purpose cover the interval $[0,1]$ by the uniform grid $\bar{\omega}_h = \{t_i = ih,\ h = 1/N, i = 0,1,...,N\}$ and denote by $\Phi_k(t), U_k(t), Y_k(t), V_k(t), Z_k(t)$ the grid functions, which are defined on the grid $\bar{\omega}_h$ and approximate the functions $\varphi_k(t), u_k(t), y_k(t), v_k(t), z_k(t)$ on this grid. We also denote by $\hat{\mu_k}$ the approximation of μ_k.

Consider now the following discrete iterative method.

1. Given

$$\Phi_0(t_i) = f(t_i, 0, 0, 0, 0),\ i = 0, ..., N; \quad \hat{\mu}_0 = 0 \tag{3.2.36}$$

2. Knowing $\Phi_k(t_i),\ i = 0, ..., N$ and $\hat{\mu}_k$ $(k = 0, 1, ...)$ compute approximately the definite integrals (3.2.33) by trapezium formulas

$$\begin{aligned}
U_k(t_i) &= \sum_{j=0}^{N} h\rho_j G_0(t_i, t_j)\Phi_k(t_j) + \hat{\mu}_k, \\
Y_k(t_i) &= \sum_{j=0}^{N} h\rho_j G_1(t_i, t_j)\Phi_k(t_j), \\
V_k(t_i) &= \sum_{j=0}^{N} h\rho_j G_2(t_i, t_j)\Phi_k(t_j), \\
Z_k(t_i) &= \sum_{j=0}^{N} h\rho_j G_3^*(t_i, t_j)\Phi_k(t_j),\ i = 0, ..., N,
\end{aligned} \tag{3.2.37}$$

where ρ_j is the weight of the trapezium formula

$$\rho_j = \begin{cases} 1/2,\ j = 0, N \\ 1,\ j = 1, 2, ..., N-1 \end{cases}$$

and

$$G_3^*(t,s) = \begin{cases} -(1-s)^2 + 1, & 0 \le s < t \le 1, \\ -(1-s)^2 + 1/2, & s = t, \\ -(1-s)^2, & 0 \le t < s \le 1. \end{cases}$$

3. Update

$$\begin{aligned}
\Phi_{k+1}(t_i) &= f(t_i, U_k(t_i), Y_k(t_i), V_k(t_i), Z_k(t_i)), \\
\hat{\mu}_{k+1} &= \sum_{j=0}^{N} h\rho_j g(t_j) U_k(t_j).
\end{aligned} \tag{3.2.38}$$

In order to get the error estimates for the approximate solution for $u(t)$ and its derivatives on the grid we need some following auxiliary results.

Proposition 3.2.6. Assume that the function $f(t, u, y, v, z)$ has all continuous partial derivatives up to second order in the domain $\mathcal{D}_M$. Then for the functions $u_k(t), y_k(t), v_k(t),\ z_k(t), k = 0, 1, ...$ constructed by the iterative method (3.2.32)-(3.2.34) there hold $z_k(t) \in C^3[0,1],\ v_k(t) \in C^4[0,1],\ y_k(t) \in C^5[0,1],\ u_k(t) \in C^6[0,1]$.

Proof. We prove the proposition by induction. For $k = 0$, by the assumption on the function f we have $\varphi_0(t) \in C^2[0,1]$ since $\varphi_0(t) = f(t,0,0,0,0)$. Taking into account the expression (3.2.17) of the function $G_3(t,s)$ we have

$$z_0(t) = \int_0^1 G_3(t,s)\varphi_0(s)ds = \int_0^t [-(1-s)^2+1]\varphi_0(s)ds - \int_t^1 (1-s)^2\varphi_0(s)ds.$$

By direct differentiation of the integrals in the right-hand side, it is easy to see that $z_0'(t) = \varphi_0(t)$. Therefore, $z_0(t) \in C^3[0,1]$. It implies $v_0(t) \in C^4[0,1],\ y_0(t) \in C^5[0,1],\ u_0(t) \in C^6[0,1]$.

Now suppose $z_k(t) \in C^3[0,1],\ v_k(t) \in C^4[0,1],\ y_k(t) \in C^5[0,1],\ u_k(t) \in C^6[0,1]$. Then, because $\varphi_{k+1}(t) = f(t,u_k(t),y_k(t),v_k(t),z_k(t))$ and the functions f by the assumption has continuous derivative in all variables up to order 2, it follows that $\varphi_{k+1}(t) \in C^2[0,1]$. Repeating the same argument as for $\varphi_0(t)$ above we obtain that $z_{k+1}(t) \in C^3[0,1],\ v_{k+1}(t) \in C^4[0,1],\ y_{k+1}(t) \in C^5[0,1],\ u_{k+1}(t) \in C^6[0,1]$. Thus, the proposition is proved. □

Proposition 3.2.7. For any function $\varphi(t) \in C^2[0,1]$ there hold the estimates

$$\int_0^1 G_n(t_i,s)\varphi(s)ds = \sum_{j=0}^N h\rho_j G_n(t_i,t_j)\varphi(t_j) + O(h^2), \quad (n=0,1,2), \tag{3.2.39}$$

$$\int_0^1 G_3(t_i,s)\varphi(s)ds = \sum_{j=0}^N h\rho_j G_3^*(t_i,t_j)\varphi(t_j) + O(h^2). \tag{3.2.40}$$

Proof. For $n = 0$ the above estimate is obvious in view of the error estimate of the trapezium formula because the function $G_0(t,s)$ defined by (3.2.11) have continuous derivatives up to second order.

In the case $n = 1,2$, although the functions $G_1(t,s), G_2(t,s)$ have not partial derivatives in respect to t continuous up to second order, they are continuous for any $0 \le t,s \le 1$. Due to this continuity the trapezium formulas also have second order accuracy. Indeed, we have for $n = 1,2$

$$\begin{aligned}
&\int_0^1 G_n(t_i,s)\varphi(s)ds = \int_0^{t_i} G_n(t_i,s)\varphi(s)ds + \int_{t_i}^1 G_n(t_i,s)\varphi(s)ds\\
&= h\big(\tfrac{1}{2}G_n(t_i,t_0)\varphi(t_0) + G_n(t_i,t_1)\varphi(t_1) + ... + G_n(t_i,t_{i-1})\varphi(t_{i-1}) + \tfrac{1}{2}G_n(t_i,t_i)\varphi(t_i)\big)\\
&+ h\big(\tfrac{1}{2}G_n(t_i,t_i)\varphi(t_i) + G_n(t_i,t_{i+1})\varphi(t_{i+1}) + ... + G_n(t_i,t_{N-1})\varphi(t_{N-1})\\
&+ \tfrac{1}{2}G_n(t_i,t_N)\varphi(t_N)\big) + O(h^2)\\
&= \sum_{j=0}^N h\rho_j G_n(t_i,t_j)\varphi(t_j) + O(h^2).
\end{aligned}$$

Thus, the estimate (3.2.39) is established. The estimate (3.2.40) is obtained using the following result, which is easily proved.

Lemma 3.2.4. Let $p(t)$ be a function having continuous derivatives up to second order in the interval $[0,1]$ except for the point $0 < t_i < 1$, where it has a jump. Denote $\lim_{t\to t_i-0} p(t) = p_i^-$, $\lim_{t\to t_i+0} p(t) = p_i^+$, $p_i = \frac{1}{2}(p_i^- + p_i^+)$. Then

$$\int_0^1 p(t)dt = \sum_{j=0}^N h\rho_j p_j + O(h^2), \tag{3.2.41}$$

where $p_j = p(t_j), j \ne i$.

□

Proposition 3.2.8. Under the assumption of Proposition 3.2.6 and the assumption that the function $g(s) \in C^2[0,1]$, for any $k = 0, 1, ...$ there hold the estimates

$$\|\Phi_k - \varphi_k\| = O(h^2), \; |\hat{\mu}_k - \mu_k| = O(h^2), \tag{3.2.42}$$

$$\begin{aligned} \|U_k - u_k\| = O(h^2), \; \|Y_k - y_k\| = O(h^2), \\ \|V_k - v_k\| = O(h^2), \; \|Z_k - z_k\| = O(h^2). \end{aligned} \tag{3.2.43}$$

where $\|.\| = \|.\|_{C(\bar{\omega}_h)}$ is the max-norm of function on the grid $\bar{\omega}_h$.

Proof. We prove the proposition by induction. For $k = 0$ we have immediately $\|\Phi_k - \varphi_k\| = 0, \; |\hat{\mu}_k - \mu_k| = 0$. Next, by the first equation in (3.2.33) and Proposition 3.2.7 we have

$$u_0(t_i) = \int_0^1 G_0(t_i, s)\varphi_0(s)ds + \mu_0 = \sum_{j=0}^{N} h\rho_j G_0(t_i, t_j)\varphi_0(t_j) + O(h^2) \tag{3.2.44}$$

for any $i = 0, ..., N$ since $\mu_0 = 0$. On the other hand, in view of the first equation in (3.2.37) and (3.2.36) we have

$$U_0(t_i) = \sum_{j=0}^{N} h\rho_j G_0(t_i, t_j)\varphi_0(t_j) \tag{3.2.45}$$

Therefore, $|U_0(t_i) - u_0(t_i)| = O(h^2)$. Consequently, $\|U_0 - u_0\| = O(h^2)$.
Similarly, we have

$$\|Y_0 - y_0\| = O(h^2), \; \|V_0 - v_0\| = O(h^2), \; \|Z_0 - z_0\| = O(h^2). \tag{3.2.46}$$

Now suppose that (3.2.42) and (3.2.43) are valid for $k \geq 0$. We shall show that these estimates are valid for $k + 1$.
Indeed, we have

$$\mu_{k+1} - \hat{\mu}_{k+1} = \sum_{j=0}^{N} h\rho_j g(t_j)\big(u_k(t_j) - U_k(t_j)\big) + O(h^2).$$

Due to the estimate $\|U_k - u_k\| = O(h^2)$ from the above estimate it follows that

$$|\mu_{k+1} - \hat{\mu}_{k+1}| = O(h^2). \tag{3.2.47}$$

Next, by the Lipschitz condition of the function f and the estimates (3.2.42) and (3.2.43) it is easy to obtain the estimate $\|\Phi_{k+1} - \varphi_{k+1}\| = O(h^2)$. Having in mind this estimate and (3.2.47) we obtain the estimate

$$\|U_{k+1} - u_{k+1}\| = O(h^2).$$

Similarly, we obtain

$$\|Y_{k+1} - y_{k+1}\| = O(h^2), \; \|V_{k+1} - v_{k+1}\| = O(h^2), \; \|Z_{k+1} - z_{k+1}\| = O(h^2).$$

Thus, by induction we have proved the proposition. □

Now combining Proposition 3.2.8 and Theorem 3.2.5 results in the following theorem.

Theorem 3.2.9. For the approximate solution of the problem (3.2.1), (3.2.2) obtained by the discrete iterative method on the uniform grid with grid size h there hold the estimates

$$\begin{aligned} &\|U_k - u\| \le \left(M_0 + \frac{1}{r}\right) p_k d + O(h^2),\ \|Y_k - u'\| \le M_1 p_k d + O(h^2), \\ &\|V_k - u''\| \le M_2 p_k d + O(h^2),\ \|Z_k - u'''\| \le M_3 p_k d + O(h^2). \end{aligned} \tag{3.2.48}$$

Proof. The first above estimate is easily obtained if representing

$$U_k(t_i) - u(t_i) = (u_k(t_i) - u(t_i)) + (U_k(t_i) - u_k(t_i))$$

and using the first estimate in Theorem 3.2.5 and the first estimate in (3.2.43). The remaining estimates are obtained in the same way. Thus, the theorem is proved. □

3.2.5. Examples

Consider some examples for confirming the validity of the obtained theoretical results and the efficiency of the proposed discrete iterative method (3.2.36)-(3.2.38). In all examples we perform the process until $\max\{\|\Phi_{k+1} - \Phi_k\|, |\hat{\mu_{k+1}} - \hat{\mu_k}|\} \le TOL$, where TOL is a given tolerance.

Example 3.2.1 (Example with exact solution)**.** Consider the problem with

$$\begin{aligned} &f = f(t,u) = -18 + \frac{1}{5}u^2 - \frac{1}{5}(\frac{5}{6} + t^3 - \frac{3}{4}t^4)^2 \\ &g(s) = 4s^4. \end{aligned}$$

It is possible to verify that the positive function

$$u(t) = \frac{5}{6} + t^3 - \frac{3}{4}t^4$$

is the exact solution of the problem.

For the given $g(s)$ we have $C_0 = \int_0^1 g(s)\ ds = \frac{4}{5}$. Taking $r = 4, M = 18.2$ we define

$$\mathcal{D}_M^+ = \{(t,u) \mid 0 \le t \le 1,\ 0 \le u \le (M_0 + \frac{1}{r})M = 4.8030\}.$$

In $\mathcal{D}_M^+$ we have $-M \le f \le 0,\ |f'_u| \le 1.9212 = L_0$. After simple calculations we obtain $q_1 = 0.8445,\ q_2 = 0.5070$. Therefore, $q = 0.8445 < 1$. Hence, by Theorem 3.2.4, the problem has a unique positive solution. It is the above exact solution. Meanwhile, it is easy to see that neither Theorem 3.1 nor Theorem 3.2 in [48] are applicable, so the existence of positive solution is not guaranteed by the authors of that paper. Below are the results of the numerical experiments with different tolerances.

In the above tables N is the number of grid points, K is the number of iterations and $Error = \|U_K - u\|$.

Table 3.1: The convergence in Example 3.2.1 for $TOL = 10^{-4}$

N	K	$Error$	N	K	$Error$
30	34	0.0065	500	34	3.9522e-04
50	34	0.0021	1000	34	3.9461e-04
100	34	3.9522e-04	1500	34	3.9413e-04
200	34	3.9522e-04	2000	34	3.9534e-04

Table 3.2: The convergence in Example 3.2.1 for $TOL = 10^{-5}$

N	K	$Error$	N	K	$Error$
30	44	0.0069	300	44	2.8711e-05
50	44	0.0025	500	44	1.6429e-05
100	44	5.8244e-04	1000	44	3.4294e-05
200	44	1.1519e-04	2000	44	3.8906e-05

Table 3.3: The convergence in Example 3.2.1 for $TOL = 10^{-6}$

N	K	$Error$	N	K	$Error$
50	54	0.0050	1000	54	2.6122e-06
100	54	6.1906e-04	2000	54	3.4403e-06
200	54	3.9533e-04	3000	54	3.4403e-06
500	54	3.9522e-04	4000	54	3.7370e-06

Remark 3.2.1. From the tables we observe that for each tolerance the number of iterations is constant and the approximate solution reaches the tolerance when h^2 ($h = 1/N$) is the same order as the tolerance. The further increase of number of grid points does not increase the accuracy of approximate solution.

This phenomenon can be explained as follows:
From Theorem 3.2.9 it is seen that the error of the actual solution, i.e., the discrete solution, consists of two terms. The first term $(M_0 + 1/r)p_k d$ is the error of the iterative method at continuous level (see Theorem 3.2.5) and the second term $O(h^2)$ is the error of discretization at each iteration. The first term depends on the iteration number k by the formula $p_k = q^k/(1-q)$, where q is determined by the nature of the boundary value problem (see Theorem 3.2.3). So, it is desired to choose appropriate h consistent with q because the choice of very small h does not increase the accuracy of approximate discrete solution. Indeed, suppose h^* is consistent with q in the sense that the quantities $O((h^*)^2)$ and $(M_0 + 1/r)p_K d$ for some K are the same as TOL. Then for any $h < h^*$ the accuracy almost remains the same. Theoretically, the number of iterations K is the minimal natural number k satisfying the inequality $(M_0 + 1/r)p_k d \leq TOL$.

Example 3.2.2 (Example 4.1 in [48]). Consider the boundary value problem

$$u''''(t) = -u^2(e^{-u} + 1), \quad 0 < t < 1,$$
$$u'(0) = u''(0) = u'(1) = 0, \; u(0) = \int_0^1 s^2 u(s)ds.$$

In this example

$$f = f(t, u) = -u^2(e^{-u} + 1), \quad g(s) = s^2.$$

So

$$C_0 = \int_0^1 s^2 ds = \frac{1}{3}.$$

Choose $M = 0.4, r = 3$ and define

$$\mathcal{D}_M^+ = \{(t,u) \mid 0 \le t \le 1,\ 0 \le u \le (M_0+1)M\},$$

where $M_0 = 0.0139$ as was computed in (3.2.18). Then it is easy to verify that

$$-M \le f(t,u) \le 0 \text{ in } \mathcal{D}_M^+$$

and $|\dfrac{\partial f}{\partial u}| \le 1.622 =: L_0$ in $\mathcal{D}_M^+$. Therefore, $q_1 = rC_0M_0 + C_0 = 0.3472$, $q_2 = L_0(M_0 + \frac{1}{r}) = 0.5633$, and due to this $0 < q < 1$. By Theorem 3.2.4, the problem has a unique nonnegative solution. Since the function $u(t) \equiv 0$ is a solution of the problem, we conclude that the unique solution of the problem is this trivial solution. The computational experiment supports this theoretical conclusion. Remark that in [48] the authors established that the problem has a positive solution. So, their result is not correct.

Example 3.2.3 (Example 4.2 in [48])**.** Consider the boundary value problem

$$\begin{aligned} u''''(t) &= -\sqrt{(1+u)} - \sin u, \quad 0 < t < 1,\\ u'(0) &= u''(0) = u'(1) = 0,\ u(0) = \int_0^1 su(s)ds. \end{aligned}$$

In this example

$$f = f(t,u) = -\sqrt{(1+u)} - \sin u, \quad g(s) = s.$$

So,

$$C_0 = \int_0^1 sds = \frac{1}{2}.$$

Choosing $r = 3$ and M sufficiently large, for example, $M = 3$, we have $-M \le f(t,u) \le 0$ in $\mathcal{D}_M^+$, where

$$\mathcal{D}_M^+ = \{(t,u) \mid 0 \le t \le 1,\ 0 \le u \le (M_0 + \frac{1}{r})M\}.$$

In this domain we can take the Lipschitz coefficient $L_0 = 1.5$. Therefore, $q_1 = q_2 = 0.5209$ and then $q = 0.5209 < 1$. Moreover, $f(t,0) = -1 \ne 0$. Hence, by Theorem 3.2.4 the problem has a unique positive solution. Remark that in [48] the authors could only conclude the existence of at least one positive solution.

The numerical computations show that the iterative method described in Section 3.2.3 converges fast. As in Example 3.2.1, the number of iterations for achieving a given tolerance is independent of the grid size. Table 3.4 reports the number of iterations in dependence on TOL.

Table 3.4: The convergence in Example 3.2.3

TOL	10^{-4}	10^{-5}	10^{-6}	10^{-8}
K	12	16	19	26

The graph of the approximate solution for $N = 100$ and $TOL = 10^{-4}$ is depicted in Figure 3.3.

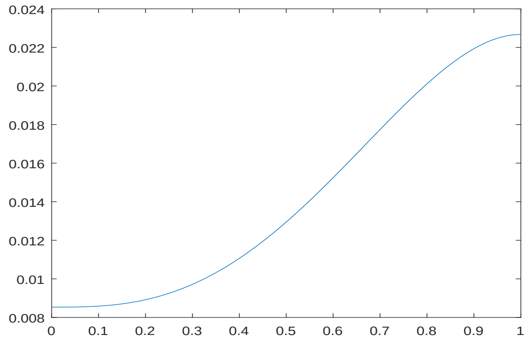

Figure 3.3: The graph of the approximate solution in Example 3.2.3.

Example 3.2.4. Consider Example 3.2.2 with

$$f = -(1 + u^2).$$

Then $\frac{-f(u)}{u} \to +\infty$ as $u \to +0$ and $u \to +\infty$. Thus, neither Theorem 3.1 or Theorem 3.2 in [48] are satisfied, so the existence of positive solution is not guaranteed.
Now apply our theory: Choose $M = 2, r = 3$, then

$$\mathcal{D}_M^+ = \{(t, u) \mid 0 \le t \le 1,\ 0 \le u \le (M_0 + \frac{1}{r})M = 0.6944\}.$$

In $\mathcal{D}_M^+$ we have $-M \le f \le 0$, $|f'_u| \le 1.3888 = L_0$. After simple calculations we obtain $q_1 = 0.3472$, $q_2 = 0.4822$. Hence, by Theorem 3.2.3, the problem has a unique nonnegative solution. Due to $f(t, 0) \ne 0, u(t) \not\equiv 0$, it is a positive solution. The performed numerical experiments also show that the number of iterations for achieving a given tolerance is independent of the grid size. Table 3.5 reports the number of iterations in dependence on TOL.

Table 3.5: The convergence in Example 3.2.4

TOL	10^{-4}	10^{-5}	10^{-6}	10^{-8}
K	7	9	12	16

The graph of the approximate solution for $N = 100$ and $TOL = 10^{-4}$ is depicted in Figure 3.4.

Example 3.2.5. Consider the problem (3.2.1)-(3.2.2) with

$$f(t, u, y, v, z) = -(\sqrt{1 + u} + \sin y + \frac{1}{3}\cos v + \sin z),\ g(s) = s.$$

It is possible to verify that all the conditions of Theorem 3.2.4 are satisfied. So, the problem has a unique positive solution.

The results on the convergence of the iterative method for this example is given in Table 3.6.

The approximate solution obtained on the grid with the number of nodes $N = 100$ and $TOL = 10^{-4}$ is depicted on Figure 3.5.

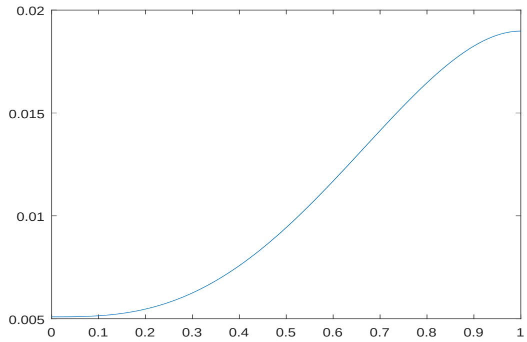

Figure 3.4: The graph of the approximate solution in Example 3.2.4.

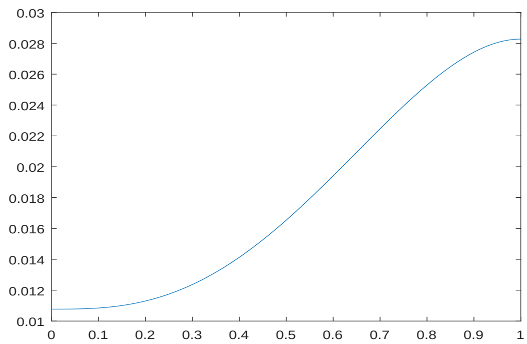

Figure 3.5: The graph of the approximate solution in Example 3.2.5.

Table 3.6: The convergence in Example 3.2.5

TOL	10^{-4}	10^{-5}	10^{-6}	10^{-8}
K	11	14	18	25

3.2.6. Conclusion

In this section, we have established the existence, uniqueness and positivity of solution of a fully fourth order nonlinear integral boundary value problem. The idea of the method used is to reduce the problem to a fixed point problem for an operator defined on pairs of functions and numbers. It is a further development of the method applied by ourselves before for other types of boundary conditions. We also study an iterative method for solving the problem at continuous level. After that we propose a discrete scheme for realizing the continuous iterative method. Our contribution also includes the analysis of total error of the approximate discrete solution, which consists of the error of the continuous iterative method and the error of discretization at each iteration. Many examples demonstrate the validity of the obtained theoretical results and efficiency of the iterative method.

The method used in this paper can be applied to other BVPs of higher order and with other boundary conditions including nonlinear boundary conditions. This is the subject of our researches in the future.

Chapter 4

Existence results and iterative method for integro-differential and functional differential equations

4.1. Existence results and iterative method for integro-differential equation

4.1.1. Introduction

In this section we consider the problem

$$
\begin{aligned}
u^{(4)}(x) = f(x, u(x), u'(x), \int_0^1 k(x,t)u(t)dt), \\
u(0) = 0,\ u(1) = 0,\ u''(0) = 0,\ u''(1) = 0,
\end{aligned}
\tag{4.1.1}
$$

where the function $f(x, u, v, z)$ and $k(x, t)$ are assumed to be continuous. This problem is an extension of the problem

$$
\begin{aligned}
y^{(4)}(x) = f(x, y(x), \int_0^1 k(x,t)y(t)dt),\ 0 < x < 1, \\
y(0) = 0,\ y(1) = 0,\ y''(0) = 0,\ y''(1) = 0
\end{aligned}
\tag{4.1.2}
$$

considered recently by Wang in [66], where by using the monotone method and a maximum principle, he constructed the sequences of functions, which converge to the extremal solutions of the problem. Remark that the presence of an extra u' in the right hand side function of (4.1.1) does not allow to use the argument in [66] to study the existence of solutions of the problem. Here, using the method developed in our previous papers [11, 13, 14, 85, 86, 88, 90] we establish the existence and uniqueness of the solution and propose an iterative method at both continuous and discrete levels for finding the solution. The second order convergence of the method is proved. The theoretical results are illustrated by some examples.

4.1.2. Existence results

Using the methodology in [11, 13, 14, 85, 86, 88, 90] we introduce the operator A defined in the space of continuous functions $C[0, 1]$ by the formula

$$
(A\varphi)(x) = f(x, u(x), u'(x), \int_0^1 k(x,t)u(t)dt),
\tag{4.1.3}
$$

where $u(x)$ is the solution of the boundary value problem

$$
\begin{aligned}
u'''' = \varphi(x),\ 0 < x < 1, \\
u(0) = u''(0) = u(1) = u''(1) = 0.
\end{aligned}
\tag{4.1.4}
$$

It is easy to verify the following lemma.

Lemma 4.1.1. If the function φ is a fixed point of the operator A, i.e., φ is the solution of the operator equation

$$A\varphi = \varphi, \tag{4.1.5}$$

where A is defined by (4.1.3)-(4.1.4) then the function $u(x)$ determined from the BVP (4.1.4) is a solution of the BVP (4.1.1). Conversely, if the function $u(x)$ is the solution of the BVP (4.1.1) then the function

$$\varphi(x) = f(x, u(x), u'(x), \int_0^1 k(x,t)u(t)dt)$$

satisfies the operator equation (4.1.5).

Due to the above lemma we shall study the original BVP (4.1.1) via the operator equation (4.1.5). Before doing this we notice that the BVP (4.1.4) has a unique solution representable in the form

$$u(x) = \int_0^1 G_0(x,s)\varphi(s)ds, \quad 0 < t < 1, \tag{4.1.6}$$

where

$$G_0(x,s) = \frac{1}{6}\begin{cases} s(x-1)(x^2 - x + s^2), & 0 \le s \le x \le 1 \\ x(s-1)(s^2 - s + x^2), & 0 \le x \le s \le 1 \end{cases} \tag{4.1.7}$$

is the Green's function of the operator $u''''(t) = 0$ associated with the homogeneous boundary conditions $u(0) = u''(0) = u(1) = u''(1) = 0$.
Differentiating both sides of (4.1.6) gives

$$u'(x) = \int_0^1 G_1(x,s)\varphi(s)ds, \tag{4.1.8}$$

where

$$G_1(x,s) = \frac{1}{6}\begin{cases} s(3x^2 - 6x + s^2 + 2), & 0 \le s \le x \le 1, \\ (s-1)(3x^2 - 2s + s^2), & 0 \le x \le s \le 1. \end{cases} \tag{4.1.9}$$

Set

$$\begin{aligned} M_0 &= \max_{0\le x\le 1} \int_0^1 |G_0(x,s)|ds, \\ M_1 &= \max_{0\le x\le 1} \int_0^1 |G_1(x,s)|ds, \\ M_2 &= \max_{0\le x\le 1} \int_0^1 |k(x,s)|ds. \end{aligned} \tag{4.1.10}$$

It is easy to obtain

$$M_0 = \frac{5}{384}, M_1 = \frac{1}{24}. \tag{4.1.11}$$

Now for any positive number M, we define the domain

$$\begin{aligned} \mathcal{D}_M = \{(x,u,v,z) \mid 0 \le x \le 1,\ |u| \le M_0 M, \\ |v| \le M_1 M,\ |z| \le M_0 M_2 M\}. \end{aligned} \tag{4.1.12}$$

As usual, we denote by $B[0,M]$ the closed ball centered at 0 with radius M in the space $C[0,1]$, i.e.,

$$B[0,M] = \{u \in C[0,1] \mid \|u\| \le M\},$$

where $\|u\| = \max_{0\le x\le 1}|u(x)|$.

Theorem 4.1.1 (Existence and uniqueness)**.** Suppose that the function $k(x,t)$ is continuous in the square $[0,1]\times[0,1]$ and there exist numbers $M>0$, $L_0, L_1, L_2 \ge 0$ such that:

(i) The function $f(x,u,v,z)$ is continuous in the domain $\mathcal{D}_M$ and $|f(x,u,v,z)| \le M$, $\forall (x,u,v,z) \in \mathcal{D}_M$.

(ii) $|f(x_2,u_2,v_2,z_2) - f(x_1,u_1,v_1,z_1)| \le L_0|u_2-u_1| + L_1|v_2-v_1| + L_2|z_2-z_1|$, $\forall (x_i,u_i,v_i,z_i) \in \mathcal{D}_M,\ i=1,2$.

(iii) $q = L_0M_0 + L_1M_1 + L_2M_0M_2 < 1$.

Then the problem (4.1.1) has a unique solution $u \in C^4[0,1]$ satisfying $|u(x)| \le M_0M$, $|u'(x)| \le M_1M$ for any $0 \le x \le 1$.

Proof. Under the assumptions of the theorem we shall prove that the operator A is a contraction mapping in the closed ball $B[O,M]$. Then the operator equation (4.1.5) has a unique solution $u \in C^{(4)}[0,1]$ and this implies the existence and uniqueness of solution of the BVP (4.1.1).

Indeed, take $\varphi \in B[O,M]$. Then the problem (4.1.4) has a unique solution of the form (4.1.6). From there and (4.1.10) we obtain $|u(x)| \le M_0\|\varphi\|$ for all $x \in [0,1]$. Analogously, we have $\|u'(x)\| \le M_1\|\varphi\|$ for all $x \in [0,1]$. Denote by K the integral operator defined by

$$(Ku)(x) = \int_0^1 k(x,t)u(t)dt.$$

Then from the last equation in (4.1.10) we have the estimate $|(Ku)(x)| \le M_0M_2\|\varphi\|$, $x \in [0,1]$. Thus, if $\varphi \in B[O,M]$, i.e., $\|\varphi\| \le M$ then for any $x \in [0,1]$ we have

$$|u(x)| \le M_0M,\ |u'(x)| \le M_1M,\ |(Ku)(x)| \le M_0M_2M.$$

Therefore, $(x, u(x), u'(x), (Ku)(x)) \in \mathcal{D}_M$. By the assumption (i) there is

$$|f(x,u(x),u'(x),(Ku)(x))| \le M \quad \forall x \in [0,1].$$

Hence, $|(A\varphi)(x)| \le M$, $\forall x \in [0,1]$ and $\|A\varphi\| \le M$. It means that A maps $B[O,M]$ into itself.

Next, take $\varphi_1, \varphi_2 \in B[O,M]$. Using the assumption (ii) and (iii) it is easy to obtain

$$\|A\varphi_2 - A\varphi_1\| \le (L_0M_0 + L_1M_1 + L_2M_0M_2)\|\varphi_2 - \varphi_1\| = q\|\varphi_2 - \varphi_1\|.$$

Since $q<1$ the operator A is a contraction in $B[O,M]$. This completes the proof of the theorem. □

Now, in order to study positive solutions of the BVP (4.1.1) we introduce the domain

$$\begin{aligned}\mathcal{D}_M^+ = \{(x,u,v,z) \mid 0 \le x \le 1,\ 0 \le u \le M_0 M,\\ |v| \le M_1 M,\ |z| \le M_0 M_2 M\},\end{aligned} \tag{4.1.13}$$

and denote

$$S_M = \{\varphi \in C[0,1], 0 \le \varphi(x) \le M\}.$$

Theorem 4.1.2 (Positivity of solution)**.** Suppose that the function $k(x,t)$ is continuous in the square $[0,1]\times[0,1]$ and there exist numbers $M > 0$, $L_0, L_1, L_2 \ge 0$ such that:

(i) The function $f(x,u,v,z)$ is continuous in the domain $\mathcal{D}_M^+$ and $0 \le f(x,u,v,z) \le M$, $\forall (x,u,v,z) \in \mathcal{D}_M^+$ and $f(x,0,0,0) \not\equiv 0$.

(ii) $|f(x_2,u_2,v_2,z_2) - f(x_1,u_1,v_1,z_1)| \le L_0|u_2 - u_1| + L_1|v_2 - v_1| + L_2|z_2 - z_1|$, $\forall (x_i,u_i,v_i,z_i) \in \mathcal{D}_M^+,\ i = 1,2$.

(iii) $q = L_0M_0 + L_1M_1 + L_2M_0M_2 < 1$.

Then the problem (4.1.1) has a unique positive solution $u \in C^4[0,1]$ satisfying $0 \le u(x) \le M_0M$, $|u'(x)| \le M_1M$ for any $0 \le x \le 1$.

Proof. Similarly to the proof of Theorem 4.1.1, where instead of $\mathcal{D}_M$ and $B[O;M]$ there stand $\mathcal{D}_M^+$ and S_M, we conclude that the problem has a nonnegative solution. Due to the condition $f(x,0,0,0) \not\equiv 0$, this solution must be positive. □

4.1.3. Numerical method

In this section we suppose that all the conditions of Theorem 4.1.1 are satisfied. Then the problem (4.1.1) has a unique solution. For finding this solution consider the following iterative method:

1. Given
$$\varphi_0(x) = f(x,0,0,0). \tag{4.1.14}$$

2. Knowing $\varphi_m(x)$ $(m = 0,1,...)$ compute
$$\begin{aligned} u_m(x) &= \int_0^1 G_0(x,t)\varphi_m(t)dt,\\ v_m(x) &= \int_0^1 G_1(x,t)\varphi_m(t)dt,\\ z_m(x) &= \int_0^1 k(x,t)u_m(t)dt. \end{aligned} \tag{4.1.15}$$

3. Update
$$\varphi_{m+1}(x) = f(x,u_m(x),v_m(x),z_m(x)). \tag{4.1.16}$$

This iterative method indeed is the successive iterative method for finding the fixed point of operator A. Therefore, it converges with the rate of geometric progression and there holds the estimate

$$\|\varphi_m - \varphi\| \leq \frac{q^m}{1-q}\|\varphi_1 - \varphi_0\| = p_m d,$$

where φ is the fixed point of the operator A and

$$p_m = \tfrac{q^m}{1-q},\ d = \|\varphi_1 - \varphi_0\|. \tag{4.1.17}$$

This estimate implies the following result of the convergence of the iterative method (4.1.14)-(4.1.16).

Theorem 4.1.3. Under the conditions of Theorem 4.1.1 the iterative method (4.1.14)-(4.1.16) converges and for the approximate solution $u_k(t)$ there hold estimates

$$\|u_m - u\| \leq M_0 p_m d,\ \|u'_m - u'\| \leq M_1 p_m d,$$

where u is the exact solution of the problem (4.1.1), p_m and d are defined by (4.1.17).

To numerically realize the above iterative method we construct a corresponding discrete iterative method. For this purpose cover the interval $[0,1]$ by the uniform grid $\bar{\omega}_h = \{x_i = ih,\ h = 1/N, i = 0, 1, ..., N\}$ and denote by $\Phi_m(x), U_m(x), V_m(x), Z_m(x)$ the grid functions, which are defined on the grid $\bar{\omega}_h$ and approximate the functions $\varphi_m(x), u_m(x), v_m(x), z_m(x)$ on this grid.

Consider now the following discrete iterative method:

1. Given
$$\Phi_0(x_i) = f(x_i, 0, 0, 0),\ i = 0, ..., N. \tag{4.1.18}$$

2. Knowing $\Phi_m(x_i),\ m = 0, 1, ...;\ i = 0, ..., N$, compute approximately the definite integrals (4.1.15) by the trapezium formulas
$$\begin{aligned} U_m(x_i) &= \sum_{j=0}^{N} h\rho_j G_0(x_i, x_j)\Phi_m(x_j), \\ V_m(x_i) &= \sum_{j=0}^{N} h\rho_j G_1(x_i, x_j)\Phi_m(x_j), \\ Z_m(x_i) &= \sum_{j=0}^{N} h\rho_j k(x_i, x_j)U_m(x_j),\ i = 0, ..., N, \end{aligned} \tag{4.1.19}$$
where ρ_j is the weight of the trapezium formula, namely
$$\rho_j = \begin{cases} 1/2,\ j = 0, N \\ 1,\ j = 1, 2, ..., N-1. \end{cases}$$

3. Update
$$\Phi_{m+1}(x_i) = f(x_i, U_m(x_i), V_m(x_i), Z_m(x_i)). \tag{4.1.20}$$

In order to get the error estimates for the approximate solution for $u(t)$ and its derivatives on the grid we need some following auxiliary results.

Proposition 4.1.4. Assume that the function $f(t,u,v,z)$ has all continuous partial derivatives up to second order in the domain $\mathcal{D}_M$ and the kernel function $k(x,t)$ also has all continuous partial derivatives up to second order in the square $[0,1]\times[0,1]$. Then for the functions $\varphi_m(x), u_m(x), v_m(x), z_m(x), m = 0,1,...,$ constructed by the iterative method (4.1.14)-(4.1.16) we have $\varphi_m(x) \in C^2[0,1]$, $u_m(x) \in C^6[0,1],\ v_m(x) \in C^5[0,1],\ z_m(x) \in C^2[0,1]$.

Proof. We prove the proposition by induction. For $k=0$, by the assumption on the function f we have $\varphi_0(t) \in C^2[0,1]$ since $\varphi_0(x) = f(x,0,0,0)$. Taking into account

$$u_0(x) = \int_0^1 G_0(x,t)\varphi_0(t)dt$$

we deduce that the function $u_0(x)$ is the solution of the BVP

$$\begin{aligned} u_0^{(4)}(x) &= \varphi_0(x), \quad x \in (0,1),\\ u_0(0) &= u_0(1) = u_0''(0) = u_0''(1) = 0. \end{aligned}$$

Therefore, $u_0(x) \in C^6[0,1]$. It implies that $v_0(x) \in C^5[0,1]$ because $v_0(x) = u_0'(x)$. Since by assumptions $k(x,t)$ has all continuous derivatives up to second order, the function $z_0(x) = \int_0^1 k(x,t)u_0(t)dt$ belongs to $C^2[0,1]$.

Now suppose $\varphi_m(x) \in C^2[0,1],\ u_m(x) \in C^6[0,1],\ v_m(x) \in C^5[0,1],\ z_m(x) \in C^2[0,1]$. Then, because $\varphi_{m+1}(x) = f(x,u_m(x),v_m(x),z_m(x))$ and the functions f by the assumption has continuous derivative in all variables up to order 2, it follows that $\varphi_{m+1}(x) \in C^2[0,1]$. Repeating the same argument as for $\varphi_0(x)$ above we obtain that $u_{m+1}(x) \in C^6[0,1],\ v_{m+1}(x) \in C^5[0,1],\ z_{m+1}(x) \in C^2[0,1]$ Thus, the proposition is proved. □

Proposition 4.1.5. For any function $\varphi(x) \in C^2[0,1]$ there holds the estimate

$$\int_0^1 G_n(x_i,t)\varphi(t)dt = \sum_{j=0}^{N} h\rho_j G_n(x_i,t_j)\varphi(t_j) + O(h^2) \quad (n=0,1). \tag{4.1.21}$$

Proof. The above estimate is obvious in view of the error estimate of the compound trapezium formula because the functions $G_n(x_i,t)$ $(n=0,1)$ are continuous at t_j and are polynomials in the intervals $[0,t_j]$ and $[t_j,1]$. □

Proposition 4.1.6. Under the assumptions of Proposition 4.1.4, for any $m = 0,1,...$ there hold the estimates

$$\|\Phi_m - \varphi_m\| = O(h^2), \quad \|U_m - u_m\| = O(h^2), \tag{4.1.22}$$

$$\|V_m - v_m\| = O(h^2),\ \|Z_m - z_m\| = O(h^2). \tag{4.1.23}$$

where $\|.\| = \|.\|_{\bar{\omega}_h}$ is the max-norm of function on the grid $\bar{\omega}_h$.

Proof. We prove the proposition by induction. For $m = 0$ we have immediately $\|\Phi_0 - \varphi_0\| = 0$. Next, by the first equation in (4.1.15) and Proposition 4.1.5 we have

$$u_0(x_i) = \int_0^1 G_0(x_i, t)\varphi_0(t)dt = \sum_{j=0}^{N} h\rho_j G_0(x_i, t_j)\varphi_0(t_j) + O(h^2) \tag{4.1.24}$$

for any $i = 0, ..., N$. On the other hand, in view of the first equation in (4.1.19) we have

$$U_0(x_i) = \sum_{j=0}^{N} h\rho_j G_0(x_i, t_j)\Phi_0(t_j). \tag{4.1.25}$$

Therefore, $|U_0(t_i) - u_0(t_i)| = O(h^2)$ because $\Phi_0(t_j) = \varphi_0(t_j) = f(t_j, 0, 0, 0)$. Consequently, $\|U_0 - u_0\| = O(h^2)$.
Similarly, we have

$$\|V_0 - v_0\| = O(h^2). \tag{4.1.26}$$

Next, by the trapezium formula we have

$$z_0(x_i) = \int_0^1 k(x_i, t)u_0(t)dt = \sum_{j=0}^{N} h\rho_j k(x_i, t_j)u_0(t_j) + O(h^2),$$

while by the third equation in (4.1.19) we have

$$Z_0(x_i) = \sum_{j=0}^{N} h\rho_j k(x_i, t_j)U_0(t_j),\ i = 0, ..., N.$$

Therefore,

$$\begin{aligned}
\Big|Z_0(x_i) - z_0(x_i)\Big| &= \Big|\sum_{j=0}^{N} h\rho_j k(x_i, t_j)(U_0(t_j) - u_0(t_j))\Big| + O(h^2) \\
&\le \sum_{j=0}^{N} h\rho_j |k(x_i, t_j)||U_0(t_j) - u_0(t_j)| + O(h^2) \\
&\le Ch^2 \sum_{j=0}^{N} h\rho_j |k(x_i, t_j)| + O(h^2) \\
&\le CC_1 h^2 \sum_{j=0}^{N} h\rho_j + O(h^2) = O(h^2)
\end{aligned}$$

because $|U_0(t_j) - u_0(t_j)| \le Ch^2$, $|k(x_i, t_j)| \le C_1$, where C, C_1 are some constants.

Now suppose that (4.1.22) and (4.1.23) are valid for $m \ge 0$. We shall show that these estimates are valid for $m+1$. By the Lipschitz condition of the function f and the estimates (4.1.22) and (4.1.23) it is easy to obtain the estimate

$$\|\Phi_{m+1} - \varphi_{m+1}\| = O(h^2).$$

Now from the first equation in (4.1.15) by Proposition 4.1.5 we have

$$u_{m+1}(x_i) = \int_0^1 G_0(x_i, t)\varphi_{m+1}(t)dt = \sum_{j=0}^{N} h\rho_j G_0(x_i, x_j)\varphi_{m+1}(x_j) + O(h^2).$$

On the other hand by the first formula in (4.1.19) we have

$$U_{m+1}(x_i) = \sum_{j=0}^{N} h\rho_j G_0(x_i, x_j)\Phi_{m+1}(x_j).$$

From this equality and the above estimates we obtain the estimate

$$\|U_{m+1} - u_{m+1}\| = O(h^2).$$

Similarly, we obtain

$$\|V_{m+1} - v_{m+1}\| = O(h^2), \ \|Z_{m+1} - z_{k+1}\| = O(h^2).$$

Thus, by induction we have proved the proposition. □

Now combining Proposition 4.1.6 and Theorem 4.1.3 results in the following theorem.

Theorem 4.1.7. Assume that all the conditions of Theorem 4.1.1 and Proposition 4.1.4 are satisfied. Then, for the approximate solution of the problem (4.1.1) obtained by the discrete iterative method on the uniform grid with grid size h there hold the estimates

$$\|U_m - u\| \le M_0 p_m d + O(h^2), \ \|V_m - u'\| \le M_2 p_m d + O(h^2). \tag{4.1.27}$$

Proof. The first above estimate is easily obtained if representing

$$U_m(t_i) - u(t_i) = (u_m(t_i) - u(t_i)) + (U_m(t_i) - u_m(t_i))$$

and using the first estimate in Theorem 4.1.3 and the second estimate in (4.1.22). The remaining estimate is obtained in the same way. Thus, the theorem is proved. □

4.1.4. Examples

Example 4.1.1. Consider the problem (4.1.1) with

$$\begin{aligned} &k(x,t) = e^x \sin(\pi t), \quad (x,t) \in [0,1] \times [0,1], \\ &f(x, u(x), u'(x), \int_0^1 k(x,t)u(t)dt) = u^2(x) \int_0^1 k(x,t)u(t)dt + u(x)u'(x) \\ &- \frac{1}{2}e^x \sin^2(\pi x) + \pi^4 \sin(\pi x) - \frac{\pi}{2}\sin(2\pi x). \end{aligned}$$

In this case

$$f(x,u,v,z) = u^2 z + uv - \frac{1}{2}e^x \sin^2(\pi x) + \pi^4 \sin(\pi x) - \frac{\pi}{2}\sin(2\pi x)$$

and $M_2 = \dfrac{2e}{\pi}$. It is possible to verify that the function $u = \sin(\pi x)$ is the exact solution of the problem. In the domain $\mathcal{D}_M$ defined by

$$\mathcal{D}_M = \{(x,u,v,z) \mid 0 \le x \le 1, \ |u| \le M_0 M, |u'| \le M_1 M, |z| \le M_0 M_2 M\}$$

we have

$$|f(x,u,v,z)| \le M_0^3 M_2 M^3 + M_0 M_1 M^2 + \pi^4 + \frac{\pi}{2} + \frac{e}{2}.$$

It is possible to verify that for $M = 113$ all the conditions of Theorem 4.1.1 are satisfied with $L_0 = 12.2010, L_1 = 1.4714, L_2 = 2.1649, q = 0.2690$. Therefore, the problem has a unique solution $u(x)$ satisfying the estimates $|u(x)| \le 1.4714, |u'(x)| \le 4.7083$. These theoretical estimates are somewhat greater than the exact estimates $|u(x)| \le 1, |u'(x)| \le \pi$.

Below we report the numerical results by the discrete iterative method (4.1.18)-(4.1.20) for the problem. In Tables 4.1 and 4.2 we use the notation $Error = \|U_m - u\|$, where u is the exact solution of the problem.

Table 4.1: The convergence in Example 4.1.1 for stopping criterion $\|U_m - u\| \le h^2$

N	h^2	m	$Error$
50	4.0000e-04	2	1.4305e-04
100	1.0000e-04	3	2.8588e-06
150	4.4444e-05	3	2.8599e-06
200	2.5000e-05	3	2.8602e-06
300	1.1111e-05	3	2.8603e-06
400	6.2500e-06	3	2.8603e-06
500	4.0000e-06	3	2.8603e-06
800	1.5625e-06	4	5.7485e-08
1000	1.0000e-06	4	5.7486e-08

It is interesting to notice that if taking stopping criterion $\|\Phi_m - \Phi_{m-1}\| \le 10^{-10}$ instead of $\|U_m - u\| \le h^2$ then we obtain better accuracy of the approximate solution with more iterations. See Table 4.2.

Table 4.2: The convergence in Example 4.1.1 for stopping criterion $\|\Phi_m - \Phi_{m-1}\| \le 10^{-10}$

N	h^2	m	$Error$
50	4.0000e-04	7	2.2152e-08
100	1.0000e-04	7	1.3831e-09
150	4.4444e-05	7	2.7279e-10
200	2.5000e-05	7	8.5995e-11
300	1.1111e-05	7	1.6618e-11
400	6.2500e-06	7	4.9447e-12
500	4.0000e-06	7	1.7567e-12
800	1.5625e-06	7	1.4588e-13
1000	1.0000e-06	7	3.3318e-13

From Table 4.2 we see that the accuracy of the approximate solution is near $O(h^4)$ although by the proved theory it is only $O(h^2)$.

Example 4.1.2 (Example 4.2 in [66]). Consider the problem

$$\begin{aligned} u^{(4)}(x) &= \sin(\pi x)[(2 - u^2(x)) \int_0^1 tu(t)dt + 1], x \in (0,1) \\ u(0) &= 0,\ u(1) = 0,\ u''(0) = 0,\ u''(1) = 0. \end{aligned} \tag{4.1.28}$$

This is the problem (4.1.1) with

$$k(x,t) = \sin(\pi x)t, \quad (x,t) \in [0,1] \times [0,1],$$

$$f(x, u(x), u'(x), \int_0^1 k(x,t)u(t)dt) = (2 - u^2(x)) \int_0^1 \sin(\pi x) t u(t)dt + \sin(\pi x).$$

So, $f(x,u,v,z) = (2-u^2)z + \sin(\pi x)$.
It is easy to see that $M_2 = \max_{0 \le x \le 1} \int_0^1 |k(x,t)|dt = \frac{1}{2}$. Since M_0 and M_1 are given by (4.1.11) we define

$$\mathcal{D}_M = \{(x,u,v,z) \mid 0 \le x \le 1,\ |u| \le \frac{5}{384}M, |v| \le \frac{1}{24}M, |z| \le \frac{5}{768}M\}. \quad (4.1.29)$$

It is possible to verify that for $M = 1.1$ all the assumptions of Theorem 4.1.1 are satisfied with $L_0 = 2.0515e-04, L_1 = 0, L_2 = 2, q = 0.0130$. Therefore, the problem (4.1.28) has a unique solution satisfying $|u(x)| \le 0.0143,\ |u'(x)| \le 0.0458$.

It is worth emphasizing that in [66] by the monotone method the author could only prove the convergence of the iterative sequences to extremal solutions of the problem but not the existence and uniqueness of solution.

Using the discrete iterative method (4.1.18)-(4.1.20) on the grid with grid step $h = 0.01$ and the stopping criterion $\|\Phi_m - \Phi_{m-1}\| \le 10^{-10}$ we found an approximate solution after 7 iterations. The graph of this approximate solution is depicted in Figure 4.1.

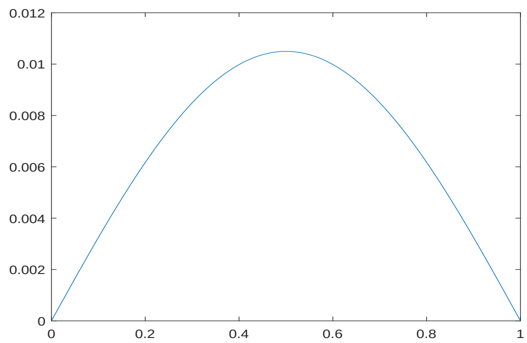

Figure 4.1: The graph of the approximate solution in Example 4.1.2.

4.1.5. Conclusion

In this section, we have established the existence and uniqueness of the solution for a fourth order nonlinear integro-differential equation with the Navier boundary conditions and proposed an iterative method at both continuous and discrete levels for finding the solution. The second order of accuracy of the discrete method has been proved. Some examples, where the exact solution is known and is not known, demonstrate the validity of the obtained theoretical results and the efficiency of the iterative method. It should be emphasized that for the example of Wang in [66] we have established the existence and uniqueness of solution and found it numerically but Wang could prove only the convergence of the iterative sequences constructed by the monotone method to extremal solutions.

The method used in this section with appropriate modifications can be applied to nonlinear integro-differential equations of any order with other boundary conditions and more complicated nonlinear terms. This is the direction of our research in the future.

4.2. Existence results and iterative method for functional differential equation

4.2.1. Introduction

In this section we propose a new approach to functional differential equations (FDE), which is different from the approach of Bica et al. [75] in 2016 for functional differential equations of even orders, where they use iterated cubic splines. Although our approach can be applied to functional differential equations of any orders with nonlinear terms containing derivatives but for simplicity we consider the FDE of the form

$$u''' = f(t, u(t), u(\varphi(t))), \quad t \in [0, a] \tag{4.2.1}$$

associated with the general boundary conditions

$$\begin{aligned} B_1[u] &= \alpha_1 u(0) + \beta_1 u'(0) + \gamma_1 u''(0) = b_1, \\ B_2[u] &= \alpha_2 u(0) + \beta_2 u'(0) + \gamma_2 u''(0) = b_2, \\ B_3[u] &= \alpha_3 u(1) + \beta_3 u'(1) + \gamma_3 u''(1) = b_3, \end{aligned} \tag{4.2.2}$$

or

$$\begin{aligned} B_1[u] &= \alpha_1 u(0) + \beta_1 u'(0) + \gamma_1 u''(0) = b_1, \\ B_2[u] &= \alpha_2 u(1) + \beta_2 u'(1) + \gamma_2 u''(1) = b_2, \\ B_3[u] &= \alpha_3 u(1) + \beta_3 u'(1) + \gamma_3 u''(1) = b_3, \end{aligned} \tag{4.2.3}$$

such that

$$Rank \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 & 0 & 0 & 0 \\ \alpha_2 & \beta_2 & \gamma_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix} = 3.$$

The function $\varphi(t)$ is assumed to be continuous and maps $[0, a]$ into itself.

Developing the unified approach for fully third order nonlinear differential equation

$$u''' = f(t, u(t), u'(t), u''(t))$$

in the previous works [13,14], in this section we establish the existence and uniqueness of solution of the problem (4.2.1)-(4.2.2) and propose an iterative method for finding the solution at both continuous and discrete levels. Some examples demonstrate the validity of obtained theoretical results and the efficiency of the proposed numerical method.

4.2.2. Existence and uniqueness of solution

Following the approach in [13, 14] (see also [11, 85]) to investigate the problem (4.2.1)-(4.2.2) we introduce the nonlinear operator A defined in the space of continuous functions $C[0, a]$ by the formula:

$$(A\psi)(t) = f(t, u(t), u(\varphi(t))), \tag{4.2.4}$$

where $u(t)$ is the solution of the problem

$$\begin{aligned} &u'''(t) = \psi(t), \quad 0 < t < a \\ &B_1[u] = b_1, B_2[u] = b_2, B_3[u] = b_3, \end{aligned} \tag{4.2.5}$$

where $B_1[u], B_2[u], B_3[u]$ are defined by (4.2.2). It is easy to verify the following

Proposition 4.2.1. If the function ψ is a fixed point of the operator A, i.e., ψ is the solution of the operator equation

$$A\psi = \psi, \tag{4.2.6}$$

where A is defined by (4.2.4)-(4.2.5) then the function $u(t)$ determined from the BVP (4.2.5) is a solution of the BVP (4.2.1)-(4.2.2). Conversely, if the function $u(x)$ is the solution of the BVP (4.2.1)-(4.2.2) then the function

$$\psi(t) = f(t, u(t), u(\varphi(t)))$$

satisfies the operator equation (4.2.6).

Now, let $G(t, s)$ be the Green function of the problem (4.2.5). Then the solution of the problem can be represented in the form

$$u(t) = g(t) + \int_0^a G(t, s)\psi(s)ds, \tag{4.2.7}$$

where $g(t)$ is the polynomial of second degree satisfying the boundary conditions

$$B_1[g] = b_1, B_2[g] = b_2, B_3[g] = b_3, \tag{4.2.8}$$

Denote

$$M_0 = \max_{0 \le t \le a} \int_0^a |G(t, s)|ds. \tag{4.2.9}$$

For any positive number M define the domain

$$\mathcal{D}_M = \Big\{(t, u, v) \mid 0 \le t \le a; |u| \le \|g\| + M_0 M; |v| \le \|g\| + M_0 M\Big\}, \tag{4.2.10}$$

where $\|g\| = \max_{0 \le t \le a} |g(t)|$.
As usual, we denote by $B[0, M]$ the closed ball of the radius M centered at 0 in the space of continuous functions $C[0, a]$.

Theorem 4.2.2. Assume that:

(i) The function $\varphi(t)$ is a continuous map from $[0, a]$ to $[0, a]$.

(ii) The function $f(t, u, v)$ is continuous and bounded by M in the domain $\mathcal{D}_M$, i.e.,

$$|f(t, u, v)| \le M \quad \forall (t, u, v) \in \mathcal{D}_M. \tag{4.2.11}$$

(iii) The function $f(t, u, v)$ satisfies the Lipschitz conditions in the variables u, v with the coefficients $L_1, L_2 \ge 0$ in $\mathcal{D}_M$, i.e.,

$$\begin{aligned} |f(t, u_2, v_2) - f(t, u_1, v_1)| \le L_1|u_2 - u_1| + L_2|v_2 - v_1| \\ \forall (t, u_i, v_i) \in \mathcal{D}_M \ (i = 1, 2) \end{aligned} \tag{4.2.12}$$

(iv)

$$q := (L_1 + L_2)M_0 < 1. \tag{4.2.13}$$

The the problem (4.2.1)-(4.2.2) has a unique solution $u(t) \in C^3[0,a]$, satisfying the estimate

$$|u(t)| \leq \|g\| + M_0 M \quad \forall t \in [0,a]. \tag{4.2.14}$$

Proof. The proof of the theorem will be done in the following steps:
First we show that the operator A is a mapping $B[0,M] \to B[0,M]$. Indeed, for any $\psi \in B[0,M]$ we have $\|\psi\| \leq M$. Let $u(t)$ be the solution of the problem (4.2.5). From (4.2.7) it follows

$$|u(t)| \leq \|g\| + M_0 M \quad \forall t \in [0,a]. \tag{4.2.15}$$

Since $0 \leq \varphi(t) \leq a$ we also have

$$|u(\varphi(t)| \leq \|g\| + M_0 M \quad \forall t \in [0,a].$$

Therefore, if $t \in [0,a]$ then $(t, u(t), u(\varphi(t))) \in \mathcal{D}_M$. By the assumption (4.2.11) we have $|f(t, u(t), u(\varphi(t)))| \leq M \quad \forall t \in [0,a]$. In view of (4.2.4) we have $|(A\psi)(t)| \leq M \quad \forall t \in [0,a]$. It means $\|A\psi\| \leq M$ or $A\psi \in B[0,M]$.
Next, we prove that A is a contraction in $B[0,M]$. Let $\psi_1, \psi_2 \in B[0,M]$ and $u_1(t), u_2(t)$ be the solutions of the problem (4.2.5), respectively. Then from the assumption (4.2.12) we obtain

$$|A\psi_2 - A\psi_1| \leq L_1|u_2(t) - u_1(t)| + L_2|u_2(\varphi(t)) - u_1(\varphi(t))|. \tag{4.2.16}$$

From the representations

$$u_i(t) = g(t) + \int_0^a G(t,s)\psi_i(s)ds, \quad (i = 1,2)$$

and (4.2.9) it is easy to obtain

$$|u_2(t) - u_1(t)| \leq M_0\|\psi_2 - \psi_1\|,$$
$$|u_2(\varphi(t)) - u_1(\varphi(t))| \leq M_0\|\psi_2 - \psi_1\|$$

Combining the above estimates and (4.2.16), in view of the assumption (4.2.13) we obtain

$$\|A\psi_2 - A\psi_1\| \leq q\|\psi_2 - \psi_1\|, \quad q < 1.$$

Thus, A is a contraction mapping in $B[0,M]$.

Therefore, the operator equation (4.2.6) has a unique solution $\psi \in B[0,M]$. By Proposition 4.2.1 the solution of the problem (4.2.5) for this right-hand side $\psi(t)$ is the solution of the original problem (4.2.1)-(4.2.2). □

Remark 4.2.1. Theorem 4.2.2 remains valid if replace the third order equation (4.2.1) by the higher order equation (0.0.3). Besides, the conditions of the theorem are weaker than the conditions (i)-(iii) in [75, page 131] because here the Lipschitz conditions should be satisfied only in a bounded domain $\mathcal{D}_M$ instead of the unbounded domain $[a,b]\times\mathbb{R}\times\mathbb{R}$ as in [75] and there always is $(L_1+L_2)M_0 \leq (L_1 + L_2)(b-a)M_G$ since $M_0 \leq (b-a)M_G$.

4.2.3. Solution method and its convergence

Consider the following iterative method:

1. Given $\psi_0 \in B[0, M]$, for example,

$$\psi_0(t) = f(t, 0, 0). \tag{4.2.17}$$

2. Knowing $\psi_k(t)$ $(k = 0, 1, ...)$ compute

$$\begin{aligned} u_k(t) &= g(t) + \int_0^a G(t,s)\psi_k(s)ds, \\ v_k(t) &= g(\varphi(t)) + \int_0^a G(\varphi(t), s)\psi_k(s)ds. \end{aligned} \tag{4.2.18}$$

3. Update

$$\psi_{k+1}(t) = f(t, u_k(t), v_k(t)). \tag{4.2.19}$$

Set

$$p_k = \frac{q^k}{1-q},\ d = \|\psi_1 - \psi_0\|. \tag{4.2.20}$$

Theorem 4.2.3 (Convergence)**.** Under the assumptions of Theorem 4.2.2 the above iterative method converges and there holds the estimate

$$\|u_k - u\| \le M_0 p_k d,$$

where u is the exact solution of the problem (4.2.1)-(4.2.2) and M_0 is given by (4.2.9).

This theorem follows straightforward from the convergence of the successive approximation method for finding fixed point of the operator A, the representations (4.2.7) and the first equation in (4.2.18).

To numerically realize the above iterative method we construct the corresponding discrete iterative method. For this purpose, we cover the interval $[0, a]$ by the uniform grid $\bar{\omega}_h = \{t_i = ih,\ h = a/N, i = 0, 1, ..., N\}$ and denote by $\Phi_k(t), U_k(t), V_k(t)$ the grid functions, which are defined on the grid $\bar{\omega}_h$ and approximate the functions $\psi_k(t), u_k(t), v_k(t)$ on this grid, respectively.

Below we describe the discrete iterative method:

1. Given

$$\Psi_0(t_i) = f(t_i, 0, 0),\ i = 0, ..., N. \tag{4.2.21}$$

2. Knowing $\Psi_k(t_i),\ k = 0, 1, ...;\ i = 0, ..., N$, compute approximately the definite integrals (4.2.18) by the trapezoidal rule

$$\begin{aligned} U_k(t_i) &= g(t_i) + \sum_{j=0}^{N} h\rho_j G(t_i, t_j)\Psi_k(t_j), \\ V_k(t_i) &= g(\xi_i) + \sum_{j=0}^{N} h\rho_j G(\xi_i, t_j)\Psi_k(t_j), i = 0, ..., N, \end{aligned} \tag{4.2.22}$$

where ρ_j are the weights

$$\rho_j = \begin{cases} 1/2, \ j = 0, N \\ 1, \ j = 1, 2, ..., N-1 \end{cases}$$

and $\xi_i = \varphi(t_i)$.

3. Update

$$\Psi_{k+1}(t_i) = f(t_i, U_k(t_i), V_k(t_i)). \tag{4.2.23}$$

Now study the convergence of the above discrete iterative method. For this purpose we need some auxiliary results.

Proposition 4.2.4. If the function $f(t, u, v)$ has all partial derivatives continuous up to second order and the function $\varphi(t)$ also has continuous derivatives up to second order then the functions $\psi_k(t), u_k(t), v_k(t)$ constructed by the iterative method (4.2.17)-(4.2.19) also have continuous derivatives up to second order.

This proposition is obvious.

Proposition 4.2.5. For any function $\psi(t) \in C^2[0, a]$ there hold the estimates

$$\int_0^a G(t_i, s)\psi(s)ds = \sum_{j=0}^{N} h\rho_j G(t_i, s_j)\psi(s_j) + O(h^2), \tag{4.2.24}$$

$$\int_0^a G(\xi_i, s)\psi(s)ds = \sum_{j=0}^{N} h\rho_j G(\xi_i, s_j)\psi(s_j) + O(h^2), \tag{4.2.25}$$

where in order to avoid possible confusion we denote $s_j = t_j$.

Proof. The validity of (4.2.24) is guaranteed by [14, Proposition 3]. Here we notice that (4.2.24) is not automatically deduced from the estimate for the composite trapezoidal rule because the function $\frac{\partial^2 G(t_i,s)}{\partial s^2}$ has discontinuity at $s = t_i$.

Now we prove the estimate (4.2.25). Since $0 \le \xi_i = \varphi(t_i) \le a$, there are two cases.

Case 1: ξ_i coincides with one node s_j of the grid $\bar{\omega_h}$, i.e., there exists $s_j \in \bar{\omega_h}$ such that $\xi_i = s_j$. Because the Green function $G(t, s)$ as a function of s, it is continuous function at $s = \xi_i$ and is a polynomial of s in the intervals $[0, \xi_i]$ and $[\xi_i, a]$, we have

$$\begin{aligned}
&\int_0^a G(\xi_i, s)\psi(s)ds = \int_0^{\xi_i} G(\xi_i, s)\psi(s)ds + \int_{\xi_i}^a G(\xi_i, s)\psi(s)ds \\
&= h\Big(\frac{1}{2}G(\xi_i, s_0)\psi(s_0) + \sum_{m=1}^{j-1} G(\xi_i, s_m)\psi(s_m) + \frac{1}{2}G(\xi_i, s_j)\psi(s_j)\Big) + O(h^2) \\
&+ h\Big(\frac{1}{2}G(\xi_i, s_j)\psi(s_j) + \sum_{m=j+1}^{N-1} G(\xi_i, s_m)\psi(s_m) + \frac{1}{2}G(\xi_i, s_N)\psi(s_N)\Big) + O(h^2) \\
&= \sum_{j=0}^{N} h\rho_j G(t_i, s_j)\psi(s_j) + O(h^2).
\end{aligned}$$

Thus, (4.2.25) is proved for Case 1.
Case 2: ξ_i lies between s_l and s_{l+1}, i.e., $s_l < \xi_i < s_{l+1}$ for some $l = \overline{0, N-1}$. In this case, we represent

$$\int_0^a G(\xi_i, s)\psi(s)ds = \int_0^{s_l} F(s)ds + \int_{s_l}^{\xi_i} F(s)ds + \int_{\xi_i}^{s_{l+1}} F(s)ds + \int_{s_{l+1}}^a F(s)ds. \quad (4.2.26)$$

Here, for short we denote $F(s) = G(\xi_i, s)\psi(s)$. Note that $F(s) \in C^2$ in $[s_l, \xi_i]$ and $[\xi_i, s_{l+1}]$. Applying the composite trapezoidal rule to the first and the last integrals in the right-hand side of (4.2.26) we obtain

$$\begin{aligned} T_1 := &\int_0^{s_l} F(s)ds + \int_{s_{l+1}}^a F(s)ds \\ = &\sum_{j=0}^{l} \rho_j^{(l-)} F(s_j) + \sum_{j=l+1}^{N} \rho_j^{(l+)} F(s_j) + O(h^2), \end{aligned} \quad (4.2.27)$$

where

$$\rho_j^{(l-)} = \begin{cases} \frac{1}{2}, & j = 0, l \\ 1, & 1 < j < l \end{cases}, \quad \rho_j^{(l+)} = \begin{cases} \frac{1}{2}, & j = l+1, N \\ 1, & l+1 < j < N. \end{cases}$$

For calculating the second and the third integrals in the right-hand side of (4.2.26) we use the trapezoidal rule

$$\begin{aligned} T_2 := &\int_{s_l}^{\xi_i} F(s)ds + \int_{\xi_i}^{s_{l+1}} F(s)ds \\ = &\frac{1}{2}\big[(F(s_l) + F(\xi_i))(\xi_i - s_l) + (F(\xi_i) + F(s_{l+1}))(s_{l+1} - \xi_i)\big] + O(h^2). \end{aligned} \quad (4.2.28)$$

Using the points s_l and s_{l+1} for linearly interpolating $F(s)$ in the point ξ_i we have

$$F(\xi_i) = F(s_l)\frac{\xi_i - s_{l+1}}{s_l - s_{l+1}} + F(s_{l+1})\frac{\xi_i - s_l}{s_{l+1} - s_l} + O(h^2).$$

From here we obtain

$$F(\xi_i)(s_{l+1} - s_l) = F(s_l)(s_{l+1} - \xi_i) + F(s_{l+1})(\xi_i - s_l) + O(h^3). \quad (4.2.29)$$

Now, transforming T_2 we have

$$T_2 = \frac{1}{2}\big[F(s_l)(\xi_i - s_l) + F(s_{l+1})(s_{l+1} - \xi_i)\big] + F(\xi_i)(s_{l+1} - s_l) + O(h^2)$$

Further, in view of (4.2.29) it is easy to obtain

$$T_2 = \frac{1}{2}h(F(s_l) + F(s_{l+1})) + O(h^3).$$

Taking into account the above estimate, (4.2.27) and (4.2.26) we have

$$\int_0^a G(\xi_i, s)\psi(s)ds = \sum_{j=0}^{N} h\rho_j G(\xi_i, s_j)\psi(s_j) + O(h^2).$$

Thus, (4.2.25) is proved for Case 2 and the proof of Proposition 4.2.5 is complete. □

Remark 4.2.2. If in Proposition 4.2.5 replace $G(t_i, s)$ and $G(\xi_i, s)$ by $|G(t_i, s)|$ and $|G(\xi_i, s)|$, respectively then we obtain the analogous estimates

$$\int_0^a |G(t_i, s)|\psi(s)ds = \sum_{j=0}^{N} h\rho_j |G(t_i, s_j)|\psi(s_j) + O(h^2), \tag{4.2.30}$$

$$\int_0^a |G(\xi_i, s)|\psi(s)ds = \sum_{j=0}^{N} h\rho_j |G(\xi_i, s_j)|\psi(s_j) + O(h^2), \tag{4.2.31}$$

Proposition 4.2.6. Under the assumptions of Theorem 4.2.2 we have the estimates

$$\|\Psi_k - \psi_k\|_{\bar{\omega_h}} = O(h^2), \tag{4.2.32}$$

$$\|U_k - u_k\|_{\bar{\omega_h}} = O(h^2), \tag{4.2.33}$$

where $\|.\|_{\bar{\omega_h}}$ is the max-norm of grid function defined on the grid $\bar{\omega_h}$.

Proof. We prove the proposition by induction. For $k = 0$ we have at once $\|\Psi_0 - \psi_0\|_{\bar{\omega_h}}$ because $\Psi_0(t_i) = f(t_i, 0, 0)$ and $\psi_0(t_i) = f(t_i, 0, 0)$, $i = \overline{0, N}$, too. Next, by (4.2.18) and Proposition 4.2.5 we have

$$\begin{aligned} u_0(t_i) &= g(t_i) + \int_0^a G(t_i, s)\psi_0(s)ds \\ &= g(t_i) + \sum_{j=0}^{N} h\rho_j G(t_i, s_j)\psi_0(s_j) + O(h^2). \end{aligned}$$

On the other hand, by (4.2.22) we have

$$U_0(t_i) = g(t_i) + \sum_{j=0}^{N} h\rho_j G(t_i, s_j)\Psi_0(s_j).$$

Therefore,

$$|U_0(t_i) - u_0(t_i)| = O(h^2).$$

It implies $\|U_0 - u_0\|_{\bar{\omega_h}} = O(h^2)$. Thus, the estimates (4.2.32) and (4.2.33) are valid for $k = 0$.

Now, suppose that these estimates are valid for $k \geq 0$. We shall show that they are valid for $k + 1$. Indeed, from (4.2.19), (4.2.23) and the Lipschitz conditions for the function $f(t, u, v)$ we have

$$\begin{aligned} |\Psi_{k+1}(t_i) - \psi_{k+1}(t_i)| &= |f(t_i, U_k(t_i), V_k(t_i)) - f(t_i, u_k(t_i), v_k(t_i))| \\ &\leq L_1|U_k(t_i) - u_k(t_i)| + L_2|V_k(t_i) - v_k(t_i)|. \end{aligned} \tag{4.2.34}$$

Now estimate $|V_k(t_i) - v_k(t_i)|$. We have by Proposition 4.2.5

$$\begin{aligned} v_k(t_i) &= g(\varphi(t_i)) + \int_0^a G(\varphi(t_i), s)\psi_k(s)ds \\ &= g(\xi_i) + \sum_{j=0}^{N} h\rho_j G(\xi_i, s_j)\psi_k(s_j) + O(h^2). \end{aligned}$$

In view of (4.2.22) we have

$$\begin{aligned} |V_k(t_i) - v_k(t_i)| &= |\sum_{j=0}^{N} h\rho_j G(\xi_i, s_j)(\Psi_k(s_j) - \psi_k(s_j))| + O(h^2) \\ &\le \sum_{j=0}^{N} h\rho_j |G(\xi_i, s_j)| \|\Psi_k - \psi_k\|_{\bar{\omega_h}} + O(h^2). \end{aligned} \tag{4.2.35}$$

Notice that (4.2.31) for $\psi(s) = 1$ gives

$$\int_0^a |G(\xi_i, s)| ds = \sum_{j=0}^{N} h\rho_j |G(\xi_i, s_j)| + O(h^2).$$

From here it follows that

$$\begin{aligned} \sum_{j=0}^{N} h\rho_j |G(\xi_i, s_j)| &= \int_0^a |G(\xi_i, s)| ds + O(h^2) \\ &\le \max_{0 \le t \le a} \int_0^1 |G(t, s)| ds + O(h^2) = M_0 + O(h^2). \end{aligned}$$

Thanks to this estimate, from (4.2.35) we obtain

$$|V_k(t_i) - v_k(t_i)| \le \|\Psi_k - \psi_k\|_{\bar{\omega_h}} + O(h^2).$$

So, due to the induction hypothesis it implies

$$\|V_k - v_k\|_{\bar{\omega_h}} = O(h^2). \tag{4.2.36}$$

Combining the induction hypothesis $\|U_k - u_k\|_{\bar{\omega_h}} = O(h^2)$ and (4.2.36), from (4.2.34) we obtain

$$\|\Psi_{k+1} - \psi_{k+1}\|_{\bar{\omega_h}} = O(h^2). \tag{4.2.37}$$

In order to prove

$$\|U_{k+1} - u_{k+1}\|_{\bar{\omega_h}} = O(h^2). \tag{4.2.38}$$

we take into account that

$$|U_{k+1}(t_i) - u_{k+1}(t_i)| \le \sum_{j=0}^{N} h\rho_j |G(t_i, s_j)| |\Psi_{k+1}(s_j) - \psi_{k+1}(s_j)| + O(h^2).$$

Doing the similar argument as above and using the proved estimate (4.2.37) it is easy to obtain

$$|U_{k+1}(t_i) - u_{k+1}(t_i)| = O(h^2),$$

or (4.2.38).
Thus, Proposition 4.2.6 is proved. □

Now combining Proposition 4.2.6 with Theorem 4.2.3 we obtain the following result.

Theorem 4.2.7. Under the assumptions of Theorem 4.2.2 for the approximate solution of the problem (4.2.1)-(4.2.2) obtained by the discrete iterative method (4.2.21)-(4.2.23) we have the estimate

$$\|U_k - u\|_{\bar{\omega_h}} \le M_0 p_k d + O(h^2),$$

where p_k and d are defined by (4.2.20).

Remark 4.2.3. If to the third order problem apply the Bica's method which uses a cubic spline interpolation procedure at each iteration then $O(h^4)$ convergence cannot be ensured because Corollary 1 in [104, p. 50] is not applicable due to the properties of the Green function for the third order equation.

Remark 4.2.4. For the discrete iterative method (4.2.17) -(4.2.19) we obtained $O(h^2)$ convergence. It is natural to think about the use of Gauss quadrature formulas to the integrals in (4.2.18) but it is impossible because the nodes of Gauss quadrature formulas do not coincide with the grid nodes, where the solution of the problem is computed.

Remark 4.2.5. The results in Section 4.2.2 and 4.2.3 are obtained for the nonlinear third order FDE with nonlinear term $f = f(t, u(t), u(\varphi(t)))$. Analogously, it is possible to obtain similar results of existence and convergence of the iterative method at continuous level for the general case

$$f = f(t, u(t), u(\varphi(t)), u'(\varphi_1(t)), u''(\varphi_2(t))),$$

where $\varphi(t), \varphi_1(t), \varphi_2(t)$ are continuous functions from $[0, a]$ to $[0, a]$. But for numerical realization of the iterative method it is needed to take attention that the second derivative $\frac{\partial^2 G(t,s)}{\partial t^2}$ of the Green function has discontinuity at the line $s = t$. In this case for computing integrals containing $\frac{\partial G(t,s)}{\partial t}$ and $\frac{\partial^2 G(t,s)}{\partial t^2}$ it is needed to use the formulas constructed in our previous work [14].

Remark 4.2.6. The iterative method developed in this section for the third order nonlinear FDE can be applied to nonlinear FDE of any order.

4.2.4. Examples

In all numerical examples below we perform the iterative method (4.2.21)-(4.2.23) until $\|\Psi_k - \Psi_{k-1}\|_{\bar{\omega}_h} \le 10^{-10}$. In the tables of results for the convergence of the iterative method $Error = \|U_K - u\|_{\bar{\omega}_h}$, K is the number of iterations performed.

Example 4.2.1. Consider the following problem

$$\begin{aligned} u'''(t) &= e^t - \frac{1}{4}u(t) + \frac{1}{4}u^2(\frac{t}{2}), \quad 0 < t < 1, \\ u(0) &= 1, \ u'(0) = 1, \ u'(1) = e \end{aligned} \tag{4.2.39}$$

with the exact solution $u(t) = e^t$. The Green function for the above problem is

$$G(t,s) = \begin{cases} \frac{s}{2}(t^2 - 2t + s), & 0 \le s \le t \le 1, \\ \frac{t^2}{2}(s-1), & 0 \le t \le s \le 1. \end{cases}$$

So, we have

$$M_0 = \max_{0 \le t \le a} \int_0^1 |G(t,s)| ds = \frac{1}{2}.$$

The second degree polynomial satisfying the boundary conditions of the problem is

$$g(t) = 1 + t + \frac{e-1}{2} t^2.$$

Therefore, $\|g\| = 2 + \dfrac{e-1}{2} = 2.7183$. In this example $f(t,u,v) = e^t - \frac{1}{4}u + \frac{1}{4}v^2$. It is easy to verify that for $M = 6.5$ we have $|f(t,u,v)| \le M$ in the domain $\mathcal{D}_M$ defined by (4.2.10). Moreover, in this domain the function $f(t,u,v)$ satisfies the Lipschitz conditions in u and v with the coefficients $L_1 = \frac{1}{4}$ and $L_2 = 1.7004$. Therefore, $q := (L_1 + L_2)M_0 = 0.16$. Thus, all the assumptions of Theorem 4.2.2 are satisfied. By the theorem, the problem (4.2.39) has a unique solution. This is the above exact solution.

The results of convergence of the iterative method (4.2.21)-(4.2.23) are given in Table 4.3. From this table we see that the results of computation support the

Table 4.3: The convergence in Example 4.2.1.

N	h^2	K	$Error$
50	4.0000e-04	3	6.1899e-05
100	1.0000e-04	3	1.5475e-05
150	4.4444e-05	3	6.877 -06
200	2.5000e-05	3	3.8688e-06
300	1.1111e-05	3	1.7195e-06
400	6.2500e-06	3	9.6721e-07
500	4.0000e-06	3	6.1901e-07
800	1.5625e-06	3	6.1901e-07
1000	1.0000e-06	3	1.5475e-07

conclusion that the accuracy of the iterative method is $O(h^2)$.

Remark 4.2.7. Theorem 4.2.7 gives sufficient conditions for convergence of the iterative method (4.2.21)-(4.2.23). In the cases when these conditions are not satisfied the iterative also can converge to some solution. For example, for the case $f(t,u,v) = e^t + u^2 + v^2 + 1$ with the same boundary conditions as in (4.2.39) the iterative method converges after 15 iterations. And for the case $f(t,u,v) = e^{2t} - u^3 + v^2 + 5$ after 16 iterations the iterative process reaches the $TOL = 10^{-10}$. Notice that the number of iterations do not depend on the grid size as in Example 4.2.1.

Example 4.2.2. Consider the following problem

$$\begin{aligned} u'''(t) &= \sin(u^2(t)) + \cos(u^2(t^2)), \quad 0 < t < 1, \\ u(0) &= 0,\ u'(0) = \pi,\ u'(1) = -\pi. \end{aligned} \tag{4.2.40}$$

For this problem $f(t,u,v) = \sin(u^2) + \cos(v^2)$ and $\varphi(t) = t^2$. It is easy to verify that all the conditions of Theorem 4.2.3 are satisfied, therefore the problem has a unique solution. Also, by Theorem 4.2.7 the iterative method (4.2.21)-(4.2.23) converges. The results of computation show that the iterative method for any number of grid points stops after 8 iterations. The graph of the approximate solution is depicted in Figure 4.2.

Example 4.2.3 (Example 5 in [76])**.** Consider the problem

$$\begin{aligned} u'''(t) &= -1 + 2u^2(t/2), \quad 0 < t < \pi, \\ u(0) &= 0,\ u'(0) = 1,\ u(\pi) = 0. \end{aligned} \tag{4.2.41}$$

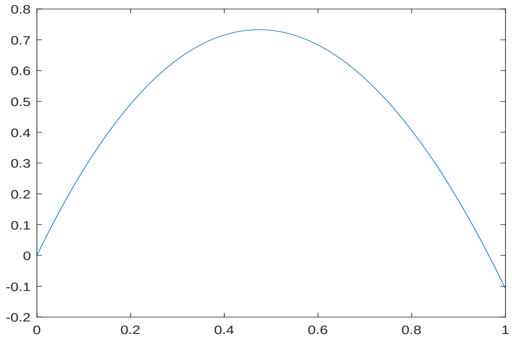

Figure 4.2: The graph of the approximate solution in Example 4.2.2.

which has exact solution $u(t) = \sin(t)$. For this problem the Green function has the form

$$G(t,s) = \begin{cases} -\dfrac{t^2(\pi-s)^2}{2\pi^2} + \dfrac{(t-s)^2}{2}, & 0 \le s \le t \le \pi, \\ -\dfrac{t^2(\pi-s)^2}{2\pi^2}, & 0 \le t \le s \le \pi \end{cases}$$

and the function $f(t,u,v) = -1 + 2v^2$.

The results of convergence of the iterative method (4.2.21)-(4.2.23) for this example are given in Table 4.4. From this table it is seen also that the numerical method has the accuracy $O(h^2)$.

Table 4.4: The convergence in Example 4.2.3.

N	h^2	K	$Error$
50	4.0000e-04	25	1.4455e-04
100	1.0000e-04	25	3.6142e-05
150	4.4444e-05	25	1.6063e-05
200	2.5000e-05	25	9.0345e-06
300	1.1111e-05	25	4.0155e-06
400	6.2500e-06	25	2.2587e-06
500	4.0000e-06	25	1.4456e-06
800	1.5625e-06	25	5.6467e-07
1000	1.0000e-06	25	3.6139e-07

4.2.5. Conclusion

In this section, we have proposed a unified approach to nonlinear functional differential equations via boundary value problems for nonlinear third order functional differential equations as a particular case. We have established the existence and uniqueness of solution and proved the convergence of order two of the discrete iterative method for finding the solution. Some examples demonstrate the validity of the theoretical results and the efficiency of the numerical method.

The proposed approach can be applied to boundary value problems for nonlinear functional differential equations of any order associated with general linear boundary conditions. It also can be applied to integro-differential equations.

General Conclusions

In this thesis, we have successfully studied the existence, uniqueness of solutions and the iterative methods for solving some nonlinear boundary value problems for some high order differential equations including integro-differential and functional differential equations. The main achievements of the thesis include:

1. The establishment of the existence, uniqueness of solutions and positive solutions for third order nonlinear BVPs and the construction of numerical methods for finding the solutions; The proposal of discrete iterative methods of second and third order accuracy for solving third order nonlinear differential equations.

2. The establishment of the existence, uniqueness of solutions and construction of iterative methods for finding the solutions for nonlinear third and fourth order differential equations with integral boundary conditions.

3. The establishment of the existence, uniqueness of solutions and construction of numerical methods for finding the solutions of nonlinear integro-differential and functional differential equations.

The validity and applicability of the theoretical results and the effectiveness of the constructed iterative methods have been confirmed by many experimental examples.

The future goals of the thesis are:

1. The further development of the above results for the case of singular right-hand sides and the case of unbounded domains.

2. The construction of iterative methods of higher order accuracy.

3. The study of the problems with nonlinear boundary conditions.

List of works of the author related to the thesis

[AL1] Q. A Dang, Q. L. Dang, A unified approach to fully third order nonlinear boundary value problems, J. Nonlinear Funct. Anal. 2020 (2020), Article ID 9 (Scopus, Q3).

[AL2] Q. A Dang, Q. L. Dang, Simple numerical methods of second- and third-order convergence for solving a fully third-order nonlinear boundary value problem, Numerical Algorithms, (2021) 87:1479-1499 (SCIE, Q1).

[AL3] Q. A Dang, Q. L. Dang, Existence results and iterative method for fully third order nonlinear integral boundary value problems, Applications of Mathematics, 66 (2021) 657-672 (SCIE, Q3).

[AL4] Q. A Dang, Q. L. Dang, A unified approach to study the existence and numerical solution of functional differential equation, Applied Numerical Mathematics 170 (2021) 208–218 (SCI, Q1).

[AL5] Q. A Dang, Q. L. Dang, Existence results and iterative method for a fully fourth-order nonlinear integral boundary value problem, Numerical Algorithms, 85 (2020) 887-907 (SCIE, Q1).

[AL6] Q. L. Dang, Q. A Dang, Existence results and numerical method for solving a fourth-order nonlinear integro-differential equation, Numerical Algorithms, 90, 563-576 (2022) (SCIE, Q1).

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