MINISTRY OF EDUCATION
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VIETNAM ACADEMY
OF SCIENCE AND TECHNOLOGY

GRADUATE UNIVERSITY OF SCIENCE AND TECHNOLOGY

Đinh Hồng Quang

ALTERNATING PROJECTION METHOD FOR FINDING A COMMON POINT OF CONVEX SETS AND APPLICATIONS

MASTER THESIS IN APPLIED MATHEMATICS

Hanoi, 2023

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Commitment

This thesis is done by my own study under the supervision of Dr. Lê Xuân Thanh and Assoc. Prof. Dr. Bùi Văn Định. It has not been defensed in any council and has not been published on any media. The results as well as the ideas of other authors are all specifically cited. I take full responsibility for my commitment.

Hanoi, September 2023

Đinh Hồng Quang

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Introduction

It is well known that Convex Optimization is a fundamental class in Global Optimization. A general form of convex optimization problems is the following

$$\begin{aligned} \min \quad & f(x), \\ \text{s.t.} \quad & x \in \cap_{i=1}^{m} C_i, \end{aligned}$$

in which $C_1, \ldots, C_m$ are convex subsets of an inner product space X and $f : X \to \mathbb{R}$ is a convex function. Given such a convex optimization problem, its feasibility is the first issue one may deal with. Stating equivalently, it is important to know whether the convex subsets $C_1, \ldots, C_m$ have a common point or not.

Alternating projection is a computational method to obtain the answer for the above question. The method is not only able to give a yes-no answer but also figures out a common point of the convex subsets in case of the yes answer. It dates back to 1950 for the first proposal of this method by von Neumann in [1], which based on his Princeton lectures on operator theory in 1935. In Chapter 13 of that book, von Neumann considers the setting in which X is a real Hilbert space, $m = 2$, and C_1, C_2 are two closed subspaces of X. The idea of von Neumann to find a point in $C_1 \cap C_2$ is described in Theorem 13.7 [1], which can be equivalently restated as follows.

Given a closed subspace C of X, and P_C is the orthogonal projection onto C, define two sequences $(x_n)_{n\geq 0}$, $(y_n)_{n\geq 0}$ of points in X by choosing an arbitrary point $x_0 \in X$ and letting

$$\begin{array}{llllll} & x_1 = P_{C_1}(x_0), & x_2 = P_{C_2}(x_1), & x_3 = P_{C1}(x_2), & x_4 = P_{C_2}(x_3), & \ldots \\ y_0 = x_0, & y_1 = P_{C_2}(y_0), & y_2 = P_{C_1}(y_1), & y_3 = P_{C_2}(y_2), & y_4 = P_{C_1}(y_3), & \ldots \end{array}$$

In words, the sequence $(x_n)_{n\geq 0}$ is obtained by starting at x_0 and alternately projecting onto C_1 and C_2, in which the first projection is onto C_1, while the sequence $(y_n)_{n\geq 0}$ is obtained by also starting at x_0 and alternately projecting onto C_2 and C_1, but the first projection is onto C_2. Then both sequences $(x_n)_{n\geq 0}$, $(y_n)_{n\geq 0}$ converge strongly to the same point $x^* = P_{C_1\cap C_2}(x_0)$, which is the projection of x_0 onto the intersection of the two subspaces.

A direction of extending von Neumann's projection method is to consider the similar setting where X is a real Hilbert space, but for $m \geq 2$ closed subspaces $C_1, \ldots, C_m$ of X. In this setting, the projection method is generalized as follows. Let $(i_n)_{n\geq 1}$ be a sequence whose elements take values in $\{1, \ldots, m\}$. Let x_0 be an arbitrary point in X, and $(x_n)_{n\geq 0}$ a sequence defined by

$$x_n = P_{C_{i_n}}(x_{n-1}) \quad \forall n \geq 1.$$

A survey on major results relating to the conditions for convergence of such sequence $(x_n)_{n\geq 0}$ is given in Section 1.2 [2]. Here is a brief summary of these results.

- Práger in 1960 proved that, when X is a finite dimensional space, the sequence (x_n) converges strongly.
- Halperin in 1962 proved the strong convergence of (x_n) when the sequence (i_n) is *periodic*.
- Amemiya and Ando in 1965 proved that we always have weak convergence of (x_n).
- Paszkiewicz in 2012 constructed an example with $m = 5$ closed subspaces in an infinite-dimensional Hilbert space, together with a starting point x_0 as well as a sequence (i_n), for which (x_n) does not converge strongly.
- Kopecká and Müller in 2014 improved the construction of Paszkiewicz in 2012 to $m = 3$.
- Kopecká and Paszkiewicz in 2017 made a further refinement by showing that for *any* infinite-dimensional Hilbert space X we can construct $m = 3$ closed subspaces such that for any starting point x_0 there is a sequence (i_n) for which (x_n) does not converge strongly.

Another direction of extending von Neumann's alternating projection method is to consider the similar setting in which X is a real Hilbert space, but $C_1, \ldots, C_m$ are closed convex subsets of X rather than closed subspaces. In this direction, Bregman in [3] proved that, if (i_n) is periodic, then the obtained sequence (x_n) converges weakly to a common point of $C_1, \ldots, C_m$ (provided that these convex subsets have non-empty intersection). In general, we do not have strong convergence of (x_n) in this setting due to a counter-example of Hundal in [4]. There, Hundal constructed $m = 2$ closed convex subsets C_1 and C_2 intersecting only at a single point, for which the sequence (x_n) does not converge strongly to that point.

In this thesis, however, we focus on the alternating projection method in the simple setting where X is a finite dimensional Euclidean space and $C_1, \ldots, C_m$ are closed convex subsets of X. In Chapter 1 we recall some preliminaries about convex sets, projection onto closed convex sets, projection onto subspaces, and projection onto intersection of hyperplanes. In Chapter 2, we first present a proof from [5] for the convergence of the method in the simplest case where $m = 2$. Then we present some variants of the alternating projection method, including periodic projection method, averaged projection method, and relaxed projection method. As methods for solving a fundamental problem in Convex Optimization, it is not surprising that the alternating projection methods are applied to solve many practical problems. Chapter 3 is devoted to discussing some interesting applications of the methods. Namely, we present the uses of the alternating projection methods in dividing a string into equal thirds and completing positive semi-definite matrices. We close the thesis by summary and remarks in conclusion part.

It is worth noting that the thesis does not contain any new results. Our main contributions in this thesis include the followings.

- We give the detail proofs for results stated in Section 1.3 and Section 1.4, which can be considered as exercises in Linear Algebra.
- We give the detail statements for the algorithms in Section 2.2 and detail proofs for their convergence, that are just mentioned briefly in the main reference [5].
- We give the detail analysis on the convergence of the algorithm in Section 3.1, which is mentioned shortly in the reference [11]. In addition, we provide a MATLAB script for experimenting the algorithm.
- We give the detail explanation and proofs for the results concerning the application in Section 3.2, that are briefly mentioned in the main reference [5]. Additionally, we provide a MATLAB script for performing the algorithm in the section.

Chapter 1

Preliminaries

In this chapter, we present some preliminaries that will be used in the sequel chapters. Namely, in Section 1.1 we recall some basic preliminaries on convex sets, in Section 1.2 we recall the definition and some useful properties of the projection onto closed convex sets, while Section 1.3 and Section 1.4 are respectively devoted to presenting some important preliminaries about the projection onto subspaces and projection onto intersection of finitely many hyperplanes. We emphasize that the results without citation in this chapter are not new, they are just simple exercises in Linear Algebra.

Throughout this chapter, X is a finite dimensional Euclidean space equipped with an inner product $\langle \cdot, \cdot \rangle$ and its induced norm $\| \cdot \|$. As well-known in linear algebra, we say that two vectors $x, y \in X$ are orthogonal (and denote $x \perp y$) if $\langle x, y \rangle = 0$. A vector $x \in X$ is called orthogonal to a subset $A \subset X$ (and denote $x \perp A$) if $x \perp y$ for any $y \in A$. The orthogonal complement of a subspace A of X is denoted by $A^{\perp}$.

1.1 Convex sets

As a well-known concept (see e.g. Definition 4.8 [6]), a subset $C \subset X$ is convex if for any $x, y \in C$ we have $\lambda x + (1 - \lambda) y \in C$ for all $\lambda \in [0, 1]$. The following proposition gives some trivial properties of convex sets, which will be used in sequel.

Proposition 1.1. *The convexity is preserved under finite intersection operator and finite Cartesian product.*

Proof. Firstly, we show that the intersection of a finite number of convex sets in X is also convex. Indeed, let $C_1, \ldots, C_m$ be convex sets in X, and $A = \cap_{i=1}^{m} C_i$.

Take $x, y \in A$ and $\lambda \in [0,1]$. Then for each $i = 1, \dots, m$ we have $x, y \in C_i$, and since C_i is convex, we have $\lambda x + (1-\lambda)y \in C_i$. So $\lambda x + (1-\lambda)y \in \cap_{i=1}^m C_i = A$, i.e. A is convex.

Secondly, we show that the Cartesian product of a finite number of convex sets in X is also convex. Indeed, let $C_1, \dots, C_m$ be convex sets in X, and $B = C_1 \times \dots \times C_m$. Let $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_m)$ be two points in B, and $\lambda \in [0,1]$. For each $i = 1, \dots, m$, since $x_i, y_i \in C_i$ and by convexity of C_i, we have $\lambda x_i + (1-\lambda)y_i \in C_i$. Thus

$$\lambda x + (1-\lambda)y = (\lambda x_1 + (1-\lambda)y_1, \dots, \lambda x_m + (1-\lambda)y_m) \in C_1 \times \dots \times C_m = B,$$

which proves the convexity of B. $\square$

1.2 Projection onto closed convex sets

To define the projection of a point onto a closed convex set, we have the following characterization.

Theorem 1.2. (see Theorem 6.1 [6]) *Let $C \subset X$ be a nonempty closed convex set and $x \in X$. A point x^* minimizes $\|x - \cdot\|$ over C if and only if $x^* \in C$ and*

$$\langle x - x^*, y - x^* \rangle \leq 0 \quad \forall y \in C. \tag{1.1}$$

Furthermore, such x^* exists uniquely.

Proof. *Sufficiency.* Assume that x^* is a minimizer of $\|x - \cdot\|$ over C. Since (1.1) obviously holds with $y = x^*$, we consider an arbitrary $y \in C \backslash \{x^*\}$. Let $\alpha \in (0,1)$. Since C is convex and $x^*, y \in C$, we have

$$x^* + \alpha(y - x^*) = \alpha y + (1-\alpha)x^* \in C.$$

By the optimality of x^*, we must have

$$\begin{aligned}\|x - x^*\|^2 &\leq \|x - (x^* + \alpha(y - x^*))\|^2 \\ &\leq \|x - x^*\|^2 + \alpha^2\|y - x^*\|^2 - 2\alpha\langle x - x^*, y - x^*\rangle,\end{aligned}$$

which implies

$$\langle x - x^*, y - x^* \rangle \leq \frac{\alpha}{2}\|x - x^*\|^2.$$

Since this inequality holds for arbitrary $\alpha \in (0,1)$, by letting $\alpha \to 0^+$ we obtain

$$\langle x - x^*, y - x^* \rangle \leq 0.$$

Hence we have (1.1).

Necessity. Let $x^* \in C$ satisfying (1.1). For any $z \in C$ such that $z \neq x^*$, we have $\|x^* - z\| > 0$ and $\langle x - x^*, z - x^* \rangle \leq 0$, therefore

$$\begin{aligned} \|x - z\|^2 - \|x - x^*\|^2 &= \|(x - x^*) + (x^* - z)\|^2 - \|x - x^*\|^2 \\ &= \|x^* - z\|^2 + 2\langle x - x^*, x^* - z \rangle \\ &> 0. \end{aligned}$$

So $\|x - z\| > \|x - x^*\|$, proving that x^* is a minimizer of $\|x - \cdot\|$ over C.

Existence. Firstly, we prove that the function $f(y) = \|x - y\|$ is continuous with respect to $y \in \mathbb{R}^n$. Indeed, let y_0 is an arbitrary point in $\mathbb{R}^n$, and $\{y^n\}_{n\in\mathbb{N}}$ a sequence of points in $\mathbb{R}^n$ converging to y^0, i.e., $\|y^n - y^0\| \to 0$ as $n \to \infty$. Since for all $n \in \mathbb{N}$ we have

$$\begin{aligned} \|y^n - y^0\| &= \|(x - y^0) - (x - y^n)\| \\ &\geq |\|x - y^0\| - \|x - y^n\|| \\ &= |f(y^0) - f(y^n)| \geq 0, \end{aligned}$$

it follows that $f(y^n) \to f(y^0)$ as $n \to \infty$. This means that f is continuous at y^0, and since y^0 is arbitrarily chosen in $\mathbb{R}^n$, it follows that f is continuous on $\mathbb{R}^n$. Let y^* be an arbitrary point in C, and define

$$C^* = \{y \in C \mid \|x - y\| \leq \|x - y^*\|\}.$$

Obviously C^* is closed and bounded subset of C, therefore C^* is compact. Furthermore, for any $y \notin C^*$ we have $\|x - y\| > \|x - y^*\|$, therefore

$$\min_{y\in C} \|x - y\| = \min_{y\in C^*} \|x - y\| = \min_{y\in C^*} f(y).$$

Since f is continuous and C^* is compact, by Bolzano-Weierstrass theorem in analysis, f reaches its minimum on C^* at some $x^* \in C^*$. It means that

$$\|x - x^*\| = \min_{y\in C^*} \|x - y\| = \min_{y\in C} \|x - y\|,$$

so the existence of x^* has been proved.

Uniqueness. Assume that y^* and z^* are minimizers of $\|x - \cdot\|$ over C. By the optimality of y^* and choosing $y = z^*$ in (1.1), we have

$$\langle x - y^*, z^* - y^* \rangle \leq 0.$$

Similarly, by the optimality of z^* and choosing $y = y^*$ in (1.1), we have

$$\langle x - z^*, y^* - z^* \rangle = \langle z^* - x, z^* - y^* \rangle \leq 0.$$

Adding side by side the two above inequalities, we obtain

$$0 \geq \langle x - y^*, z^* - y^* \rangle + \langle z^* - x, z^* - y^* \rangle = \|z^* - y^*\|^2 \geq 0.$$

Thus, we have $y^* = z^*$. $\square$

Thanks to the uniqueness of x^* in Theorem 1.2, we can define the projection of a point $x \in X$ onto a nonempty closed convex set $C \subset X$ to be the point $\operatorname{argmin}_{y \in C} \|x - y\|$, denoted by $P_C(x)$. The condition (1.1) gives us a variational characterization of the projection:

$$\langle x - P_C(x), y - P_C(x) \rangle \leq 0 \quad \forall y \in C. \tag{1.2}$$

Figure 1.1 illustrates the characterization in $\mathbb{R}^2$ with the usual inner product. It convinces us that for any $y \in C$, the angle θ between the vectors $x - P_C(x)$ and $y - P_C(x)$ must be obtuse. Since $\cos\theta \leq 0$, we have

$$\langle x - P_C(x), y - P_C(x) \rangle = \|x - P_C(x)\| \|y - P_C(x)\| \cos\theta \leq 0.$$

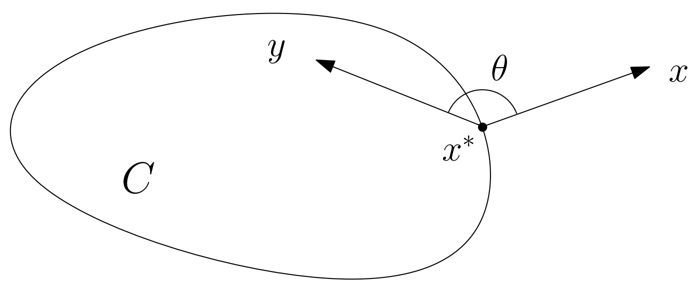

Figure 1.1: Projecting a point onto a closed convex set in $\mathbb{R}^2$.

Thanks to the optimality of x^* in Theorem 1.2, we can define the distance from a point $x \in X$ to a closed convex set $C \subset X$ by

$$\operatorname{dist}(x, C) := \|x - P_C(x)\|.$$

Concerning projection onto closed convex sets, the following Pythagoras type assertion is useful when proving the convergence of alternating projection algorithms. Figure 1.2 helps us to have an intuition on this result.

Lemma 1.3. (see Theorem V.1.1 [7]) *Let C be a closed convex set in X, $\overline{x} \in C$, and $x \in X$. Then we have*

$$\|x - \overline{x}\|^2 \geq \|x - P_C(x)\|^2 + \|P_C(x) - \overline{x}\|^2.$$

Proof. By Theorem 1.2 we have

$$\langle x - P_C(x), \overline{x} - P_C(x)\rangle \leq 0.$$

Therefore, we obtain

$$\begin{aligned}\|x - \overline{x}\|^2 &= \|x - P_C(x) + P_C(x) - \overline{x}\|^2\\ &= \|x - P_C(x)\|^2 + \|P_C(x) - \overline{x}\|^2 + 2\langle x - P_C(x), P_C(x) - \overline{x}\rangle\\ &\geq \|x - P_C(x)\|^2 + \|P_C(x) - \overline{x}\|^2.\end{aligned}$$

as desired. □

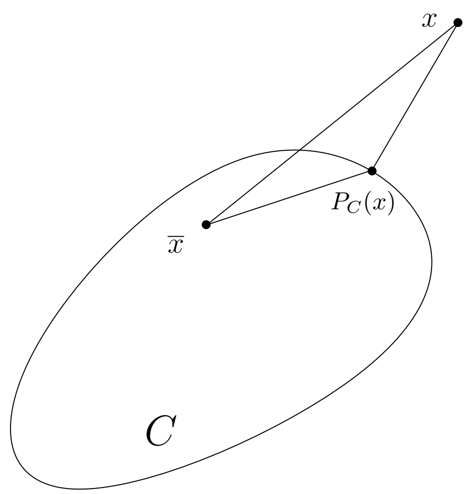

Figure 1.2: Illustration of Pythagoras type assertion in Lemma 1.3.

1.3 Projection onto subspaces

This section aims to characterize the projection onto subspaces of X. In this section, $\overline{x}$ is an arbitrary point in X and C is a subspace of X. Since C is a subspace of the finite dimensional Euclidean space X, it is closed and convex, and therefore, thanks to Theorem 1.2, $P_C(\overline{x})$ exists uniquely. Similar to the variational characterization (1.2), the following lemma gives us a characterization of the projection point on a subspace.

Lemma 1.4. *For any $c \in C$ we have*

$$\langle \overline{x} - P_C(\overline{x}), c\rangle = 0.$$

Proof. By Theorem 1.2, for any $c \in C$ we have

$$\langle \overline{x} - P_C(\overline{x}), c - P_C(\overline{x})\rangle \leq 0,$$

or equivalently

$$\langle \overline{x} - P_C(\overline{x}), c\rangle \leq \langle \overline{x} - P_C(\overline{x}), P_C(\overline{x})\rangle =: \alpha. \tag{1.3}$$

Since $\overline{x}$ and C are specified, α is a constant. Since C is a subspace, we have $\lambda c \in C$ for any $\lambda \in \mathbb{R}$. Since (1.3) holds for any $c \in C$, it also holds for λc, i.e., for any $\lambda \in \mathbb{R}$ we have

$$\lambda\langle \overline{x} - P_C(\overline{x}), c\rangle = \langle \overline{x} - P_C(\overline{x}), \lambda c\rangle \leq \alpha.$$

Since the above inequality holds for all $\lambda \in \mathbb{R}$, we must have $\alpha = 0$ and $\langle \overline{x} - P_C(\overline{x}), c\rangle = 0$. In particular, we obtain the claim of the lemma. $\square$

The following proposition gives us a characterization of the projection mapping onto a subspace.

Proposition 1.5. *Provided that C is a subspace of X, the projection mapping P_C is a linear transformation which is idempotent and self-adjoint.*

Proof. The statement of the proposition means that we have to establish the following claims.

Claim 1: P_C is a linear transformation.
Claim 2: P_C is idempotent, i.e., $P_C^2 = P_C$.
Claim 3: P_C is self-adjoint, i.e., $\langle x, P_C(y)\rangle = \langle P_C(x), y\rangle$ for any $x, y \in X$.

To prove Claim 1, we need to show that

$$P_C(x + y) = P_C(x) + P_C(y) \qquad \forall x, y \in X, \tag{1.4}$$

$$P_C(\lambda x) = \lambda P_C(x) \qquad \forall \lambda \in \mathbb{R}, x \in X. \tag{1.5}$$

Indeed, for any $x, y \in X$ we have

$$x = x - P_C(x) + P_C(x), \quad y = y - P_C(y) + P_C(y).$$

By definition, $P_C(x) \in C$ and $P_C(y) \in C$. Thanks to Lemma 1.4 we have $x - P_C(x) \in C^\perp$ and $y - P_C(y) \in C^\perp$. Since C and $C^\perp$ are subspaces of X, we have

$$x - P_C(x) + y - P_C(y) \in C^\perp, \quad P_C(x) + P_C(y) \in C.$$

These facts, together with $x + y = (x - P_C(x) + y - P_C(y)) + (P_C(x) + P_C(y))$, lead to (1.4). In addition, since $x - P_C(x) \in C^\perp$ and $C^\perp$ is a subspace of X, for any $\lambda \in \mathbb{R}$ we have

$$\lambda\,(x - P_C(x)) \in C^\perp. \tag{1.6}$$

Similarly, since $P_C(x) \in C$ and C is a subspace of X, for any $\lambda \in \mathbb{R}$ we have

$$\lambda P_C(x) \in C. \tag{1.7}$$

From (1.6), (1.7), and the fact that $\lambda x = \lambda\,(x - P_C(x)) + \lambda P_C(x)$, we have $P_C(\lambda x) = \lambda P_C(x)$, which proves (1.5).

Claim 2 is obvious. Indeed, since $P_C(x) \in C$ for any $x \in X$, we have $P_C^2(x) = P_C(P_C(x)) = P_C(x)$, which means that $P_C^2 = P_C$.

To prove Claim 3, we note that for any $x, y \in C$ we have $P_C(x) \in C$ and $P_C(y) \in C$. By Lemma 1.4 we have

$$\langle x - P_C(x), P_C(y)\rangle = 0 \quad \text{and} \quad \langle y - P_C(y), P_C(x)\rangle = 0.$$

Consequently, we obtain

$$\langle x, P_C(y)\rangle = \langle P_C(x), y\rangle = \langle P_C(x), P_C(y)\rangle,$$

which is the proof for Claim 3. □

It is important to know that the converse of the above proposition is also true in the following sense.

Proposition 1.6. *If $f : X \to X$ is a linear transformation that is idempotent and self-adjoint, then it coincides the projection mapping onto*

$$\mathrm{range}(f) := \{y \in X \mid y = f(x) \textit{ for some } x \in X\}.$$

Proof. Let $U := \mathrm{range}(f)$ and $W = \ker(f) := \{x \in X \mid f(x) = \mathbf{0}\}$. We first show that $X = U \oplus W$, i.e., $U \cap W = \{\mathbf{0}\}$ and any $x \in X$ can be represented as the sum of a vector in U with a vector in W. Indeed, let $z \in U \cap W$. Since $z \in W$, we have

$$f(z) = \mathbf{0}. \tag{1.8}$$

Since $z \in U$, we have $z = f(y)$ for some $y \in X$. Therefore we obtain

$$f(z) = f(f(y)) = f^2(y) = f(y) = z. \tag{1.9}$$

The third equality above is due to the idempotent property of f. By (1.8) and (1.9), we obtain $z = \mathbf{0}$. Hence $U \cap W = \{\mathbf{0}\}$. Furthermore, any $x \in X$ admits the following representation:

$$x = f(x) + (x - f(x)) \in U + W,$$

in which $f(x) \in U$ and $x - f(x) \in W$. The former inclusion is due to the definition of the set U, while the latter inclusion follows from

$$f(x - f(x)) = f(x) - f^2(x) = f(x) - f(x) = \mathbf{0}.$$

Here, the first equality is due to the linearity of f, while the second equality is due to the idempotent property of f. Thus, we have completed the proof for $X = U \oplus W$.

We will show furthermore that the above direct sum is indeed an orthogonal one, i.e., $X = U \oplus^{\perp} W$. Indeed, take arbitrary vectors $x \in U$ and $y \in W$. Since $x \in U$, there exists $z \in U$ such that $x = f(z)$. Since $y \in W$, we have $f(y) = \mathbf{0}$. Therefore, using self-adjoint property of f, we have

$$\langle x, y \rangle = \langle f(z), y \rangle = \langle z, f(y) \rangle = \langle z, \mathbf{0} \rangle = 0.$$

Since x and y are chosen arbitrarily in U and W respectively, it follows that $U \perp W$. Thus $X = U \oplus^{\perp} W$.

The orthogonal direct sum $X = U \oplus^{\perp} W$ means that f is the orthogonal projection on the subspace U, as claimed in the proposition. □

1.4 Projection onto intersection of hyperplanes

This section aims to give an explicit formula for the projection of a point onto the intersection of a finite number of hyperplanes. Throughout this section, let $\overline{x}$ be a given point in the Euclidean space X and let $C \subset X$ defined by

$$C := \{x \in X \mid \langle a_i, x \rangle = \alpha_i, i = 1, \dots, m\}, \tag{1.10}$$

where $a_1, \dots, a_m \in X$ and $\alpha_1, \dots, \alpha_m \in \mathbb{R}$ are given. If for each $i = 1, \dots, m$ we define

$$H_i = \{x \in X \mid \langle a_i, x \rangle = \alpha_i\},$$

then $H_i (i = 1, \dots, m)$ are hyperplanes in X, and therefore

$$C = \cap_{i=1}^{m} H_i$$

is in fact the intersection of these hyperplanes. The following proposition shows that it makes sense to consider the projection $P_C(\overline{x})$.

Proposition 1.7. *The set C defined in* (1.10) *is closed and convex.*

Proof. *Convexity.* We first show that the set $H_1 = \{x \in X \mid \langle a_1, x\rangle = \alpha_1\}$ is convex. Indeed, let $x_1, x_2 \in H_1$, $\lambda \in [0,1]$, and let $y := \lambda x_1 + (1-\lambda)x_2$. Since $x_1 \in H_1$, we have $\langle a_1, x\rangle = \alpha_1$. Similarly, since $x_2 \in H_1$, we have $\langle a_1, x_2\rangle = \alpha_1$. Then we have

$$\begin{aligned}
\langle a_1, y\rangle &= \langle a_1, \lambda x_1 + (1-\lambda)x_2\rangle \\
&= \lambda\langle a_1, x_1\rangle + (1-\lambda)\langle a_1, x_2\rangle \\
&= \lambda\alpha_1 + (1-\lambda)\alpha_1 \\
&= \alpha_1.
\end{aligned}$$

So y is also in H_1 by definition, which means that H_1 is convex. By similar arguments, we also have H_i is convex for each $i = 2, \dots, m$. Therefore, thanks to Proposition 1.1, $C = \cap_{i=1}^m H_i$ is convex.

Closedness. Let us first revise the set H_1 defined above. Consider the following mapping:

$$\begin{aligned}
g_1 : X &\to \mathbb{R} \\
x &\mapsto \langle a_1, x\rangle.
\end{aligned}$$

Since the inner product is continuous with respect to each of its components, g_1 is continuous. Therefore, the set

$$H_1 = \{x \in X \mid \langle a_1, x\rangle = \alpha_1\} = g_1^{-1}(\{\alpha_1\})$$

is closed, since it is the preimage of the closed set $\{\alpha_1\}$ via the continuous mapping g_1. Similarly, the set $H_i (i = 2, \dots, m)$ are also closed. Thus we have

$$C = \cap_{i=1}^m H_i$$

is closed, since it is intersection of closed sets. □

For the computation of $P_C(\overline{x})$, we introduce the following two sets.

$$\begin{aligned}
C^0 &= \{x \in X \mid \langle a_i, x\rangle = 0, i = 1, \dots, m\}, \\
C^\dagger &= \operatorname{span}(a_1, \dots, a_m).
\end{aligned}$$

In words, $C^\dagger$ is the subspace of X spanned by $a_1, \dots, a_m$, while C^0 is the set of vectors $x \in X$ that are orthogonal to each of $a_1, \dots, a_m$, i.e., $x \perp a_i$ for all $i = 1, \dots, m$. The following properties of these sets will be useful in computing the projection $P_C(\overline{x})$.

Proposition 1.8. *(i) The set C defined in* (1.10) *is affine.*

(ii) The set C^0 is the subspace of X which is parallel to the set C in the sense that $C^0 = C - \{\overline{y}\}$ for any fixed $\overline{y} \in C$.

(iii) $(C^0)^\perp = C^\dagger$.

Proof. (i) Let $x_1, x_2 \in C$, $\lambda \in \mathbb{R}$, and $y = \lambda x_1 + (1-\lambda)x_2$. Since $x_1 \in C$, we have

$$\langle a_i, x_1 \rangle = \alpha_i \qquad \forall i = 1, \dots, m.$$

Similarly, since $x_2 \in C$, we have

$$\langle a_i, x_2 \rangle = \alpha_i \qquad \forall i = 1, \dots, m.$$

Therefore, for each $i = 1, \dots, m$ we obtain

$$\begin{aligned} \langle a_i, y \rangle &= \langle a_i, \lambda x_1 + (1-\lambda)x_2 \rangle \\ &= \lambda \langle a_i, x_1 \rangle + (1-\lambda)\langle a_i, x_2 \rangle \\ &= \lambda \alpha_i + (1-\lambda)\alpha_i \\ &= \alpha_i. \end{aligned}$$

Hence $y \in C$ by definition, which means that C is affine.

(ii) Firstly, we show that C^0 is a subspace of X. Indeed, let $x_1, x_2 \in C^0$, $\lambda_1, \lambda_2 \in \mathbb{R}$, and let $y = \lambda_1 x_1 + \lambda_2 x_2$. Since $x_1, x_2 \in C^0$, by definition we have

$$\begin{aligned} \langle a_i, x_1 \rangle &= 0, \qquad & \forall i = 1, \dots, m, \\ \langle a_i, x_2 \rangle &= 0, \qquad & \forall i = 1, \dots, m. \end{aligned}$$

Therefore, for each $i = 1, \dots, m$ we obtain

$$\langle a_i, y \rangle = \langle a_i, \lambda_1 x_1 + \lambda_2 x_2 \rangle = \lambda_1 \langle a_i, x_1 \rangle + \lambda_2 \langle a_i, x_2 \rangle = \lambda_1 \cdot 0 + \lambda_2 \cdot 0 = 0,$$

which means that $y \in C^0$. This proves that C^0 is a subspace of X.

Secondly, we prove $C^0 = C - \{\overline{y}\}$ by showing $C - \{\overline{y}\} \subseteq C^0$ and $C^0 \subseteq C - \{\overline{y}\}$. To show the former inclusion, apart from the fixed $\overline{y} \in C$, let x be an arbitrary vector in C. Since both $x, \overline{y}$ are in C, by definition we have

$$\begin{aligned} \langle a_i, x \rangle &= \alpha_i, \qquad & \forall i = 1, \dots, m, \\ \langle a_i, \overline{y} \rangle &= \alpha_i, \qquad & \forall i = 1, \dots, m. \end{aligned}$$

So for each $i = 1, \dots, m$ we obtain

$$\langle a_i, x - \overline{y} \rangle = \langle a_i, x \rangle - \langle a_i, \overline{y} \rangle = \alpha_i - \alpha_i = 0.$$

Hence $x - \overline{y} \in C^0$. Since x is chosen arbitrarily in C, we have $C - \{\overline{y}\} \subseteq C^0$.

To show the inverse inclusion $C^0 \subseteq C - \{\overline{y}\}$, for any $z \in C^0$ let $x = \overline{y} + z$. Then $z = x - \overline{y}$. Furthermore, since $\overline{y} \in C$ and $z \in C^0$, by definition we have

$$\begin{aligned} \langle a_i, \overline{y} \rangle &= \alpha_i, & \forall i &= 1, \dots, m, \\ \langle a_i, z \rangle &= 0, & \forall i &= 1, \dots, m. \end{aligned}$$

So for each $i = 1, \dots, m$ we obtain

$$\langle a_i, x \rangle = \langle a_i, \overline{y} + z \rangle = \langle a_i, \overline{y} \rangle + \langle a_i, z \rangle = \alpha_i + 0 = \alpha_i.$$

Hence $x \in C$, and consequently, $z = x - \overline{y} \in C - \{\overline{y}\}$. Since z is chosen arbitrarily in C^0, we obtain $C^0 \subseteq C - \{\overline{y}\}$ as desired.

(iii) For each $i = 1, \dots, m$, let

$$H_i^0 = \{x \in X \mid \langle a_i, x \rangle = 0\}.$$

Then, on one hand we have $(H_i^0)^\perp = \operatorname{span}(a_i)$ for each $i = 1, \dots, m$, and on the other hand we have $C^0 = \cap_{i=1,\dots,m} H_i^0$. Therefore, we obtain

$$\begin{aligned} (C^0)^\perp &= (\cap_{i=1,\dots,m} H_i^0)^\perp \\ &= (H_1^0)^\perp + \ldots + (H_m^0)^\perp \\ &= \operatorname{span}(a_1) + \ldots + \operatorname{span}(a_m) \\ &= \operatorname{span}(a_1, \dots, a_m) \\ &= C^\dagger. \end{aligned}$$

□

We are now ready to state the main result in this subsection.

Theorem 1.9. *The projection $P_C(\overline{x})$ of a given point $\overline{x} \in X$ onto the set C defined by* (1.10) *is given by*

$$P_C(\overline{x}) = \overline{x} - \sum_{i=1}^{m} \beta_i a_i,$$

in which the coefficients $\beta_1, \dots, \beta_m$ are found from

$$\begin{bmatrix} \langle a_1, a_1 \rangle & \langle a_1, a_2 \rangle & \dots & \langle a_1, a_m \rangle \\ \langle a_2, a_1 \rangle & \langle a_2, a_2 \rangle & \dots & \langle a_2, a_m \rangle \\ \vdots & \vdots & \dots & \vdots \\ \langle a_m, a_1 \rangle & \langle a_m, a_2 \rangle & \dots & \langle a_m, a_m \rangle \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{bmatrix} = \begin{bmatrix} \langle a_1, \overline{x} \rangle - \alpha_1 \\ \langle a_2, \overline{x} \rangle - \alpha_2 \\ \vdots \\ \langle a_m, \overline{x} \rangle - \alpha_m \end{bmatrix}. \tag{1.11}$$

Proof. We first prove that the vector $\overline{x} - P_C(\overline{x})$ is orthogonal to C^0. Indeed, let y be any vector in C. By Theorem 1.2, we have

$$\langle \overline{x} - P_C(\overline{x}), y - P_C(\overline{x}) \rangle \leq 0. \tag{1.12}$$

Recall from Proposition 1.8 (i) that C is an affine set. Since y and $P_C(\overline{x})$ are both in the affine set C, their affine combinations

$$\{\lambda P_C(\overline{x}) + (1-\lambda)y \mid \lambda \in \mathbb{R}\}$$

are also in C. By choosing $\lambda = 2$, we obtain

$$z := 2P_C(\overline{x}) - y \in C.$$

Again, by Theorem 1.2 we have

$$0 \geq \langle \overline{x} - P_C(\overline{x}), z - P_C(\overline{x}) \rangle = \langle \overline{x} - P_C(\overline{x}), 2P_C(\overline{x}) - y - P_C(\overline{x}) \rangle = \langle \overline{x} - P_C(\overline{x}), P_C(\overline{x}) - y \rangle.$$

This inequality, together with (1.12), leads to

$$\langle \overline{x} - P_C(\overline{x}), y - P_C(\overline{x}) \rangle = 0.$$

Since y and $P_C(\overline{x})$ are both in C, it follows from Proposition 1.8 (ii) that the vector $y - P_C(\overline{x})$ is in C^0. Since y is chosen arbitrarily in C, it follows from the above equality that $\overline{x} - P_C(\overline{x})$ is orthogonal to C^0.

Since $(\overline{x} - P_C(\overline{x})) \perp C^0$, by Proposition 1.8 (iii) we have

$$\overline{x} - P_C(\overline{x}) \in (C^0)^\perp = C^\dagger = \operatorname{span}(a_1, \ldots, a_m).$$

Therefore, there exists $\beta_1, \ldots, \beta_m \in \mathbb{R}$ such that

$$\overline{x} - P_C(\overline{x}) = \beta_1 a_1 + \ldots + \beta_m a_m. \tag{1.13}$$

For each $i = 1, \ldots, m$, by taking the inner product of both sides of (1.13) with a_i we have

$$\langle a_i, \overline{x} - P_C(\overline{x}) \rangle = \langle a_i, \beta_1 a_1 + \ldots + \beta_m a_m \rangle,$$

or equivalently

$$\langle a_i, a_1 \rangle \beta_1 + \ldots + \langle a_i, a_m \rangle \beta_m = \langle a_i, \overline{x} \rangle - \langle a_i, P_C(\overline{x}) \rangle.$$

Note that $\langle a_i, P_C(\overline{x}) \rangle = \alpha_i$ due to the fact that $P_C(\overline{x}) \in C$ and due to the definition of C. So we have

$$\langle a_i, a_1 \rangle \beta_1 + \ldots + \langle a_i, a_m \rangle \beta_m = \langle a_i, \overline{x} \rangle - \alpha_i, \qquad \forall i = 1, \ldots, m,$$

that are written in matrix form exactly as (1.11). This fact, together with (1.13), verifies the statement of this theorem. $\square$

Chapter 2

Alternating projection methods

Throughout this chapter, X is a finite dimensional Euclidean space equipped with an inner product $\langle \cdot, \cdot \rangle$ and its induced norm $\| \cdot \|$. In Section 2.1, we describe the basic version of the alternating projection algorithm for finding a common point of two closed convex sets in X and show a proof for the convergence of this algorithm. In Section 2.2 we present some simple variants of this algorithm, including periodic projection algorithm (Section 2.2.1), averaged projection algorithm (Section 2.2.2), and relaxed projection algorithm (Section 2.2.3).

2.1 Basic version

In this section, we study the simplest version of alternating projection method which is applied to find a common point of two closed convex subsets of the universal space X. The contents of this section are based on Section 1 and Section 2 of [5]. The method is described as follows.

Algorithm 1 Basic alternating projection algorithm

Input: Two closed convex subsets C_1 and C_2 of X such that $C_1 \cap C_2 \neq \emptyset$.
Output: A common point of C_1 and C_2.

1: Take an arbitrary point $a \in X$.
2: $x_0 = P_{C_1}(a)$.
3: **for** $k = 0, 1, 2, \ldots$ **do**
4: $\quad y_k = P_{C_2}(x_k)$
5: $\quad x_{k+1} = P_{C_1}(y_k)$
6: **end for**

The following theorem proves the convergence of Algorithm 1. Figure 2.1 illustrates an example of Algorithm 1 for two closed convex sets in $\mathbb{R}^2$.

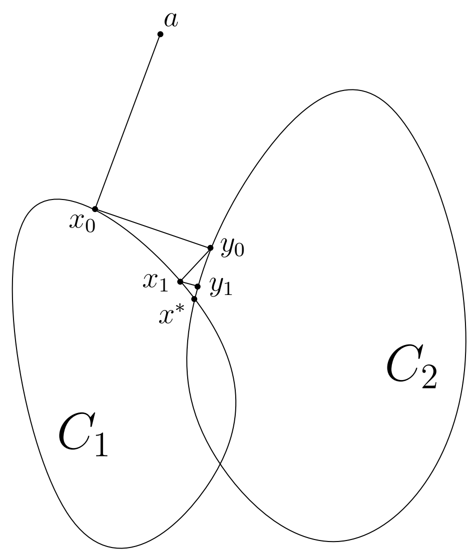

Figure 2.1: Illustration of Algorithm 1 in $\mathbb{R}^2$.

Theorem 2.1. (see [5]) *Provided that $C_1 \cap C_2 \neq \emptyset$, both sequences $(x_k)_{k\geq 0}$ and $(y_k)_{k\geq 0}$ generated by Algorithm 1 converge to a common point of C_1 and C_2.*

Proof. Let $\overline{x}$ be any point in $C_1 \cap C_2$. For convenience, we establish the following claims.

Claim 1: Each projection brings the point closer to $\overline{x}$ (this is so-called Fejér property).

Claim 2: The sequence $(x_k)_{k\geq 0}$ has an accumulation point $x^* \in C_1$.

Claim 3: The point x^* in Claim 2 also belongs to C_2.

To prove Claim 1, we first show that y_k is closer to $\overline{x}$ than x_k, i.e.

$$\|x_k - \overline{x}\| \geq \|y_k - \overline{x}\|. \tag{2.1}$$

Indeed, by applying Lemma 1.3 to the points $x_k \in X$, $y_k = P_{C_2}(x_k)$, $\overline{x} \in C_2$, we obtain

$$\|x_k - \overline{x}\|^2 \geq \|x_k - y_k\|^2 + \|y_k - \overline{x}\|^2,$$

or equivalently

$$\|x_k - \overline{x}\|^2 - \|x_k - y_k\|^2 \geq \|y_k - \overline{x}\|^2, \tag{2.2}$$

which implies (2.1). To complete the proof of Claim 1, we furthermore show that x_{k+1} is closer to $\overline{x}$ than y_k, i.e.

$$\|y_k - \overline{x}\| \geq \|x_{k+1} - \overline{x}\|. \tag{2.3}$$

Indeed, by applying Lemma 1.3 to the points $y_k \in X$, $x_{k+1} = P_{C_1}(y_k)$, $\overline{x} \in C_1$, we have

$$\|y_k - \overline{x}\|^2 \geq \|y_k - x_{k+1}\|^2 + \|x_{k+1} - \overline{x}\|^2,$$

or equivalently,

$$\|y_k - \overline{x}\|^2 - \|y_k - x_{k+1}\|^2 \geq \|x_{k+1} - \overline{x}\|^2, \tag{2.4}$$

which implies (2.3).

To prove Claim 2, we note that

$$\|x_k - \overline{x}\| \geq \|y_k - \overline{x}\| \geq \|x_{k+1} - \overline{x}\|, \tag{2.5}$$

by (2.1) and (2.3). In particular we have

$$\|x_k - \overline{x}\| \leq \|x_0 - \overline{x}\|, \qquad \forall k = 0, 1, 2, \dots,$$

which means that $(x_k)_{k\geq 0}$ is bounded. Since the underlying space X is finitely dimensional, it follows that the sequence $(x_k)_{k\geq 0}$ has an accumulation point x^*. Note that all x_k's are in C_1 and C_1 is closed, so we have $x^* \in C_1$. Thus Claim 2 is proved.

We now prove Claim 3. It follows from (2.5) that the sequence

$$\|x_0 - \overline{x}\|,\ \|y_0 - \overline{x}\|,\ \|x_1 - \overline{x}\|,\ \|y_1 - \overline{x}\|,\ \dots, \|x_k - \overline{x}\|,\ \|y_k - \overline{x}\|,\ \dots$$

is decreasing. Obviously all elements in this sequence are non-negative, i.e. this sequence is bounded below by 0. Hence it is convergent. We conclude from this fact and (2.2), (2.4) that

$$\|x_k - y_k\| \to 0 \quad \text{and} \quad \|y_k - x_{k+1}\| \to 0. \tag{2.6}$$

Since $y_k = P_{C_2}(x_k)$, we have

$$\operatorname{dist}(x_k, C_2) = \|x_k - y_k\|,$$

which, together with (2.6), implies

$$\operatorname{dist}(x_k, C_2) \to 0. \tag{2.7}$$

By Claim 2, $(x_k)_{k\geq 0}$ has a subsequence converging to x^*. This fact, together with (2.7) and closedness of C_2, leads to $x^* \in C_2$. This is what we state in Claim 3.

Now we are ready to prove the theorem. By Claim 2 and Claim 3 we have $x^* \in C_1 \cap C_2$. Since $\overline{x}$ is taken arbitrarily in $C_1 \cap C_2$, we can choose

$\overline{x} = x^*$. Then, as stated in the beginning of the proof for Claim 3, we have that both $(\|x_k - x^*\|)_{k\geq 0}$ and $(\|y_k - x^*\|)_{k\geq 0}$ converge to the same limit. Since a subsequence of $(x_k)_{k\geq 0}$ converges to x^*, $(\|x_k - x^*\|)_{k\geq 0}$ has a subsequence converging to 0. Since the whole sequence $(\|x_k - x^*\|)_{k\geq 0}$ is convergent, it follows that

$$\|x_k - x^*\| \to 0 \quad \text{and} \quad \|y_k - x^*\| \to 0.$$

So $x_k \to x^*$ and $y_k \to x^*$, i.e., both $(x_k)_{k\geq 0}$ and $(y_k)_{k\geq 0}$ converge to the point $x^* \in C_1 \cap C_2$. $\square$

Remark 2.2. We do not claim that Algorithm 1 returns a common point of C_1 and C_2 after a finite number of iterations. Generally, we can only claim (as stated in Theorem 2.1) that the algorithm produces sequences $(x_k)_{k\geq 0} \subset C_1$ and $(y_k)_{k\geq 0} \subset C_2$ converging to a common point of these two sets. However, if two consecutive projection points coincide, say $x_k = y_k$ for some $k \in \mathbb{N}$, then they belong to $C_1 \cap C_2$ since $x_k \in C_1$ and $y_k \in C_2$. In this case, the sequel projection points also coincide with x_k, so we can terminate the algorithm at iteration k. An example for this case is given in the next remark. $\square$

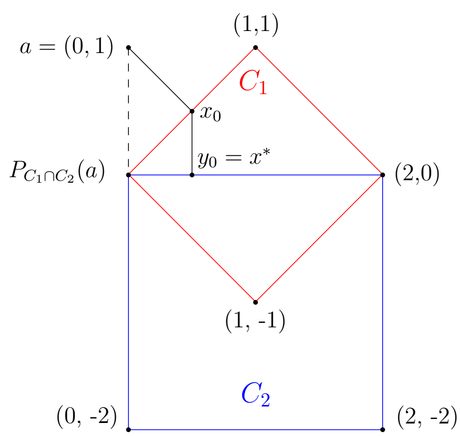

Figure 2.2: Illustration for Remark 2.2 and Remark 2.3.

Remark 2.3. The point x^* specified in the proof of Theorem 2.1 needs not to be $P_{C_1\cap C_2}(a)$. An example to illustrate this fact is given in Figure 2.2. In this example, $X = \mathbb{R}^2$ with the usual inner product. The set C_1 is the square whose vertices are $(0, 0)$, $(1, 1)$, $(2, 0)$, $(1, -1)$, and the set C_2 is the square whose vertices are $(0, 0)$, $(2, 0)$, $(2, -2)$, $(0, -2)$. The intersection $C_1 \cap C_2$ is then the triangular whose vertices are $(0, 0)$, $(2, 0)$, $(1, -1)$. The starting point a is taken as $(0, 1)$. Applying Algorithm 1 to this example, with simple calculations we obtain $x_0 = (0.5, 0.5)$, $y_0 = (0.5, 0) \in C_1 \cap C_2$, and then $x_k = y_k = y_0$ for all $k \geq 1$. Thus, $x^* = (0.5, 0)$. However, the projection of the starting point a on $C_1 \cap C_2$ is $(0, 0)$, which clearly differs from x^*. $\square$

In relation with the fact stated in Remark 2.3, we have the following proposition. It can be considered as an immediate consequence of the result by von Neumann in Theorem 13.7 [1], which states that the proposition still holds true in the setting where X is Hilbert space.

Proposition 2.4. *Let C_1 and C_2 be two closed subspaces of X such that $C_1 \cap C_2 \neq \emptyset$. Then both sequences $(x_k)_{k\geq 0}$, $(y_k)_{k\geq 0}$ generated by Algorithm 1 converge to $P_{C_1 \cap C_2}(a)$.*

2.2 Some simple variants

2.2.1 Periodic projection algorithm

The idea of periodic projection algorithm was first proposed by Halperin in [8] to find a common point of $m \geq 2$ closed subspaces in Hilbert space. This variant is so-called sequential projection or cyclic projection algorithm. In [5] the authors shortly mention that this variant of the basic alternating projection algorithm also works when finding a common point of $m \geq 2$ closed convex sets in the finite dimensional Euclidean space X. In the latter setting, the variant can be described more precisely as follows.

Algorithm 2 Periodic projection algorithm

Input: m closed convex subsets $C_1, \ldots, C_m$ of X such that $\cap_{i=1}^m C_i \neq \emptyset$.
Output: A point in $\cap_{i=1}^m C_i$.

Take an arbitrary point $a \in X$.
$x_0^1 = P_{C_1}(a)$.
for $i = 2, \ldots, m$ **do**
    $x_0^i = P_{C_i}(x_0^{i-1})$
end for
for $k = 0, 1, 2, \ldots$ **do**
    $x_{k+1}^1 = P_{C_1}(x_k^m)$.
    for $i = 2, \ldots, m$ **do**
        $x_{k+1}^i = P_{C_i}(x_{k+1}^{i-1})$
    end for
end for

Roughly speaking, the periodic alternating projection algorithm starts at an arbitrary point in X, and projecting on C_1, then C_2, …, then C_m, and repeating the cycle of m projections. In the description of this algorithm (Algorithm 2), for each projection point x_k^i, its superscript corresponds to the index of the convex set where it belongs to, while its subscript corresponds to the iteration in which it is generated. Figure 2.3 illustrates an example of Algorithm 2 for three convex sets in $\mathbb{R}^2$.

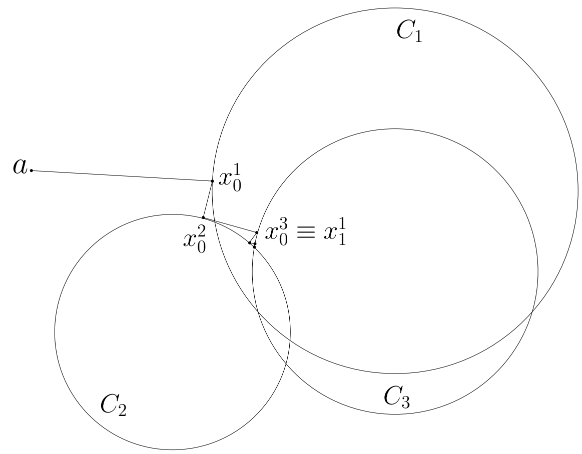

Figure 2.3: Illustration of Algorithm 2 in case of three non-empty intersection convex sets in $\mathbb{R}^2$.

The convergence of Algorithm 2 is stated in the following theorem, which is analogous to Theorem 2.1.

Theorem 2.5. (see [5]) *The sequences $(x_k^1)_{k\geq 0}$, $\ldots$, $(x_k^m)_{k\geq 0}$ generated by Algorithm 2 converge to a common point of the sets $C_1, \ldots, C_m$.*

Proof. For simplicity and clarity, we give the proof in case $m = 3$. The proof in case $m \geq 4$ can be done similarly. This proof is analogous to the proof of Theorem 2.1.

Let $\overline{x}$ be any point in $C := C_1 \cap C_2 \cap C_3$. We establish the following claims.

Claim 1: The projection points satisfy Fejér property, i.e., each projection brings the point closer to $\overline{x}$.

Claim 2: The sequence $(x_k^1)_{k\geq 0}$ has an accumulation point $x^* \in C_1$.

Claim 3: The point x^* in Claim 2 is also in C_2 and C_3.

Proving Claim 1 means that we have to show the followings.

- The projection point x_k^2 is closer to $\overline{x}$ than x_k^1, i.e.
$$\|x_k^2 - \overline{x}\| \leq \|x_k^1 - \overline{x}\|. \tag{2.8}$$
- The projection point x_k^3 is closer to $\overline{x}$ than x_k^2, i.e.
$$\|x_k^3 - \overline{x}\| \leq \|x_k^2 - \overline{x}\|. \tag{2.9}$$

- The projection point x^1_{k+1} is closer to $\overline{x}$ than x^3_k, i.e.

$$\|x^1_{k+1} - \overline{x}\| \le \|x^3_k - \overline{x}\|. \tag{2.10}$$

To see (2.8), we apply Lemma 1.3 to the points $x^1_k \in X$, $x^2_k = P_{C_2}(x^1_k)$, and $\overline{x} \in C_2$ and get

$$\|x^1_k - \overline{x}\|^2 \ge \|x^1_k - x^2_k\|^2 + \|x^2_k - \overline{x}\|^2,$$

or equivalently,

$$\|x^2_k - \overline{x}\|^2 \le \|x^1_k - \overline{x}\|^2 - \|x^1_k - x^2_k\|^2, \tag{2.11}$$

which implies (2.8). By similar arguments, we obtain

$$\|x^3_k - \overline{x}\|^2 \le \|x^2_k - \overline{x}\|^2 - \|x^2_k - x^3_k\|^2, \tag{2.12}$$

which implies (2.9), and

$$\|x^1_{k+1} - \overline{x}\|^2 \le \|x^3_k - \overline{x}\|^2 - \|x^3_k - x^1_{k+1}\|^2, \tag{2.13}$$

which implies (2.10). So we complete the proof for Claim 1.

For the proof of Claim 2, we note from (2.8)-(2.10) that

$$\|x^1_k - \overline{x}\| \ge \|x^2_k - \overline{x}\| \ge \|x^3_k - \overline{x}\| \ge \|x^1_{k+1} - \overline{x}\|. \tag{2.14}$$

Particularly, we have

$$\|x^1_0 - \overline{x}\| \ge \|x^1_k - \overline{x}\| \quad \forall k = 0, 1, 2, \ldots$$

So the sequence $(x^1_k)_{k\ge 0}$ is bounded and therefore it has an accumulation point x^* (since the underlying space X is finite dimensional). Note that all points in this sequence are in C_1, so we have $x^* \in C_1$ by closedness of C_1. This completes the proof for Claim 2.

To prove Claim 3, we note that by (2.14) the sequence

$$\|x^1_0 - \overline{x}\|, \|x^2_0 - \overline{x}\|, \|x^3_0 - \overline{x}\|, \|x^1_1 - \overline{x}\|, \|x^2_1 - \overline{x}\|, \|x^3_1 - \overline{x}\|, \ldots$$

decreases, hence it is convergent since it is bounded below by 0. By the convergence of this sequence and (2.11)-(2.13), we have

$$\|x^1_k - x^2_k\| \to 0, \quad \|x^2_k - x^3_k\| \to 0, \quad \|x^3_k - x^1_{k+1}\| \to 0. \tag{2.15}$$

Since $x^2_k = P_{C_2}(x^1_k)$, we have

$$\mathrm{dist}(x^1_k, C_2) = \|x^1_k - x^2_k\|$$

which, together with (2.15), implies

$$\operatorname{dist}(x_k^1, C_2) \to 0. \tag{2.16}$$

By Claim 2, $(x_k^1)_{k\geq 0}$ has a subsequence $(x_{k_i}^1)_{i\geq 0}$ converging to x^*. Hence, on one hand, (2.16) implies that $x^* \in C_2$ since C_2 is closed. On the other hand, by (2.15) it follows that the subsequence $(x_{k_i}^2)_{i\geq 0}$ also converges to x^*. Since $x_k^3 = P_{C_3}(x_k^2)$, we have

$$\operatorname{dist}(x_k^2, C_3) = \|x_k^2 - x_k^3\|,$$

which, together with (2.15), implies

$$\operatorname{dist}(x_k^2, C_3) \to 0. \tag{2.17}$$

Since $(x_k^2)_{k\geq 0}$ has the subsequence $(x_{k_i}^2)_{i\geq 0}$ converging to x^* and C_3 is closed, (2.17) implies that $x^* \in C_3$. This completes the proof of Claim 3.

Now we are ready to prove the theorem. By Claim 2 and Claim 3 we get $x^* \in C_1 \cap C_2 \cap C_3$. Since $\overline{x}$ is chosen arbitrarily in $C_1 \cap C_2 \cap C_3$, we can take $\overline{x} = x^*$. Then, as shown in the beginning of the proof for Claim 3, all sequences $(\|x_k^1 - x^*\|)_{k\geq 0}$, $(\|x_k^2 - x^*\|)_{k\geq 0}$, and $(\|x_k^3 - x^*\|)_{k\geq 0}$ converge to the same limit. Since $(x_k^1)_{k\geq 0}$ has the subsequence $(x_{k_i}^1)_{i\geq 0}$ converging to x^*, $(\|x_k^1 - x^*\|)_{k\geq 0}$ has the subsequence $(\|x_{k_i}^1 - x^*\|)_{i\geq 0}$ converging to 0. Since the whole sequence $(\|x_k^1 - x^*\|)_{k\geq 0}$ is convergent, we conclude that

$$\|x_k^1 - x^*\| \to 0.$$

Recall that the sequences $(\|x_k^1 - x^*\|)_{k\geq 0}$, $(\|x_k^2 - x^*\|)_{k\geq 0}$, and $(\|x_k^3 - x^*\|)_{k\geq 0}$ converge to the same limit, so we have

$$\|x_k^2 - x^*\| \to 0 \quad \text{and} \quad \|x_k^3 - x^*\| \to 0.$$

Thus $x_k^1 \to x^*$, $x_k^2 \to x^*$, and $x_k^3 \to x^*$. This proves the theorem, as x^* has been shown to be a common point of C_1, C_2, C_3. □

Remark 2.6. As stated in the description of Algorithm 2, in each iteration of this algorithm we sequentially project onto the input convex sets in the order $C_1, C_2, \ldots, C_m$. By the equal role of these sets, we can sequentially project onto them in any order C_{σ_1}, C_{σ_2}, …, C_{σ_m}, in which $(\sigma_1, \sigma_2, \ldots, \sigma_m)$ is a permutation of $(1, 2, \ldots, m)$, provided that this order is the same in every iteration. □

Remark 2.7. It is stated in the end of Section 3 in [5] that the order of projections onto the input convex sets does not need to be periodic. We can project the current point onto any of the sets the point is not in, provided that each input convex set is infinitely projected onto. □

2.2.2 Averaged projection algorithm

The variant presented in this subsection was proposed by Cimmino in [9]. Here we give a detailed discussion on this variant.

The variant also starts with an arbitrary point in the underlying space X, but then simultaneously projecting on the involved convex sets. The average of the projection points is then chosen as the starting point for the next iteration. This variant is described more precisely as follows.

Algorithm 3 Averaged projection algorithm for finding a common point of two closed convex sets

Input: Two closed convex subsets C_1 and C_2 of X such that $C_1 \cap C_2 \neq \emptyset$.
Output: A common point of C_1 and C_2.

1: Take an arbitrary point $a \in X$.
2: $x_0 = P_{C_1}(a)$.
3: $y_0 = P_{C_2}(a)$.
4: $z_0 = (x_0 + y_0)/2$.
5: **for** $k = 0, 1, 2, \ldots$ **do**
6: $\quad x_{k+1} = P_{C_1}(z_k)$
7: $\quad y_{k+1} = P_{C_2}(z_k)$
8: $\quad z_{k+1} = (x_{k+1} + y_{k+1})/2$
9: **end for**

Figure 2.4 illustrates an example of Algorithm 3 for two convex sets in $\mathbb{R}^2$.

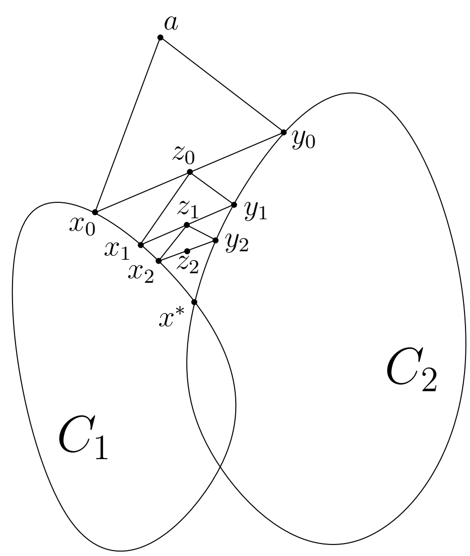

Figure 2.4: Illustration of Algorithm 3 in case of two non-empty intersection convex sets in $\mathbb{R}^2$.

For convenience of proving the convergence of Algorithm 3, we first need the following proposition. For the statement of the proposition, we equip on the space X^2 the usual inner product $\langle\cdot,\cdot\rangle_{X^2}$ defined by

$$\langle(x_1,x_2),(y_1,y_2)\rangle_{X^2} = \langle x_1,y_1\rangle + \langle x_2,y_2\rangle,$$

in which $x_1,x_2,y_1,y_2 \in X$, and the induced norm $\|\cdot\|_{X^2}$ determined by

$$\|(x_1,x_2)\|_{X^2} = \sqrt{\|x_1\|^2+\|x_2\|^2}.$$

Proposition 2.8. *Let $C = C_1 \times C_2$, and $u = (u_1,u_2) \in X^2$. Then we have*

$$P_C(u) = (P_{C_1}(u_1), P_{C_2}(u_2)).$$

Proof. Since Cartesian product of closed sets is also closed and note that C_1, C_2 are closed sets, it follows that C is closed. The convexity of C follows from the convexity of C_1, C_2 and Proposition 1.1. So C is a closed convex set in X^2, and therefore $P_C(u)$ exists uniquely.

Since C_1, C_2 are closed convex set, $P_{C_1}(u_1)$ and $P_{C_2}(u_2)$ exist uniquely. By definition we have

$$\begin{aligned}
\|u-P_C(u)\|^2_{X^2} &= \min_{(x,y)\in C_1\times C_2} \|(u_1,u_2)-(x,y)\|^2_{X^2} \\
&= \min_{(x,y)\in C_1\times C_2} \left(\|u_1-x\|^2+\|u_2-y\|^2\right) \\
&= \min_{x\in C_1}\|u_1-x\|^2 + \min_{y\in C_2}\|u_2-y\|^2 \quad (\text{since } u_1\in C_1 \text{ and } u_2\in C_2) \\
&= \|u_1-P_{C_1}(u_1)\|^2 + \|u_2-P_{C_2}(u_2)\|^2.
\end{aligned}$$

The last equality holds when $x = P_{C_1}(u_1)$ and $y = P_{C_2}(u_2)$. So the minimum in the first equality is achieved at $(x,y) = (P_{C_1}(u_1), P_{C_2}(u_2))$, and therefore we obtain the claim of the proposition. □

We furthermore need the following proposition.

Proposition 2.9. *Let $D = \{(x,y)\in X^2 \mid x=y\}$ and $u=(u_1,u_2)\in X^2$. Then D is a closed convex set in X^2 and $P_D(u) = ((u_1+u_2)/2, (u_1+u_2)/2)$.*

Proof. Let $(u^k)_{k\geq 0} = (u_1^k, u_2^k)_{k\geq 0}$ be a sequence of points in D converging to $\overline{u} = (\overline{u}_1, \overline{u}_2)$. Then we have $u_1^k \to \overline{u}_1$ and $u_2^k \to \overline{u}_2$ as k tends to infinity. For every index k, since $(u_1^k, u_2^k)\in D$, we have $u_1^k = u_2^k$ by definition of D. Thus $\overline{u}_1 = \overline{u}_2$, and therefore $\overline{u} = (\overline{u}_1,\overline{u}_2)\in D$. So D is closed.

Let $v=(v_1,v_2)$ and $w=(w_1,w_2)$ be two points in D, and $\lambda\in[0,1]$. Since v and w are in D, we have $v_1=v_2$ and $w_1=w_2$. So $\lambda v_1+(1-\lambda)w_1 = \lambda v_2+(1-\lambda)w_2$, and consequently

$$\lambda v + (1-\lambda)w = (\lambda v_1 + (1-\lambda)w_1, \lambda v_2 + (1-\lambda)w_2) \in D.$$

So D is convex.

Now, for any $(x, x) \in D$ we have

$$\begin{aligned} \|(u_1, u_2) - (x, x)\|_{X^2}^2 &= \|(u_1 - x, u_2 - x)\|_{X^2}^2 \\ &= \|u_1 - x\|^2 + \|x - u_2\|^2 \\ &\geq \frac{1}{2}\|(u_1 - x) + (x - u_2)\|^2 \\ &= \frac{1}{2}\|u_1 - u_2\|^2. \end{aligned} \tag{2.18}$$

The equality in (2.18) happens when $u_1 - x = x - u_2$, or equivalently, when $x = (u_1 + u_2)/2$. So the distance from $u = (u_1, u_2)$ to D is

$$\min_{(x,x) \in D} \|(u_1, u_2) - (x, x)\|_{X^2} = \frac{\sqrt{2}}{2}\|u_1 - u_2\|,$$

which is attained at $x = x^* = (u_1 + u_2)/2$. Therefore we have

$$P_D(u) = (x^*, x^*) = ((u_1 + u_2)/2, (u_1 + u_2)/2).$$

□

We are now ready for the proof of the convergence of Algorithm 3, which is stated in the following theorem.

Theorem 2.10. (see [5]) *The three sequences $(x_k)_{k\geq 0}$, $(y_k)_{k\geq 0}$, $(z_k)_{k\geq 0}$ generated by Algorithm 3 converge to the same point which is in $C_1 \cap C_2$.*

Proof. By Proposition 2.8 and Proposition 2.9, the sets $C = C_1 \times C_2$ and $D = \{(x, y) \in X^2 \mid x = y\}$ are closed and convex. Note that $C_1 \cap C_2 \neq \emptyset$, so any point of form (x, x) with $x \in C_1 \cap C_2$ are in both C and D. Thus $C \cap D \neq \emptyset$.

Let us apply the basic alternating projection algorithm (Algorithm 1) to these two sets with the starting point $(a, a) \in X^2$. The first projection of this algorithm in this setting is $P_C(a, a)$, and by Proposition 2.8 we have $P_C(a, a) = (P_{C_1}(a), P_{C_2}(a)) = (x_0, y_0)$, in which x_0 and y_0 are defined in the description of Algorithm 3. The second projection of this algorithm in this setting is $P_D(x_0, y_0)$, and by Proposition 2.9 we have

$$P_D(x_0, y_0) = ((x_0 + y_0)/2, (x_0 + y_0)/2) = (z_0, z_0),$$

in which z_0 is also defined in the description of Algorithm 3. In this way, the next two projection points are respectively

$$P_C(z_0, z_0) = (P_{C_1}(z_0), P_{C_2}(z_0)) = (x_1, y_1),$$

and

$$P_D(x_1, y_1) = ((x_1 + y_1)/2, (x_1 + y_1)/2) = (z_1, z_1),$$

in which x_1, y_1, z_1 are defined in the description of Algorithm 3. Continuing this process, Algorithm 1 in this setting generates two sequences $(x_k, y_k)_{k\geq 0} \subset C_1 \times C_2$ and $(z_k, z_k)_{k\geq 0} \subset D$, where x_k, y_k, z_k are defined in the description of Algorithm 3.

Since $C \cap D \neq \emptyset$, Theorem 2.1 certifies that these two sequences converge to a point $(x^*, y^*) \in C \times D$. Since $(x^*, y^*) \in C = C_1 \times C_2$, we have $x^* \in C_1$ and $y^* \in C_2$. Since $(x^*, y^*) \in D$, we have $x^* = y^*$. So x^* is also in C_2, and therefore $x^* \in C_1 \cap C_2$. Furthermore, since

$$(x_k, y_k) \to (x^*, x^*) \quad \text{and} \quad (z_k, z_k) \to (x^*, x^*),$$

we have $x_k \to x^*$, $y_k \to x^*$, and $z_k \to x^*$ as $k \to +\infty$. As a conclusion, $x^* \in C_1 \cap C_2$ is the common limit of the three sequences $(x_k)_{k\geq 0}$, $(y_k)_{k\geq 0}$, $(z_k)_{k\geq 0}$. This proves the theorem. □

Remark 2.11. The idea of Algorithm 3 can be extend to the case of $m \geq 2$ closed convex sets, as described in Algorithm 4. Similar to the proof of Theorem 2.10, by applying Algorithm 1 (the basic alternating projection algorithm) to the two closed convex sets

$$C = C_1 \times C_2 \times \ldots \times C_m,$$
$$D = \{(x_1, x_2, \ldots, x_m) \in X^m \mid x_1 = x_2 = \ldots = x_m\},$$

then applying Theorem 2.1, we can prove that the sequences $(x_k^1)_{k\geq 0}$, $(x_k^2)_{k\geq 0}$, $\ldots$, $(x_k^m)_{k\geq 0}$, and $(z_k)_{k\geq 0}$ generated by Algorithm 4 converge to a common points of $C_1, C_2, \ldots, C_m$. □

Remark 2.12. In each iteration of the averaged versions of alternating projection algorithm, the number of projections equals the number of input convex sets. This fact is also true for the basic version and the periodic version of alternating projection algorithm. However, it is worth noting that the projections in each iteration of the averaged versions can be computed ***in parallel***, while for the basic and periodic versions the projections need to be computed sequentially. □

Algorithm 4 Averaged projection algorithm for finding a common point of $m \geq 2$ closed convex sets

Input: m closed convex subsets $C_1, \ldots, C_m$ of X such that $\cap_{i=1}^m C_i \neq \emptyset$.
Output: A point in $\cap_{i=1}^m C_i$.

```
Take an arbitrary point a ∈ X.
for i = 1, ..., m do
    x_0^i = P_{C_i}(a)
end for
z_0 = (1/m) ∑_{i=1}^m x_0^i.
for k = 0, 1, 2, ... do
    for i = 1, ..., m do
        x_{k+1}^i = P_{C_i}(z_k)
    end for
    z_{k+1} = (1/m) ∑_{i=1}^m x_k^i.
end for
```

2.2.3 Relaxed projection algorithm

This variant of the basic alternating projection algorithm is shortly mentioned in Section 3 [5]. For simplicity and clarity, we consider the relaxed projection algorithm applied to find a common point of $m = 2$ closed convex sets in the finite dimensional Euclidean space X. It is worth noting that, as remarked at the end of this subsection, the relaxed projection algorithm can be applied to the case with $m \geq 3$. For the description of the algorithm, we need the following concept.

Definition 2.13. (Relaxed projection, see [10]) *Let $C \subset X$ be a closed convex set and $\alpha \in (0, 2)$. The relaxed projection (with parameter α) of a point $x \in X$ onto C is*

$$R_{C,\alpha}(x) = x + \alpha(P_C(x) - x) = \alpha P_C(x) + (1 - \alpha)x. \tag{2.19}$$

By (2.19) we obtain

$$R_{C,\alpha}(x) - x = \alpha(P_C(x) - x), \tag{2.20}$$

$$R_{C,\alpha}(x) - P_C(x) = (1 - \alpha)(x - P_C(x)). \tag{2.21}$$

For convenience, we shortly call $R_{C,\alpha}(x)$ the α-relaxed projection of x onto C. By definition, when $\alpha = 1$ we have $R_{C,\alpha}(x) = P_C(x)$, i.e., the 1-relaxed projection is nothing but the usual projection. Figure 2.5 illustrates $R_{C,\alpha}(x)$ in $\mathbb{R}^2$ with $\alpha \in (0, 1)$, and Figure 2.6 illustrates $R_{C,\alpha}(x)$ in $\mathbb{R}^2$ with $\alpha \in (1, 2)$.

Intuitively, in case $\alpha \in (0, 1)$ we step only the fraction α from x to $P_C(x)$, while in case $\alpha \in (1, 2)$ we go farther than the projection $P_C(x)$. Therefore, we also call $R_{C,\alpha}(x)$ an under projection of x onto C when $\alpha \in (0,1)$, and an over projection of x onto C when $\alpha \in (1,2)$.

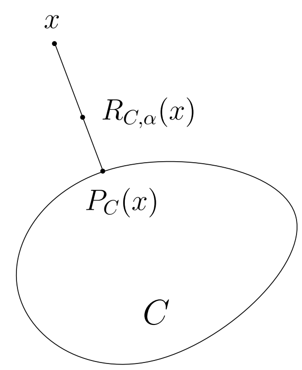

Figure 2.5: Under projection.

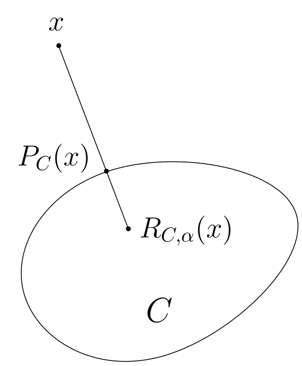

Figure 2.6: Over projection.

The relaxed projection algorithm for finding a point in the intersection of $m = 2$ closed convex sets is described as follows.

Algorithm 5 Relaxed projection algorithm for finding a common point of two closed convex sets

---

Input: Two closed convex subsets C_1 and C_2 of X such that $C_1 \cap C_2 \neq \emptyset$, and $\alpha \in (0,2)$.
Output: A common point of C_1 and C_2.

```
Take an arbitrary point a ∈ X.
x_0 = R_{C_1,α}(a).
for k = 0, 1, 2, ... do
    y_k = R_{C_2,α}(x_k)
    x_{k+1} = R_{C_1,α}(y_k)
end for
```

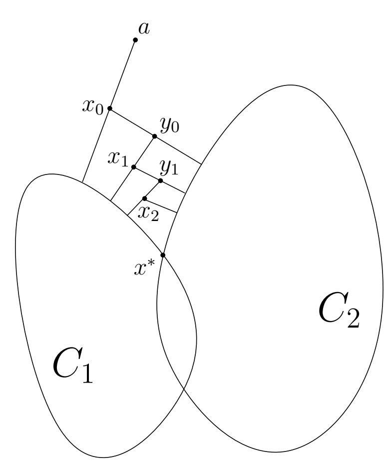

Figure 2.7: Illustration of under projection algorithm.

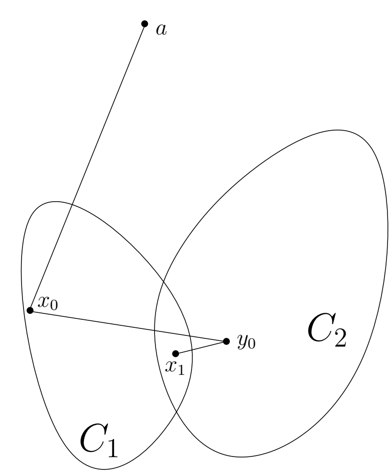

Figure 2.8: Illustration of over projection algorithm.

As a remark, Algorithm 5 is the same as Algorithm 1, except that the projections are replaced by the relaxed projections. When $\alpha = 1$, Algorithm 5 is exactly Algorithm 1. Figure 2.7 illustrates an example of Algorithm 5 with $\alpha \in (0, 1)$ and Figure 2.8 illustrates an example of Algorithm 5 with $\alpha \in (1, 2)$.

To prove the convergence of Algorithm 5, we follow the scheme of proof of Theorem 2.1 which is for the convergence of the basic alternating projection algorithm. According to this scheme, the first step to prove Fejér property of the sequence of points generated by Algorithm 5, i.e., to show that each relaxed projection brings the point closer to a point in $C_1 \cap C_2$ fixed in advance. For convenience of presenting the proof for this property, the following analogous results of Lemma 1.3 are useful.

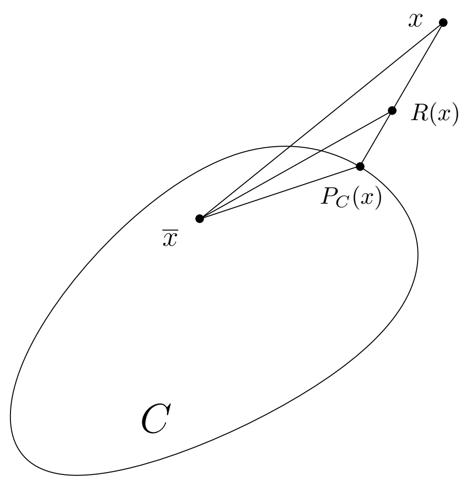

Figure 2.9: Illustration for Lemma 2.14.

Lemma 2.14. (see [10]) *Let $C \subset X$ be a closed convex set, $x \in X$, $\overline{x} \in C$, and $\alpha \in (0, 1)$. Then we have*

$$\|x - \overline{x}\|^2 \geq \alpha^2 \|x - P_C(x)\|^2 + \|R_{C,\alpha}(x) - \overline{x}\|^2.$$

Proof. Since C and α are specified, for simplicity of presenting this proof, we denote $R(x) := R_{C,\alpha}(x)$. We first show that

$$\langle x - R(x), \overline{x} - R(x) \rangle \leq 0. \tag{2.22}$$

Indeed, we have

$$\begin{aligned} &\langle x - R(x), \overline{x} - R(x) \rangle \\ = &\langle x - R(x), \overline{x} - P_C(x) + P_C(x) - R(x) \rangle \end{aligned}$$

$$
\begin{aligned}
&= \langle x - R(x), \overline{x} - P_C(x)\rangle + \langle x - R(x), P_C(x) - R(x)\rangle \\
&= \langle \alpha(x - P_C(x)), \overline{x} - P_C(x)\rangle + \langle \alpha(x - P_C(x)), (\alpha - 1)(x - P_C(x))\rangle \qquad (2.23)\\
&= \alpha\langle x - P_C(x), \overline{x} - P_C(x)\rangle + \alpha(1-\alpha)\|x - P_C(x)\|^2. \qquad (2.24)
\end{aligned}
$$

The equality (2.23) is due to (2.20) and (2.21). By (1.2), the inner product in the first term of (2.24) is non-positive. Since $\alpha \in (0, 1)$, the second term of (2.24) is also non-positive. Hence (2.22) follows, and consequently we have

$$
\begin{aligned}
\|x - \overline{x}\|^2 &= \|x - R(x) + R(x) - \overline{x}\|^2 \\
&= \|x - R(x)\|^2 + \|R(x) - \overline{x}\|^2 + 2\langle x - R(x), R(x) - \overline{x}\rangle \\
&\geq \|x - R(x)\|^2 + \|R(x) - \overline{x}\|^2 \\
&= \alpha^2\|x - P_C(x)\|^2 + \|R(x) - \overline{x}\|^2,
\end{aligned}
$$

as desired, where the last equality is due to (2.20). □

Lemma 2.15. (see [10]) *Let $C \subset X$ be a closed convex set, $x \in X$, $\overline{x} \in C$, and $\alpha \in (1, 2)$. Then we have*

$$
\|x - \overline{x}\|^2 \geq \left(1 - (1-\alpha)^2\right) \|x - P_C(x)\|^2 + \|R_{C,\alpha}(x) - \overline{x}\|^2.
$$

Proof. Similar to the proof of the previous lemma, we denote $R(x) := R_{C,\alpha}(x)$ for simplicity. Keeping (2.21) in mind, we have

$$
\begin{aligned}
&\|R(x) - \overline{x}\|^2 \\
&= \|R(x) - P_C(x) + P_C(x) - \overline{x}\|^2 \\
&= \|R(x) - P_C(x)\|^2 + \|P_C(x) - \overline{x}\|^2 + 2\langle R(x) - P_C(x), P_C(x) - \overline{x}\rangle \\
&= (1-\alpha)^2\|x - P_C(x)\|^2 + \|P_C(x) - \overline{x}\|^2 + 2(1-\alpha)\langle x - P_C(x), P_C(x) - \overline{x}\rangle \\
&= (1-\alpha)^2\|x - P_C(x)\|^2 + \|P_C(x) - \overline{x}\|^2 + 2(\alpha-1)\langle x - P_C(x), \overline{x} - P_C(x)\rangle \\
&\leq (1-\alpha)^2\|x - P_C(x)\|^2 + \|P_C(x) - \overline{x}\|^2. \qquad (2.25)
\end{aligned}
$$

The inequality (2.25) is due to $\alpha > 1$ and (1.2). By Lemma 1.3 we have

$$
\|x - P_C(x)\|^2 + \|P_C(x) - \overline{x}\|^2 \leq \|x - \overline{x}\|^2. \qquad (2.26)
$$

Adding side by side (2.25) and (2.26), then subtracting the term $\|P_C(x) - \overline{x}\|^2$ in both sides, we obtain

$$
\|R(x) - \overline{x}\|^2 + \|x - P_C(x)\|^2 \leq (1-\alpha)^2\|x - P_C(x)\|^2 + \|x - \overline{x}\|^2,
$$

which is equivalent to the inequality in the statement of this lemma. □

As a summary of Lemma 2.14 and Lemma 2.15, we have

$$\|x-\overline{x}\|^2 \geq \beta\|x-P_C(x)\|^2+\|R_{C,\alpha}(x)-\overline{x}\|^2, \tag{2.27}$$

in which

$$\beta = \begin{cases} \alpha^2 & \text{if } \alpha \in (0,1), \\ 1-(1-\alpha)^2 & \text{if } \alpha \in (1,2). \end{cases} \tag{2.28}$$

It is worth noting that $\beta > 0$. We are now ready for the proof of convergence of the relaxed projection algorithm.

Theorem 2.16. (see [5]) *The sequences $(x_k)_{k\geq 0}$ and $(y_k)_{k\geq 0}$ generated by Algorithm 5 converge to a common point of C_1 and C_2.*

Proof. In case $\alpha = 1$, Algorithm 5 is exactly the basic alternating projection algorithm (Algorithm 1) whose convergence is proved in Theorem 2.1. In the following we consider $\alpha \in (0,2)\backslash\{1\} = (0,1)\cup(1,2)$ and follow the scheme of proof of Theorem 2.1.

Take an arbitrary $\overline{x} \in C_1 \cap C_2$. We will go through the following claims.

Claim 1: Each relaxed projection brings the point closer to $\overline{x}$, i.e. the points generated by Algorithm 5 follow Fejér property.

Claim 2: The sequence $(x_k)_{k\geq 0}$ has an accumulation point $x^* \in C_2$.

Claim 3: The point x^* in Claim 2 also belongs to C_1. □

We first prove Claim 1. Proving this claim means that we have to show

$$\|x_k-\overline{x}\| \geq \|y_k-\overline{x}\|, \tag{2.29}$$

i.e., y_k is closer to $\overline{x}$ than x_k, and to show

$$\|y_k-\overline{x}\| \geq \|x_{k+1}-\overline{x}\|, \tag{2.30}$$

i.e., x_{k+1} is closer to $\overline{x}$ than y_k. Indeed, by applying (2.27) to the set C_2 and the points $\overline{x} \in C_2$, $x_k \in X$, $y_k = R_{C_2,\alpha}(x_k)$ we obtain

$$\|x_k-\overline{x}\|^2 \geq \beta\|x_k-P_{C_2}(x_k)\|^2+\|y_k-\overline{x}\|^2,$$

in which $\beta > 0$ is defined by (2.28). This is equivalent to

$$\|x_k-\overline{x}\|^2 - \beta\|x_k-P_{C_2}(x_k)\|^2 \geq \|y_k-\overline{x}\|^2, \tag{2.31}$$

which implies (2.29). Similarly, by applying (2.27) to the set C_1 and the points $\overline{x} \in C_1$, $y_k \in X$, $x_{k+1} = R_{C_1,\alpha}(y_k)$ we obtain

$$\|y_k-\overline{x}\|^2 \geq \beta\|y_k-P_{C_1}(y_k)\|^2+\|x_{k+1}-\overline{x}\|^2,$$

or equivalently

$$\|y_k - \overline{x}\|^2 - \beta\|y_k - P_{C_1}(y_k)\|^2 \geq \|x_{k+1} - \overline{x}\|^2, \tag{2.32}$$

which implies (2.30). This completes the proof of Claim 1.

Now we go to Claim 2. By (2.29) and (2.30) we have

$$\|x_0-\overline{x}\| \geq \|y_0-\overline{x}\| \geq \|x_1-\overline{x}\| \geq \|y_1-\overline{x}\| \geq \ldots \geq \|x_k-\overline{x}\| \geq \|y_k-\overline{x}\| \geq \ldots \tag{2.33}$$

On one hand, the sequence in (2.33) is decreasing and all of its elements are non-negative. Therefore this sequence is convergent. Together with (2.31), (2.32) and note that $\beta > 0$, this fact implies

$$\|x_k - P_{C_2}(x_k)\| \to 0 \quad \text{and} \quad \|y_k - P_{C_1}(y_k)\| \to 0, \tag{2.34}$$

and hence we have

$$\mathrm{dist}(x_k, C_2) \to 0 \quad \text{and} \quad \mathrm{dist}(y_k, C_1) \to 0. \tag{2.35}$$

On the other hand, by (2.33) we have $\|x_0 - \overline{x}\| \geq \|x_k - \overline{x}\|$ for all $k \geq 0$, i.e., $(x_k)_{k\geq 0}$ is a bounded sequence. Note that the underlying space X is finite dimensional, therefore this sequence has an accumulation point x^*. This fact, together with (2.35) and the closedness of C_2, leads to $x^* \in C_2$. This proves Claim 2.

To prove Claim 3, we apply (2.20) to the set C_2 and the points $x_k \in X$, $y_k = R_{C_2,\alpha}(x_k)$ and get

$$y_k - x_k = \alpha(P_{C_2}(x_k) - x_k).$$

Therefore we have

$$\|y_k - x_k\| = \alpha\|P_{C_2}(x_k) - x_k\|,$$

which, together with (2.34), leads to

$$\|y_k - x_k\| \to 0. \tag{2.36}$$

By combining (2.36) with the fact that $(x_k)_{k\geq 0}$ has an accumulation point x^*, we conclude that $(y_k)_{k\geq 0}$ also admits x^* as an accumulation point. This fact, together with (2.35) and the closedness of C_1, implies that $x^* \in C_1$. This completes the proof of Claim 3.

Having Claim 1 - Claim 3 proved, the convergence of Algorithm 5 can be shown by the same arguments as in the proof of Theorem 2.1. More precisely, we have $x^* \in C_1 \cap C_2$ by Claim 2 and Claim 3. Since $\overline{x}$ is taken arbitrarily in $C_1 \cap C_2$, we can choose $\overline{x} = x^*$. Then, from (2.33) we have that both $(\|x_k - x^*\|)_{k\geq 0}$ and $(\|y_k - x^*\|)_{k\geq 0}$ converge to the same limit. Since a subsequence of $(x_k)_{k\geq 0}$ converges to x^*, $(\|x_k - x^*\|)_{k\geq 0}$ has a subsequence converging to 0. Since the whole sequence $(\|x_k - x^*\|)_{k\geq 0}$ is convergent, it follows that

$$\|x_k - x^*\| \to 0 \quad \text{and} \quad \|y_k - x^*\| \to 0.$$

So $x_k \to x^*$ and $y_k \to x^*$, i.e., both $(x_k)_{k\geq 0}$ and $(y_k)_{k\geq 0}$ converge to the point $x^* \in C_1 \cap C_2$. $\square$

Remark 2.17. As the basic alternating projection algorithm (Algorithm 1) can be extended to the periodic projection algorithm (Algorithm 2), the relaxed projection algorithm (Algorithm 5) can also be extended to find an intersection point of $m \geq 2$ closed convex sets (see Algorithm 6 below). The convergence of Algorithm 6 can be proved using the idea in the proofs of Theorem 2.5 and Theorem 2.16. $\square$

Algorithm 6 Relaxed projection algorithm for finding a common point of $m \geq 2$ closed convex sets.

Input: m closed convex subsets $C_1, \ldots, C_m$ of X such that $\cap_{i=1}^m C_i \neq \emptyset$, and $\alpha \in (0, 2)$.
Output: A point in $\cap_{i=1}^m C_i$.

Take an arbitrary point $a \in X$.
$x_0^1 = R_{C_1,\alpha}(a)$.
for $i = 2, \ldots, m$ **do**
  $x_0^i = R_{C_i,\alpha}(x_0^{i-1})$
end for
for $k = 0, 1, 2, \ldots$ **do**
  $x_{k+1}^1 = R_{C_1,\alpha}(x_k^m)$.
  for $i = 2, \ldots, m$ **do**
    $x_{k+1}^i = R_{C_i,\alpha}(x_{k+1}^{i-1})$
  end for
end for

Remark 2.18. We cannot choose $\alpha = 0$ or $\alpha = 2$ in Algorithm 5 as well as Algorithm 6. If we choose $\alpha = 0$, then $R_{C,\alpha}(x) = x$ for any $x \in X$, i.e., $R_{C,\alpha}$ is the identical mapping. In this case, every 0-relaxed projection point of the algorithms coincides the starting point a, hence the algorithms do not give us an intersection point of the involved convex sets if the starting point is not in the intersection. If $\alpha = 2$, the example illustrated in Figure 2.10 shows that the sequence of 2-relaxed projection points generated by Algorithm 5 is cyclic (similar example can be constructed for Algorithm 6). In this example, C_1 is the x-axis and C_2 is the y-axis in $\mathbb{R}^2$, while the starting point is chosen as

$a = (1, 1)$. The first generated projection points in this case are

$$x_0 = (1, -1), \quad y_0 = (-1, -1), \quad x_1 = (-1, 1), \quad y_1 = (1, 1) = a.$$

So after each two iterations, the projection point returns to the starting point, hence the sequence of relaxed projection points in this case does not converge.

$\square$

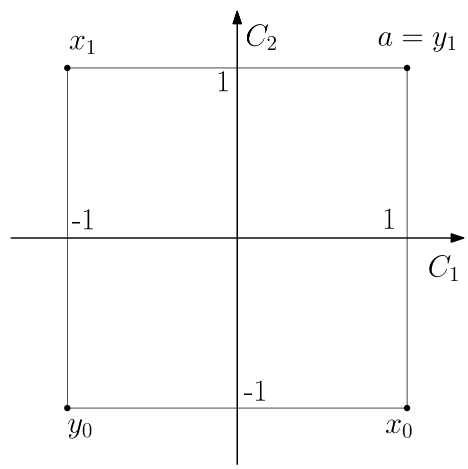

Figure 2.10: Example for Remark 2.18.

Chapter 3

Some selected applications

We have discussed the basic alternating projection algorithm as well as some simple variants of this algorithm in Chapter 2. In the present chapter, we show how the algorithms can be applied to solve two interesting problems, including dividing a string into equal thirds (Section 3.1) and completing positive semi-definite matrices (Section 3.2).

3.1 Dividing a string into equal thirds

This section is written on the base of the discussion of Don Burkholder presented in [11].

3.1.1 Problem statement and algorithm

The concerned problem is simply stated as follows.

> *Given a string of finite length, how to divide it into three parts of equal length.*

In discussions with the authors of [11], Burkholder proposed the following procedure to solve this problem. In the initial step, we straighten the string from left to right, and attach two small stickers to two arbitrary positions on the string. The sticker on the left is called the 'left sticker', and the one on the right is called the 'right sticker'. We then perform an iterative process, in which each iteration is made up by executing the following two steps.

Step 1:

(a) Fold over the right end of the string to touch the left sticker.

(b) Hold the right end of the string at the left sticker, and slide the right sticker to the right until it reaches the right end of the formed loop.

(c) Unfold the string.

Step 2:

(a) Fold over the left end of the string to touch the right sticker.

(b) Hold the left end of the string at the right sticker, and slide the left sticker to the left until it reaches the left end of the formed loop.

(c) Unfold the string.

Figure 3.1 illustrates the actions in Step 1, while Figure 3.2 illustrates the actions in Step 2. In these figures, the notation L stands for the left sticker, while the notation R stands for the right sticker.

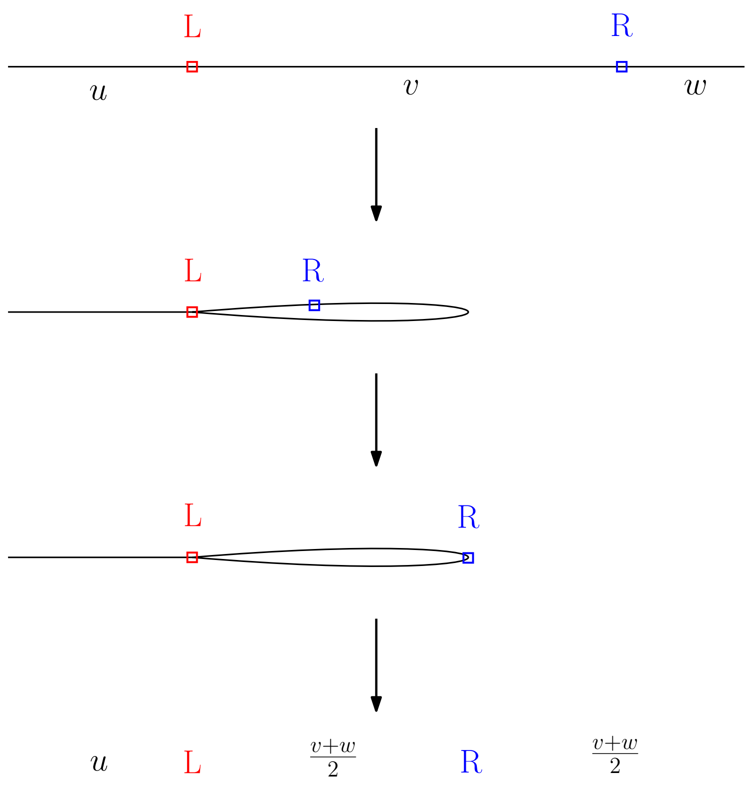

Figure 3.1: Illustration of Step 1 of Burkholder's procedure.

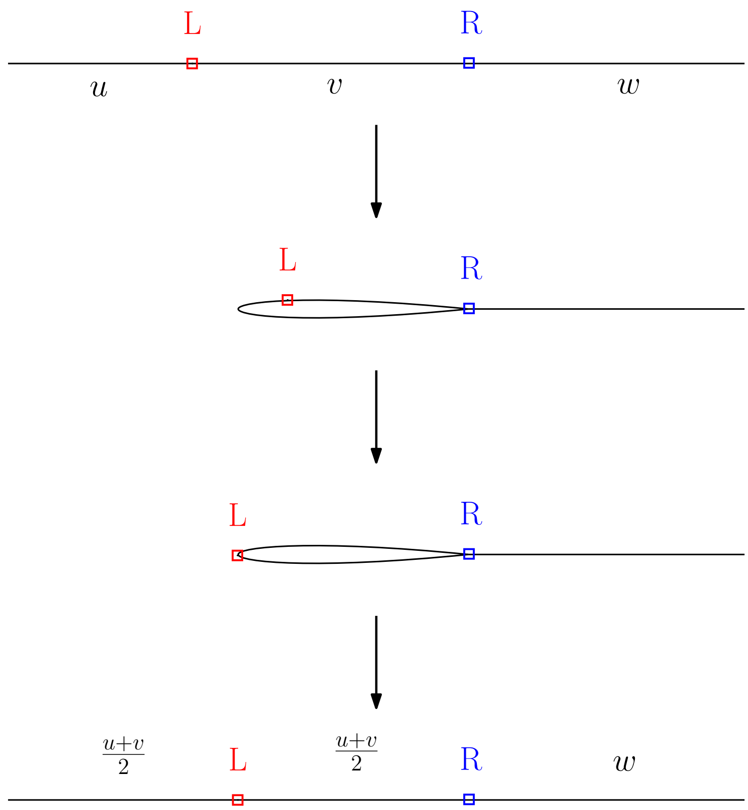

Figure 3.2: Illustration of Step 2 of Burkholder's procedure.

3.1.2 Convergence analysis

Charmingly, the basic alternating projection algorithm can be applied to give an elegant proof for the convergence and correctness of Burkholder's procedure.

Theorem 3.1. (see Section 3.1 [2]) *Iteration by iteration in Burkholder's procedure, the positions of the left (resp., right) sticker converge to the one third (resp., two thirds) of the total length of the string.*

Proof. To see the connection with the alternating projections, let us consider the string before performing each step of a sample iteration in Burkholder's procedure. Assume that, at the beginning of the procedure, the two stickers divide the string into three parts whose lengths are u, v, w respectively from left to right. After performing Step 1, the right sticker is moved to the middle of the part from the left sticker to the right end of the string, hence the lengths of the three parts become u, $\frac{v+w}{2}$, $\frac{v+w}{2}$ respectively. If we denote by $x = (u, v, w)^t$ the vector of lengths of three parts of the string, then the application of Step 1 corresponds to the linear transformation

$$P_1(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} u \\ \frac{v+w}{2} \\ \frac{v+w}{2} \end{bmatrix}.$$

The matrix of this linear transformation is

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Similarly, after performing Step 2, the left sticker is moved to the middle of the part from the left end of the string to the right sticker, hence the lengths of the three parts become $\frac{u+v}{2}$, $\frac{u+v}{2}$, w respectively. Thus the application of Step 2 corresponds to the linear transformation

$$P_2(x) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \frac{u+v}{2} \\ \frac{u+v}{2} \\ w \end{bmatrix}.$$

The matrix of this linear transformation is

$$M_2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $C_1 = P_1(\mathbb{R}^3) = \text{range}(P_1)$. Since P_1 is a linear transformation, it is well-known from linear algebra that C_1 is a subspace of $\mathbb{R}^3$, hence it is also closed. Similarly, let $C_2 = P_2(\mathbb{R}^3) = \text{range}(P_2)$, then C_2 is also a closed subspace of $\mathbb{R}^3$. Note that $M_1^2 = M_1 = M_1^t$, i.e., M_1 is idempotent and self-adjoint, so by Proposition 1.6 the corresponding linear transformation P_1 is the (orthogonal) projection mapping onto the subspace C_1. Similarly, since $M_2^2 = M_2 = M_2^t$, the corresponding linear transformation P_2 is the (orthogonal) projection mapping onto the subspace C_2. Therefore, *Burkholder's procedure is nothing but the basic alternating projection algorithm (Algorithm 1) applied to two closed subspaces C_1 and C_2 of $\mathbb{R}^3$.* By Proposition 2.4, the projection points generated by the algorithm converge to the projection $P_{C_1 \cap C_2}(x)$ of the starting point $x = (u, v, w)^t$ onto the subspace $C_1 \cap C_2$.

We now find an explicit formula for $C_1 \cap C_2$. Since P_1 is the projection onto the closed subspace C_1, we have

$$y = (y_1, y_2, y_3)^t \in C_1 \quad \Leftrightarrow \quad P_1(y) = y \quad \Leftrightarrow \quad \begin{bmatrix} y_1 \\ \frac{y_2+y_3}{2} \\ \frac{y_2+y_3}{2} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad \Leftrightarrow \quad y_2 = y_3.$$

Similarly, since P_2 is the projection onto the closed subspace C_2, we have

$$y \in C_2 \quad \Leftrightarrow \quad P_2(y) = y \quad \Leftrightarrow \quad \begin{bmatrix} \frac{y_1+y_2}{2} \\ \frac{y_1+y_2}{2} \\ y_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad \Leftrightarrow \quad y_1 = y_2.$$

Thus we have

$$y \in C_1 \cap C_2 \quad \Leftrightarrow \quad P_1(y) = P_2(y) = y \quad \Leftrightarrow \quad y_1 = y_2 = y_3,$$

and therefore

$$C_1 \cap C_2 = \{(a, a, a)^t \mid a \in \mathbb{R}\}. \tag{3.1}$$

By (3.1) we have

$$P_{C_1 \cap C_2}(x) = \text{argmin}_{a \in \mathbb{R}} \|(u, v, w) - (a, a, a)\|.$$

Note that

$$\begin{aligned} \|(u, v, w) - (a, a, a)\|^2 &= (u - a)^2 + (v - a)^2 + (w - a)^2 \\ &= 3a^2 - 2(u + v + w)a + (u^2 + v^2 + w^2) =: f(a) \end{aligned}$$

is a convex quadratic function on $a \in R$, and that

$$f'(a) = 6a - 2(u + v + w) = 0 \quad \Leftrightarrow \quad a = \frac{u + v + w}{3},$$

the function f attains its minimum at $a = \frac{u+v+w}{3}$. Hence we obtain

$$P_{C_1 \cap C_2}(x) = \left(\frac{u + v + w}{3}, \frac{u + v + w}{3}, \frac{u + v + w}{3}\right)^t.$$

It means that the lengths of the three parts of the string divided by the two stickers converge to $\frac{u+v+w}{3}$. Equivalently, the positions of the stickers converge to one third and two thirds of the total length of string. $\square$

We did some numerical experiments to have an intuition on how fast are convergences of the stickers' positions. For the experiments, we assume without loss of generality that the total length of the string is 3 meters. Each experiment starts by choosing the initial positions for the two stickers, and reporting the initial lengths u, v, w of the three parts of the string divided by the two stickers (from left to right). Given the initial value of (u, v, w), we performed the iteration process in Burkholder's procedure until the deviation of the left sticker from the one third of the string and the deviation of the right sticker from the two thirds of the string are both less than 10^{-3} m, i.e.,

$$d := \max\{|u-1|, |u+v-2|\} \leq 0.001. \tag{3.2}$$

We implemented the above procedure in MATLAB R2020a and executed on a PC Intel(R) Core(TM) i7-6700HQ CPU 2*2.60GHz, RAM 16GB. Our code is given below. The last line of this code is to print out the following information: the number of iterations needed to achieve the desired deviation of the stickers from their desired positions, the values of u, v, w, and the deviation after terminating the process. The initial values of u, v, w are given in lines 3-5 of the code and can be changed by users.

```
1   % Initial lengths of three parts divided by two stickers from left to right
2   % (total length of the string is 3 meters)
3   u = 0.25;
4   v = 0.25;
5   w = 2.5;
6
7   % Initial maximum deviation of the stickers from their desired positions
8   d = max(abs(u - 1), abs(u + v - 2));
9
10  % Perform Burkholder's procedure until the maximum deviation of the
11  % stickers from their desired positions less than 1mm
12  iter = 0;
13  while d > 0.001
14      % Step 1
15      v = (v + w)/2;
16      w = v;
17      % Step 2
18      u = (u + v)/2;
19      v = u;
20      % Update the deviation value
21      d = max(abs(u - 1), abs(u + v - 2));
22      % Increase the number of iterations
23      iter = iter + 1;
24  end
25
26  % Output after termination
27  fprintf('After %d iterations: u = %f, v = %f, w = %f, d = %f\n', iter, u, v, w,
        d);
```

We tested with 4 following different initial values of $x = (u, v, w)$:

$$x_1 = (0.5, 2, 0.5), \quad x_2 = (0, 1, 2), \quad x_3 = (1.25, 1.75, 0), \quad x_4 = (0.25, 0.25, 2.5).$$

Figure 3.3 shows how the maximum deviation d defined in (3.2) changes with respect to the number of iterations, according to the 4 above initial values.

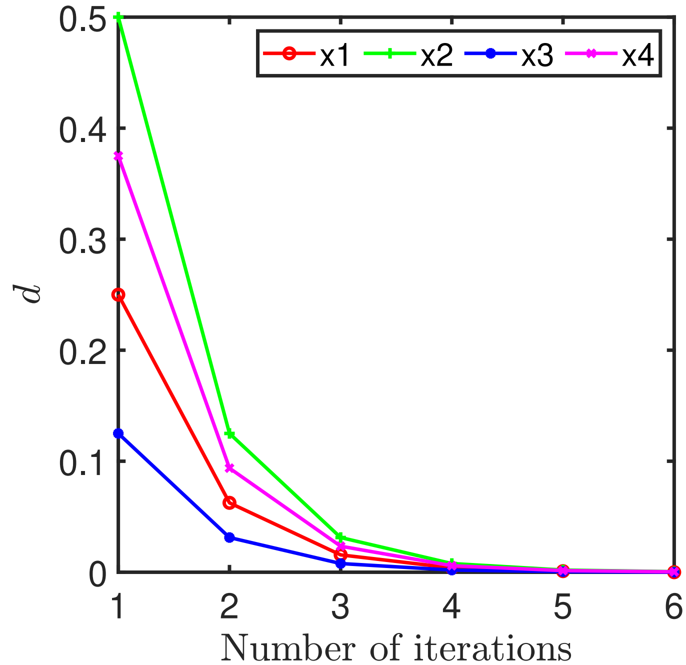

Figure 3.3: Maximum deviation d with respect to the number of iterations and initial values.

We can do some further analysis to know how many iterations are required for achieving the desired maximum deviation d. As proved above, the application of Step 1 in Burkholder's procedure corresponds to the linear transformation $P_1(x) = M_1 x$, while the application of Step 2 in Burkholder's procedure corresponds to the linear transformation $P_2(x) = M_2 x$. Therefore, the application of n iterations in Burkholder's procedure corresponds to the linear transformation

$$(P_2 \circ P_1)^n(x) = (M_2 M_1)^n x = M^n x,$$

in which

$$M = M_2 M_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

The well-known diagonalization procedure in linear algebra applied to M gives us

$$M = QDQ^{-1}$$

in which

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & -2 & 1 \end{bmatrix}, \quad Q^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

Therefore, we have

$$\begin{aligned} M^n x = QD^nQ^{-1}x &= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{4^n} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{3}\frac{1}{4^n} + \frac{1}{3} & \frac{-1}{3}\frac{1}{4^n} + \frac{1}{3} & \frac{-1}{3}\frac{1}{4^n} + \frac{1}{3} \\ \frac{2}{3}\frac{1}{4^n} + \frac{1}{3} & \frac{-1}{3}\frac{1}{4^n} + \frac{1}{3} & \frac{-1}{3}\frac{1}{4^n} + \frac{1}{3} \\ \frac{-4}{3}\frac{1}{4^n} + \frac{1}{3} & \frac{2}{3}\frac{1}{4^n} + \frac{1}{3} & \frac{2}{3}\frac{1}{4^n} + \frac{1}{3} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{3}\frac{1}{4^n}u - \frac{1}{3}\frac{1}{4^n}v - \frac{1}{3}\frac{1}{4^n}w + \frac{1}{3}(u+v+w) \\ \frac{2}{3}\frac{1}{4^n}u - \frac{1}{3}\frac{1}{4^n}v - \frac{1}{3}\frac{1}{4^n}w + \frac{1}{3}(u+v+w) \\ -\frac{4}{3}\frac{1}{4^n}u + \frac{2}{3}\frac{1}{4^n}v + \frac{2}{3}\frac{1}{4^n}w + \frac{1}{3}(u+v+w) \end{bmatrix}. \end{aligned}$$

It means that, after n iterations of Burkholder's procedure, the distance between the left end of the string and the position of the left sticker is

$$\frac{2}{3}\frac{1}{4^n}u - \frac{1}{3}\frac{1}{4^n}v - \frac{1}{3}\frac{1}{4^n}w + \frac{1}{3}(u+v+w),$$

and hence the deviation of the position of the left sticker from the one third of the length of the string is

$$\begin{aligned} d_n^\ell &= \left|\frac{2}{3}\frac{1}{4^n}u - \frac{1}{3}\frac{1}{4^n}v - \frac{1}{3}\frac{1}{4^n}w\right| \\ &= \left|\frac{2}{3}\frac{1}{4^n}u - \frac{1}{3}\frac{1}{4^n}(v+w)\right| \\ &= \left|\frac{2}{3}\frac{1}{4^n}u - \frac{1}{3}\frac{1}{4^n}(c-u)\right| \\ &= \frac{1}{4^n}\left|u - \frac{1}{3}c\right|, \end{aligned}$$

in which $c = u+v+w$ is the given length of the string. Note that $u \in [0, c]$, so d_n^ℓ attains its maximum when $u = c$, at which we have

$$\max_{u\in[0,c]} d_n^\ell = \frac{2}{3 \cdot 4^n}c.$$

Similarly, after n iterations of Burkholder's procedure, the distance between the right end of the string and the position of the right sticker is

$$-\frac{4}{3}\frac{1}{4^n}u+\frac{2}{3}\frac{1}{4^n}v+\frac{2}{3}\frac{1}{4^n}w+\frac{1}{3}(u+v+w),$$

and hence the deviation of the position of the right sticker from the two thirds of the length of the string is

$$\begin{aligned}
d_n^r &= \left|-\frac{4}{3}\frac{1}{4^n}u+\frac{2}{3}\frac{1}{4^n}v+\frac{2}{3}\frac{1}{4^n}w\right| \\
&= \left|\frac{4}{3}\frac{1}{4^n}u-\frac{2}{3}\frac{1}{4^n}v-\frac{2}{3}\frac{1}{4^n}w\right| \\
&= \left|\frac{4}{3}\frac{1}{4^n}u-\frac{2}{3}\frac{1}{4^n}(v+w)\right| \\
&= \left|\frac{4}{3}\frac{1}{4^n}u-\frac{2}{3}\frac{1}{4^n}(c-u)\right| \\
&= \frac{2}{4^n}\left|u-\frac{1}{3}c\right|,
\end{aligned}$$

which attains its maximum when $u=c$, at which we have

$$\max_{u\in[0,c]} d_n^r=\frac{4}{3\cdot 4^n}c.$$

So the maximum deviation of the stickers from their desired positions after n iterations of Burkholder's procedure is

$$d_n=\max\{d_n^\ell,d_n^r\}=\frac{4}{3\cdot 4^n}c=\frac{1}{3\cdot 4^{n-1}}c.$$

If we would like to achieve the maximum deviation of $\varepsilon>0$, then the number n of iterations must satisfy

$$\frac{1}{3\cdot 4^{n-1}}c\le\varepsilon \quad\Leftrightarrow\quad n\ge\left\lceil \log_4\frac{4c}{3\varepsilon}\right\rceil.$$

For instance, when $c=3$ and $\varepsilon=0.001$, we must execute at least $\left\lceil \log_4 \frac{4c}{3\varepsilon}\right\rceil=6$ iterations.

3.2 Completing positive semi-definite matrices

This section is written on the base of Section 4 in [5]. Throughout this section, X is the Euclidean space $\mathbf{S}^n$ of symmetric matrices of size n. On the space X we equip the well-known Frobenius inner product which is defined as follows:

$$\langle U, V\rangle_F = \mathrm{Tr}(UV),$$

in which $U, V \in X$, while $\mathrm{Tr}(UV)$ is the trace of the product of U and V. The induced norm, so-called Frobenius norm, is defined as

$$\|U\|_F = \sqrt{\langle U, U\rangle_F} = \sqrt{\mathrm{Tr}(U^2)} = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} u_{ij}^2},$$

in which $U = (u_{ij})_{n\times n}$. We use notation $\mathbf{S}^n_+$ for the set of positive semi-definite matrices in $\mathbf{S}^n$. More precisely, $U \in \mathbf{S}^n_+$ means that $U \in \mathbf{S}^n$ and $x^t U x \geq 0$ for all $x \in \mathbb{R}^n$. We also use the notation $U \succcurlyeq \mathbf{0}$ to mean $U \in \mathbf{S}^n_+$ if n is specified from the context.

3.2.1 Problem statement and reformulation

We are interested in the positive semi-definite matrix completion problem, which is stated simply as follows.

> *Consider a matrix $\bar{A} \in \mathbf{S}^n$ with some entries are fixed and the others are unspecified. Determine the values of the unspecified entries so that $\bar{A} \in \mathbf{S}^n_+$.*

To be precise, since $\bar{A} \in \mathbf{S}^n$, i.e. $\bar{A}$ is symmetric, it is sufficient to know the information of the upper triangular part of $\bar{A}$ in order to complete the matrix. For that reason, let $I \subsetneq \{1, \ldots, n\} \times \{1, \ldots, n\}$ be the indices of the fixed entries *in the upper triangular part* (including the diagonal) of $\bar{A}$. For each $(i, j) \in I$, let $\overline{a}_{ij}$ be the given value of the entry on the i-th row and the j-th column of $\bar{A}$, and due to the symmetry of $\bar{A}$, $\overline{a}_{ji} = \overline{a}_{ij}$ is the given value for the entry on the j-th row and the i-th column of $\bar{A}$. Our goal is to determine the values for unspecified entries of $\bar{A}$, i.e., entries a_{ij} with $(i, j) \notin I$ and $(j, i) \notin I$, such that $\bar{A}$ is positive semi-definite.

Example 3.2. To illustrate our arguments in the sequel, we take the following matrix from Section 4 in [5]:

$$\bar{A} = \begin{bmatrix} 4 & 3 & ? & 2 \\ 3 & 4 & 3 & ? \\ ? & 3 & 4 & 3 \\ 2 & ? & 3 & 4 \end{bmatrix} \tag{3.3}$$

as an example, in which the question marks stand for the unspecified entries. For this example, we have

$$I = \{(1,1),(1,2),(1,4),(2,2),(2,3),(3,3),(3,4),(4,4)\}.$$

□

We first reformulate the stated problem in a form that is convenient for applying the alternating projection methods. The reformulation bases on the following two observations.

- In the first observation, let D_i be the square matrix of size n whose all entries are 0 except that the i-th entry on the diagonal equals 1, and $U = (u_{ij}) \in X$. Then we have $D_i \in \mathbf{S}^n$ and $\mathrm{Tr}(D_i U) = u_{ii}$.
- In the second observation, for $1 \leq i \neq j \leq n$, let M_{ij} be the square matrix of size n whose all entries are 0 except that the entry on the i-th row and the j-th column, as well as the entry on the j-th row and i-th column, equals 1. Let $U = (u_{ij}) \in X$. Then we have $M_{ij} \in \mathbf{S}^n$ and $\mathrm{Tr}(M_{ij}U) = u_{ij} + u_{ji}$.

Now, let $i \in \{1, \dots, n\}$ such that $(i,i) \in I$. Using the first observation, the fact that $\overline{a}_{ii}$ is the given value for the i-th entry on the diagonal of matrix $\bar{A}$ is equivalent to the following condition

$$\mathrm{Tr}(D_i \bar{A}) = \overline{a}_{ii}.$$

Similarly, let $(i,j) \in I$ with $i \neq j$. Using the second observation, the fact that $\overline{a}_{ij}$ is the given value for the entry on the i-th row and the j-th column of $\bar{A}$ is equivalent to the following condition

$$\mathrm{Tr}(M_{ij}\bar{A}) = \overline{a}_{ij} + \overline{a}_{ji} = 2\overline{a}_{ij}.$$

Therefore, the positive semi-definite matrix completion problem stated above can be reformulated as follows.

(PSDcompletion) *Find* $A \in \mathbf{S}^n$ *such that*

$$\begin{aligned} A &\in \mathbf{S}^n_+, && \\ \mathrm{Tr}(D_i A) &= \overline{a}_{ii} && \forall (i,i) \in I, \\ \mathrm{Tr}(M_{ij} A) &= 2\overline{a}_{ij} && \forall (i,j) \in I, i \neq j. \end{aligned}$$

Example 3.3. For illustration, the positive semi-definite matrix completion problem for the matrix $\bar{A}$ in (3.3) can be stated explicitly as follows.

Find $A \in \mathbf{S}^4$ such that

$$\begin{aligned} A &\in \mathbf{S}^4_+, \\ \mathrm{Tr}(D_i A) &= 4 && \forall i \in \{1,2,3,4\}, \\ \mathrm{Tr}(M_{12} A) &= 6, \\ \mathrm{Tr}(M_{14} A) &= 4, \\ \mathrm{Tr}(M_{23} A) &= 6, \\ \mathrm{Tr}(M_{34} A) &= 6, \end{aligned}$$

in which

$$D_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, D_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, D_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, D_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$M_{12} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, M_{14} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$M_{23} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, M_{34} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

□

In the next subsection, we will show that $(PSDcompletion)$ is in fact a problem of finding a matrix in the intersection of two closed convex sets in $\mathbf{S}^n$, hence we can apply alternating projection method to solve this problem.

3.2.2 Solution approach

The $(PSDcompletion)$ problem can be rewritten in the following general form.

$$(SDPfeasibility) \qquad \textit{Find } A \in \mathbf{S}^n \textit{ such that}$$
$$A \in \mathbf{S}^n_+,$$

$$\operatorname{Tr}(U_i A) = \alpha_i \qquad \forall i = 1, \ldots, m,$$

in which $U_1, \ldots, U_m \in \mathbf{S}^n$ and $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$ are given. Indeed, by choosing U_i to be the matrices D_i and M_{ij}, and choosing α_i to be the corresponding values $\overline{a}_{ii}$ and $2\overline{a}_{ij}$, $(SDPfeasibility)$ becomes $(PSDcompletion)$. As a remark, in the language of Frobenius inner product, $(SDPfeasibility)$ is stated as follows.

$$\begin{aligned} &\text{Find } A \in \mathbf{S}^n \text{ such that} \\ &\quad A \succcurlyeq \mathbf{0}, \\ &\langle U_i, A \rangle_F = \alpha_i \qquad \forall i = 1, \ldots, m, \end{aligned}$$

which gives us an intuition that this problem is nothing but the problem of finding a feasible solution to a semi-definite program. The following two lemmas exploit special structures of $(SDPfeasibility)$.

Lemma 3.4. (see [5]) *The set* $C_1 := \mathbf{S}^n_+$ *is a closed convex subset of* $\mathbf{S}^n$*.*

Proof. The fact $\mathbf{S}^n_+ \subsetneq \mathbf{S}^n$ is trivial by definition of these two sets. We will prove the convexity and closedness of $\mathbf{S}^n_+$.

Convexity. Let $U, V \in \mathbf{S}^n_+$ and $\lambda \in [0, 1]$. Let $W = \lambda U + (1 - \lambda)V$. Since U and V are symmetric, we have

$$W^t = \lambda U^t + (1 - \lambda)V^t = \lambda U + (1 - \lambda)V = W,$$

which means that $W \in \mathbf{S}^n$. Furthermore, since U and V are positive semi-definite, for any $x \in \mathbb{R}^n$ we have

$$x^t U x \geq 0 \quad \text{and} \quad x^t V x \geq 0,$$

which, together with the facts that $\lambda \geq 0$ and $1 - \lambda \geq 0$, implies

$$x^t W x = x^t(\lambda U + (1 - \lambda)V)x = \lambda x^t U x + (1 - \lambda)x^t V x \geq 0.$$

Hence $W \succcurlyeq \mathbf{0}$, which, together with the fact that $W \in \mathbf{S}^n$, leads to $W \in \mathbf{S}^n_+$. This proves the convexity of $\mathbf{S}^n_+$.

Closedness. For each fixed $x \in \mathbb{R}^n$, the mapping

$$\begin{aligned} f_x : \mathbf{S}^n &\to \mathbb{R} \\ U &\mapsto x^t U x \end{aligned}$$

is a linear transformation. Indeed, for $U, V \in \mathbf{S}^n$ and $\lambda, \mu \in \mathbb{R}$ we have

$$f_x(\lambda U + \mu V) = x^t(\lambda U + \mu V)x = \lambda x^t U x + \mu x^t V x = \lambda f_x(U) + \mu f_x(V),$$

which means the linearity of f_x. Since the space $\mathbf{S}^n$ is of finite dimension (in fact its dimension is $n(n+1)/2$), f_x is continuous. Therefore, the set

$$f_x^{-1}([0,+\infty)) = \{U \in \mathbf{S}^n \mid x^t U x \geq 0\}$$

is closed, since it is the preimage of the closed set $[0,+\infty)$ via the continuous mapping f_x. Consequently, we have

$$\mathbf{S}^n_+ = \{U \in \mathbf{S}^n \mid x^t U x \geq 0 \ \forall x \in \mathbb{R}^n\} = \cap_{x \in \mathbb{R}^n} \{U \in \mathbf{S}^n \mid x^t U x \geq 0\}$$

is closed, since it is intersection of closed sets in $\mathbf{S}^n$. □

Lemma 3.5. (see [5]) *The set* $C_2 := \{A \in \mathbf{S}^n \mid \mathrm{Tr}(U_i A) = \alpha_i, i = 1, \ldots, m\}$ *is a closed convex set in* $\mathbf{S}^n$*.*

Proof. This lemma can be viewed as an immediate consequence of Proposition 1.7 by taking $X = \mathbf{S}^n$, the inner product is Frobenius one, and $a_i := U_i$ for $i = 1, \ldots, m$. □

With the sets C_1, C_2 defined respectively in Lemma 3.4 and Lemma 3.5, the problem $(SDPfeasibility)$ can be reformulated equivalently as the problem of finding $A \in C_1 \cap C_2$. As proved in these two lemmas, C_1 and C_2 are closed convex subsets of the space $\mathbf{S}^n$. Therefore, we can apply the basic alternating projection algorithm (Algorithm 1) to solve this problem. Precisely, in the current setting, the algorithm reads as follows.

Algorithm 7 Alternating projection algorithm for solving $(SDPfeasibility)$

Input: Two closed convex subsets C_1 and C_2 of $X = \mathbf{S}^n$ defined in Lemma 3.4 and Lemma 3.5.
Output: A matrix in $C_1 \cap C_2$, if exists.

```
1: Take an initial matrix A_0 ∈ X.
2: V_0 = P_{C_1}(A_0).
3: for k = 0, 1, 2, ... do
4:     W_k = P_{C_2}(V_k)
5:     V_{k+1} = P_{C_1}(W_k)
6: end for
```

In the setting of Algorithm 7, Theorem 2.1 verifies that the sequences $(V_k)_{k\geq 0}$ and $(W_k)_{k \geq 0}$ generated by this algorithm converge to a matrix A^* in $C_1 \cap C_2$ if the intersection is nonempty, in the sense that

$$\|V_k - A^*\|_F \to 0 \quad \text{and} \quad \|W_k - A^*\|_F \to 0.$$

From computational point of view, there are three issues concerning the above algorithm.

- Issue 1: how to choose the initial matrix A_0.
- Issue 2: how to compute the projection of an iterate on C_1.
- Issue 3: how to compute the projection of an iterate on C_2.

The first issue can be easily solved by simply initialize with $A_0 = \bar{A}$, taking the unspecified entries as 0. Here, $\bar{A}$ is the matrix in the statement of the positive semi-definite matrix completion problem. The last issue has been already solved thanks to Theorem 1.9. Indeed, concerning this issue, we have the following result.

Proposition 3.6. (see [5]) *The projection of the iterate V_k onto the closed convex set C_2 is given by*

$$P_{C_2}(V_k) = V_k - \sum_{i=1}^{m} \beta_i U_i, \tag{3.4}$$

in which the coefficients $\beta_1, \ldots, \beta_m$ are found form

$$\begin{bmatrix} \mathrm{Tr}(U_1U_1) & \mathrm{Tr}(U_1U_2) & \ldots & \mathrm{Tr}(U_1U_m) \\ \mathrm{Tr}(U_2U_1) & \mathrm{Tr}(U_2U_2) & \ldots & \mathrm{Tr}(U_2U_m) \\ \vdots & \vdots & \ldots & \vdots \\ \mathrm{Tr}(U_mU_1) & \mathrm{Tr}(U_mU_2) & \ldots & \mathrm{Tr}(U_mU_m) \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{bmatrix} = \begin{bmatrix} \mathrm{Tr}(U_1V_k) - \alpha_1 \\ \mathrm{Tr}(U_2V_k) - \alpha_2 \\ \vdots \\ \mathrm{Tr}(U_mV_k) - \alpha_m \end{bmatrix}. \tag{3.5}$$

Proof. This proposition can be viewed as an immediate consequence of Theorem 1.9 by taking $X = \mathbf{S}^n$, the inner product is Frobenius one, $a_i := U_i$ for $i = 1, \ldots, m$, the set $C := C_2$, and $\overline{x} := V_k$. □

The following example continues Example 3.3 and illustrates the above proposition.

Example 3.7. For the positive semi-definite matrix completion problem which is reformulated in Example 3.3, we can set $U_1 := D_1$, $U_2 := D_2$, $U_3 := D_3$, $U_4 := D_4$, $U_5 := M_{12}$, $U_6 := M_{14}$, $U_7 := M_{23}$, $U_8 := M_{34}$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 := 4$, $\alpha_5 := 6$, $\alpha_6 := 4$, $\alpha_7 := 6$, $\alpha_8 := 6$. So, for $V_k = (v_{ij})_{4\times 4} \in \mathbf{S}^4$ being an iterate of

Algorithm 7 applied to this problem, the equation (3.5) becomes

$$\begin{bmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&2&0&0&0\\0&0&0&0&0&2&0&0\\0&0&0&0&0&0&2&0\\0&0&0&0&0&0&0&2\end{bmatrix}\begin{bmatrix}\beta_1\\\beta_2\\\beta_3\\\beta_4\\\beta_5\\\beta_6\\\beta_7\\\beta_8\end{bmatrix}=\begin{bmatrix}v_{11}-4\\v_{22}-4\\v_{33}-4\\v_{44}-4\\v_{21}+v_{12}-6\\v_{41}+v_{14}-4\\v_{32}+v_{23}-6\\v_{43}+v_{34}-6\end{bmatrix}.$$

Note that $v_{12}=v_{21}$, $v_{14}=v_{41}$, $v_{23}=v_{32}$, $v_{34}=v_{43}$ since V_k is symmetric, therefore this equation has the following unique solution

$$\beta_1=v_{11}-4,\quad \beta_2=v_{22}-4,\quad \beta_3=v_{33}-4,\quad \beta_4=v_{44}-4,$$

$$\beta_5=v_{12}-3,\quad \beta_6=v_{14}-2,\quad \beta_7=v_{23}-3,\quad \beta_8=v_{34}-3.$$

Therefore, in the setting of the problem in Example 3.3, the formula (3.4) can be written more explicitly as follows:

$$\begin{aligned}P_{C_2}(V_k)&=V_k-\sum_{i=1}^{8}\beta_i U_i\\&=(v_{ij})_{4\times 4}-(v_{11}-4)D_1-(v_{22}-4)D_2-(v_{33}-4)D_3-(v_{44}-4)D_4\\&\quad-(v_{12}-3)M_{12}-(v_{14}-2)M_{14}-(v_{23}-3)M_{23}-(v_{34}-3)M_{34}\\&=\begin{bmatrix}v_{11}&v_{12}&v_{13}&v_{14}\\v_{12}&v_{22}&v_{23}&v_{24}\\v_{13}&v_{23}&v_{33}&v_{34}\\v_{14}&v_{24}&v_{34}&v_{44}\end{bmatrix}-\begin{bmatrix}v_{11}-4&v_{12}-3&v_{13}&v_{14}-2\\v_{12}-3&v_{22}-4&v_{23}-3&v_{24}\\v_{13}&v_{23}-3&v_{33}-4&v_{34}-3\\v_{14}-2&v_{24}&v_{34}-3&v_{44}-4\end{bmatrix}\\&=\begin{bmatrix}4&3&v_{13}&2\\3&4&3&v_{24}\\v_{13}&3&4&3\\2&v_{24}&3&4\end{bmatrix}.\end{aligned}$$

From the above computation, it is worth noting that $P_{C_2}(V_k)$ has the same entries as the fixed entries of the given matrix $\bar{A}$. In other words, the projection onto C_2 is extremely easy: we simply set the corresponding fixed entries of the iterate back to the fixed values of the original matrix $\bar{A}$. $\square$

It is left to consider Issue 3 mentioned above. The following two lemmas will be useful for our discussion on this issue. They can be considered as simple exercises in Linear Algebra.

Lemma 3.8. *Let $Z = (z_{ij})_{n\times n} \in \mathbf{S}^n_+$. Then all diagonal entries of Z are non-negative.*

Proof. For each $i = 1, \dots, n$, let x^i be the i-th unit vector in $\mathbb{R}^n$. More precisely, the i-th entry of x^i equals 1 while the other entries are 0. Since $Z \in \mathbf{S}^n_+$, we must have $x^tZx \geq 0$ for all $x \in \mathbb{R}^n$. By choosing $x = x^i$, we have

$$0 \leq (x^i)^t Z x^i = z_{ii}.$$

Therefore, all diagonal entries of Z are non-negative. □

Lemma 3.9. *Let*

$$Z = \begin{bmatrix} Z_1 & Z_3 \\ Z_3^t & Z_2 \end{bmatrix}$$

be a matrix in $\mathbf{S}^n$, in which $Z_1 \in \mathbb{R}^{\ell\times\ell}$ and $Z_2 \in \mathbb{R}^{(n-\ell)\times(n-\ell)}$ are symmetric, $Z_3 \in \mathbb{R}^{\ell\times(n-\ell)}$ for some $\ell \in \{1, \dots, n\}$. Then we have

$$\|Z\|_F^2 = \|Z_1\|_F^2 + \|Z_2\|_F^2 + 2\mathrm{Tr}(Z_3Z_3^t).$$

Proof. In this proof we denote for simplicity by $\mathbf{0}$ the matrix whose all entries are 0 and whose size is specified from the context. We have

$$\begin{aligned}
&\|Z\|_F^2 \\
=& \left\| \begin{bmatrix} Z_1 & Z_3 \\ Z_3^t & Z_2 \end{bmatrix} \right\|_F^2 \\
=& \left\| \begin{bmatrix} Z_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & Z_2 \end{bmatrix} + \begin{bmatrix} \mathbf{0} & Z_3 \\ Z_3^t & \mathbf{0} \end{bmatrix} \right\|_F^2 \\
=& \left\| \begin{bmatrix} Z_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right\|_F^2 + \left\| \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & Z_2 \end{bmatrix} \right\|_F^2 + \left\| \begin{bmatrix} \mathbf{0} & Z_3 \\ Z_3^t & \mathbf{0} \end{bmatrix} \right\|_F^2 \\
&+2 \left\langle \begin{bmatrix} Z_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & Z_2 \end{bmatrix} \right\rangle_F + 2 \left\langle \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & Z_2 \end{bmatrix}, \begin{bmatrix} \mathbf{0} & Z_3 \\ Z_3^t & \mathbf{0} \end{bmatrix} \right\rangle_F + 2 \left\langle \begin{bmatrix} \mathbf{0} & Z_3 \\ Z_3^t & \mathbf{0} \end{bmatrix}, \begin{bmatrix} Z_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right\rangle_F .
\end{aligned}$$

Let us compute in detail each term in the last sum. For the first term we have

$$\left\| \begin{bmatrix} Z_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right\|_F^2 = \mathrm{Tr}\left(\begin{bmatrix} Z_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} Z_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right) = \mathrm{Tr}\left(\begin{bmatrix} Z_1^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right) = \mathrm{Tr}(Z_1^2) = \|Z_1\|_F^2.$$

Similarly, for the second term we have

$$\left\| \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & Z_2 \end{bmatrix} \right\|_F^2 = \mathrm{Tr}\left(\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & Z_2 \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & Z_2 \end{bmatrix} \right) = \mathrm{Tr}\left(\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & Z_2^2 \end{bmatrix} \right) = \mathrm{Tr}(Z_2^2) = \|Z_2\|_F^2.$$

For the third term we have

$$\begin{aligned} \left\| \begin{bmatrix} \mathbf{0} & Z_3 \\ Z_3^t & \mathbf{0} \end{bmatrix} \right\|_F^2 &= \mathrm{Tr}\left(\begin{bmatrix} \mathbf{0} & Z_3 \\ Z_3^t & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} & Z_3 \\ Z_3^t & \mathbf{0} \end{bmatrix} \right) \\ &= \mathrm{Tr}\left(\begin{bmatrix} Z_3 Z_3^t & \mathbf{0} \\ \mathbf{0} & Z_3^t Z_3 \end{bmatrix} \right) = \mathrm{Tr}(Z_3 Z_3^t) + \mathrm{Tr}(Z_3^t Z_3). \end{aligned}$$

For the fourth term we have

$$\left\langle \begin{bmatrix} Z_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & Z_2 \end{bmatrix} \right\rangle_F = \mathrm{Tr}\left(\begin{bmatrix} Z_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & Z_2 \end{bmatrix} \right) = \mathrm{Tr}\left(\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right) = 0.$$

For the fifth term we have

$$\left\langle \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & Z_2 \end{bmatrix}, \begin{bmatrix} \mathbf{0} & Z_3 \\ Z_3^t & \mathbf{0} \end{bmatrix} \right\rangle_F = \mathrm{Tr}\left(\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & Z_2 \end{bmatrix} \begin{bmatrix} \mathbf{0} & Z_3 \\ Z_3^t & \mathbf{0} \end{bmatrix} \right) = \mathrm{Tr}\left(\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ Z_2 Z_3^t & \mathbf{0} \end{bmatrix} \right) = 0.$$

Similarly, for the sixth term we have

$$\left\langle \begin{bmatrix} \mathbf{0} & Z_3 \\ Z_3^t & \mathbf{0} \end{bmatrix}, \begin{bmatrix} Z_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right\rangle_F = \mathrm{Tr}\left(\begin{bmatrix} \mathbf{0} & Z_3 \\ Z_3^t & \mathbf{0} \end{bmatrix} \begin{bmatrix} Z_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right) = \mathrm{Tr}\left(\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ Z_3^t Z_1 & \mathbf{0} \end{bmatrix} \right) = 0.$$

In summary, we obtain

$$\|Z\|_F^2 = \|Z_1\|_F^2 + \|Z_2\|_F^2 + \mathrm{Tr}(Z_3 Z_3^t) + \mathrm{Tr}(Z_3^t Z_3).$$

It is a trivial property of trace function that $\mathrm{Tr}(Z_3 Z_3^t) = \mathrm{Tr}(Z_3^t Z_3)$. Thus, in conclusion, we have

$$\|Z\|_F^2 = \|Z_1\|_F^2 + \|Z_2\|_F^2 + 2\mathrm{Tr}(Z_3 Z_3^t).$$

$\square$

The following proposition gives us an answer for the second issue mentioned above. Before stating the proposition, note that W_k is an iterate of Algorithm 7, so it is a real symmetric matrix. As well-known from linear algebra, all eigenvalues of W_k are real numbers. Let $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$ be eigenvalues of W_k. Furthermore, W_k admits an orthogonal diagonalization $W_k = QDQ^t$, in which $D = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ is the diagonal matrix whose entries on its diagonal are eigenvalues of W_k, and Q is an orthogonal matrix whose columns are normalized eigenvectors $q_1, \ldots, q_n$ respectively corresponding to these eigenvalues.

Proposition 3.10. (see [5]) *Let*

$$W_k = QDQ^t \tag{3.6}$$

be the orthogonal diagonalization of W_k. Then we have

$$P_{C_1}(W_k) = P_{\mathbf{S}^n_+}(W_k) = QD^+Q^t, \tag{3.7}$$

in which D^+ is obtained from the diagonal matrix D by replacing negative entries by 0.

Proof. If all eigenvalues of W_k are non-negative, then on one hand we have W_k is positive semi-definite, and on the other hand we have $D^+ = D$ by definition of D^+. In this case we have $W_k \in \mathbf{S}^n_+$ and therefore $P_{\mathbf{S}^n_+}(W_k) = W_k$. This adapts (3.7), since $D^+ = D$ and consequently

$$QD^+Q^t = QDQ^t = W_k = P_{\mathbf{S}^n_+}(W_k).$$

It is left to consider the case that not all eigenvalues of W_k are non-negative. Let σ_ℓ is the smallest eigenvalue of W_k that is non-negative, then we have $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_\ell \geq 0$ and $0 > \sigma_{\ell+1} \geq \ldots \geq \sigma_n$. By definition, the projection point $P_{\mathbf{S}^n_+}(W_k)$ is the unique solution to the following problem

$$\min_{Y \in \mathbf{S}^n_+} \|W_k - Y\|_F,$$

which is equivalent to

$$\min_{Y \in \mathbf{S}^n_+} \|W_k - Y\|^2_F.$$

Consider the objective function in the latter problem. Since Q is an orthogonal matrix, we have $QQ^t = Q^tQ = E_n$ which is the identity matrix of size n, and therefore we obtain

$$\begin{aligned}
\|W_k - Y\|^2_F &= \|QDQ^t - Y\|^2_F \\
&= \|QDQ^t - QQ^tYQQ^t\|^2_F \\
&= \|Q(D - Q^tYQ)Q^t\|^2_F \\
&= \mathrm{Tr}\left(Q(D - Q^tYQ)Q^t \cdot Q(D - Q^tYQ)Q^t\right) \\
&= \mathrm{Tr}\left(Q(D - Q^tYQ)^2Q^t\right) \\
&= \mathrm{Tr}\left(QQ^t(D - Q^tYQ)^2\right) \\
&= \mathrm{Tr}\left((D - Q^tYQ)^2\right) \\
&= \|D - Q^tYQ\|^2_F
\end{aligned}$$

$$= \|Q^tYQ - D\|_F^2. \tag{3.8}$$

For convenience, we decompose the diagonal matrix D into blocks as follows:

$$D = \begin{bmatrix} D_1 & \mathbf{0} \\ \mathbf{0} & D_2 \end{bmatrix},$$

in which $D_1 = \text{diag}(\sigma_1, \ldots, \sigma_\ell)$ is the diagonal matrix whose entries on its diagonal are non-negative eigenvalues of W_k, and $D_2 = \text{diag}(\sigma_{\ell+1}, \ldots, \sigma_n)$ is the diagonal matrix whose entries on its diagonal are negative eigenvalues of W_k. We do the similar decomposition to Q^tYQ. For the ease of representation, let $B = (b_{ij})_{n\times n} = Q^tYQ$. Since $Y \in \mathbf{S}_+^n$, so is B. In particular, B is symmetric, therefore we can decompose B into blocks as follows:

$$B = \begin{bmatrix} B_1 & B_3 \\ B_3^t & B_2 \end{bmatrix},$$

in which $B_1 \in \mathbb{R}^{\ell\times\ell}$ and $B_2 \in \mathbb{R}^{(n-\ell)\times(n-\ell)}$ are symmetric, $B_3 \in \mathbb{R}^{\ell\times(n-\ell)}$. It is worth to remind that, by Lemma 3.8, the diagonal entries of B_1 and B_2 are non-negative.

Return to (3.8), we have

$$\begin{aligned} \|W_k - Y\|_F^2 &= \|Q^tYQ - D\|_F^2 \\ &= \|B - D\|_F^2 \\ &= \left\| \begin{bmatrix} B_1 - D_1 & B_3 \\ B_3^t & B_2 - D_2 \end{bmatrix} \right\|_F^2 \\ &= \|B_1 - D_1\|_F^2 + \|B_2 - D_2\|_F^2 + 2\text{Tr}(B_3B_3^t). \end{aligned} \tag{3.9}$$

The last equality is due to Lemma 3.9. In the last sum above, the first term is with respect to B_1, the second term is with respect to B_2, and the last term is with respect to B_3. So we can minimize these terms independently.

Concerning the first term, we have

$$\|B_1 - D_1\|_F^2 = \text{Tr}\left((B_1 - D_1)^2\right) = \sum_{i=1}^{\ell}(b_{ii} - \sigma_i)^2 + \sum_{\substack{i,j\in\{1,\ldots,\ell\} \\ i\neq j}} b_{ij}^2.$$

We aim to minimize this term with respect to B_1, given the facts that both $D_1 = \text{diag}(\sigma_1, \ldots, \sigma_\ell)$ and B_1 have non-negative entries on their diagonals. It is trivial to see that the minimum is attained when $b_{ii} = \sigma_i$ for all $i = 1, \ldots, \ell$

and $b_{ij} = 0$ for all $i, j \in \{1, \dots, \ell\}$ with $i \neq j$. Equivalently, the first term attains its minimum when $B_1 = D_1$.

Concerning the second term, we have

$$\|B_2 - D_2\|_F^2 = \operatorname{Tr}\left((B_2 - D_2)^2\right) = \sum_{i=\ell+1}^{n} (b_{ii} - \sigma_i)^2 + \sum_{\substack{i,j \in \{\ell+1,\dots,n\} \\ i \neq j}} b_{ij}^2.$$

We aim to minimize this term with respect to B_2, given the facts that B_2 has non-negative diagonal entries, and that $D_2 = \operatorname{diag}(\sigma_{\ell+1}, \dots, \sigma_n)$ has negative entries on its diagonal. It is trivial to see that the minimum is attained when $b_{ii} = 0$ for all $i = \ell + 1, \dots, n$ and $b_{ij} = 0$ for all $i, j \in \{\ell + 1, \dots, n\}$ with $i \neq j$. Equivalently, the second term attains its minimum when $B_2 = \mathbf{0}$.

Concerning the third term, we have

$$\operatorname{Tr}(B_3 B_3^t) = \sum_{\substack{i=1,\dots,\ell \\ j=\ell+1,\dots,n}} b_{ij}^2.$$

So it is trivial to see that the third term attains it minimum when $b_{ij} = 0$ for all $i = 1, \dots, \ell$ and $j = \ell + 1, \dots, n$, or in short, when $B_3 = \mathbf{0}$.

Combining the above arguments and (3.9), we conclude that $\|W_k - Y\|_F^2$ attains its minimum when

$$Q^t Y Q = B = \begin{bmatrix} D_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = D^+.$$

The last equality is due to the definition of D^+. Equivalently, we have $Y = QD^+Q^t \in \mathbf{S}_+^n$ minimizes $\|W_k - Y\|_F^2$, so (3.7) is verified. $\square$

3.2.3 Numerical experiments

To see how Algorithm 7 performs, we tested it on the positive semi-definite matrix completion problem in Example 3.2, which is reformulated in Example 3.3. We implemented this algorithm by using MATLAB R2020a and executed on a PC Intel(R) Core(TM) i7-6700HQ CPU 2*2.60GHz, RAM 16GB. Our MATLAB code is given below. In our code, we compute in each iteration the value

$$d_k^1 = \|W_k - V_{k+1}\|_F,$$

which is the distance of the iterate W_k to the set $C_1 = \mathbf{S}_+^n$, and the value

$$d_k^2 = \|V_k - W_k\|_F,$$

which is the distance of the iterate V_k to the set C_2 defined in Lemma 3.5. We terminated the MATLAB program when both distances d_k^1 and d_k^2 are less than 0.001. The last line of our code is to print out the last iterate before terminating the program.

```
% (Symmetric) matrix to be completed as a positive semi-definite one.
% Zeros stand for unspecified entries.
% This matrix is also the initial point to start alternating projections.
Abar = [4 3 0 2; 3 4 3 0; 0 3 4 3; 2 0 3 4];

% Open file to record the computational results
fID = fopen('.\PSDcompletion.dat', 'w');

%% First iteration
% Initiate counting the number of iterations
iter = 1;
% Projection onto the positive semi-definite cone
[P, D] = eig(Abar);       % Orthogonal diagonalization of iterate Abar
Dplus = max(D, 0);        % Set nagative eigenvalues to 0
Vk = P * Dplus * P';      % Projection point of Abar onto the cone
% Projection onto the intersection of hyperplanes
Wk = Vk;
for i = 1:4
    for j = 1:4
        if Abar(i,j) > 0
            Wk(i,j) = Abar(i,j);
        end
    end
end
% Distance of each iterate to its corresponding projection set
dC1 = norm(Abar - Vk, 'fro');
dC2 = norm(Vk - Wk, 'fro');
% Print necessary information to file
    fprintf(fID, '%d  %f  %f \n', iter, dC1, dC2);

%% Alternating projections until the maximum distance of iterates
%% to their corresponding projection sets is less than 0.001
while or(dC1 > 0.001, dC2 > 0.001)
    % Record the order of current iteration
    iter = iter + 1;
    % Projection onto the positive semi-definite cone
    [P, D] = eig(Wk);           % Orthogonal diagonalization of iterate Wk
    Dplus = max(D, 0);          % Set nagative eigenvalues to 0
    Vk = P * Dplus * P';        % Projection point of Abar onto the cone
    dC1 = norm(Wk - Vk, 'fro'); % Distance from iterate Wk to the cone
    % Projection onto the intersection of hyperplanes
    Wk = Vk;
    for i = 1:4
        for j = 1:4
            if Abar(i,j) > 0
                Wk(i,j) = Abar(i,j);
            end
```

```
            end
        end
        % Distance from Vk to the intersection of hyperplanes
        dC2 = norm(Vk - Wk, 'fro');
        % Print necessary information to file
        fprintf(fID, '%d  %f  %f \n', iter, dC1, dC2);
    end

    %% Print the last iterate before terminating the program
    Wk
```

After 30 iterations, our MATLAB program above printed out the following matrix

$$\begin{bmatrix} 4.0000 & 3.0000 & 1.5851 & 2.0000 \\ 3.0000 & 4.0000 & 3.0000 & 1.5851 \\ 1.5851 & 3.0000 & 4.0000 & 3.0000 \\ 2.0000 & 1.5851 & 3.0000 & 4.0000 \end{bmatrix}$$

which means that all unspecified entries of the original matrix $\overline{A}$ given in (3.3) converge to the same value 1.5851. In other words, the above matrix is a positive semi-definite (approximate) completion of the original matrix $\overline{A}$.

To have an intuition on how fast the iterates generated by Algorithm 7 converge to the desired solution, we plot in Figure 3.4 the values of the two distances d_k^1 and d_k^2 with respect to the number of iterations. It can be seen from the figure that the distances quickly converge to 0 in an exponential rate, hence the iterates also converge quickly to the desired solution.

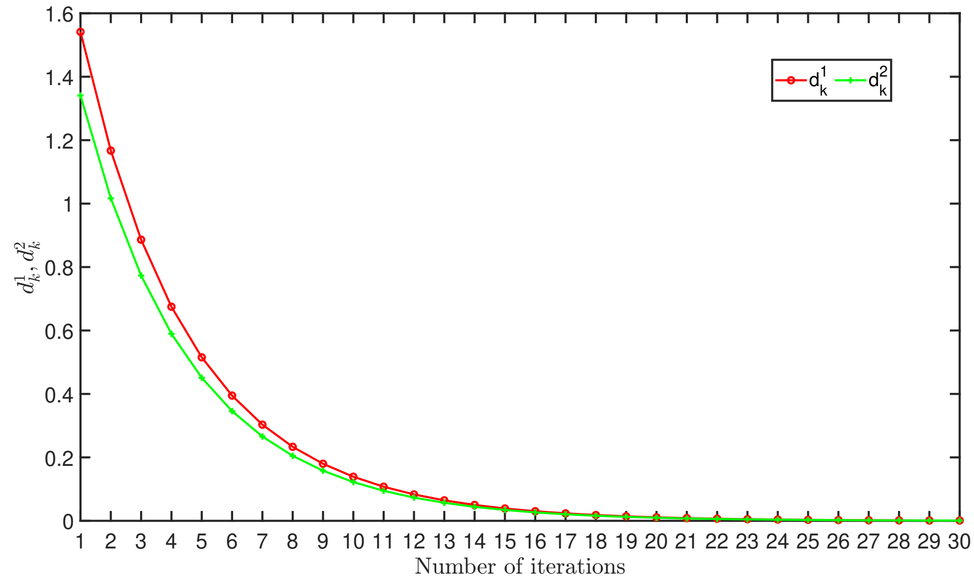

Figure 3.4: Plot of d_k^1 and d_k^2 with respect to the number of iterations.

Conclusions

In this thesis, we have studied the alternating projection method to find a common point of finitely many closed convex sets in a finite dimensional Euclidean space. More precisely, in Chapter 1 we recall some preliminaries about convex sets, projection onto closed convex sets, projection onto subspaces, and projection onto intersection of hyperplanes. In Chapter 2, we first describe the method in the simplest setting with two closed convex sets, and present a detailed proof for the convergence of the method in this setting. Then we describe three variants of the alternating projection method together with detailed proofs for their convergence, namely:

- periodic projection method to find a common point of $m \geq 2$ closed convex sets, in which the key idea is to project sequentially and periodically onto the involved convex sets;
- averaged projection method to find a common point of $m = 2$ closed convex sets (which can be generalized for $m \geq 2$), in which the key idea is to project on each involved convex sets and take the average of the projection points to be the next iterate;
- relaxed projection method to find a common point of $m = 2$ closed convex sets (which can also be generalized for $m \geq 2$), in which the key idea is to replace the usual projection by under projection or over projection.

Chapter 3 presents two charming applications of the alternating projection method. Namely, we present the uses of the alternating projection methods in dividing a string into equal thirds and completing positive semi-definite matrices. We also present some numerical experiments for these applications to see the performance and evaluate the efficiency of the method. It is worth noting that the variants and applications of the alternating projection method are not limited to the presented ones in this thesis. Therefore, studying other variants and applications of this method is in our plans in future.

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