MINISTRY OF EDUCATION
AND TRAINING

VIETNAM ACADEMY OF SCIENCE
AND TECHNOLOGY

GRADUATE UNIVERSITY OF SCIENCE AND TECHNOLOGY

DANG QUANG LONG

THE EXISTENCE, UNIQUENESS AND ITERATIVE METHODS FOR SOME NONLINEAR BOUNDARY VALUE PROBLEMS OF ORDINARY DIFFERENTIAL EQUATIONS

DOCTORAL DISSERTATION
ON APPLIED MATHEMATICS

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Hanoi – 2024

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DECLARATION OF AUTHORSHIP

I hereby declare that this thesis was carried out by myself under the guidance and supervision of Prof. Dr.Sc. Nguyen Dong Anh. The results in it are original, genuine and have not been published by any other author. The numerical experiments performed in MATLAB are honest and precise. The joint-authored publications have been granted permission to be used in this thesis by the co-authors.

The author
Dang Quang Long

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The author

List of Notations and Abbreviations

BVP	Boundary value problem
ODE	Ordinary differential equation
IDE	Integro-differential equation
FDE	Functional differential equation
i.e.	That is
TOL	Tolerance
$u'(t)$	The first derivative of the function $u(t)$
$u''(t)$	The second derivative of the function $u(t)$
$u'''(t)$	The third derivative of the function $u(t)$
$\mathbb{R}$	The set of real numbers
$\mathbb{R}^+$	The set of non-negative real numbers
$\mathbb{R}^-$	The set of non-positive real numbers
$\mathbb{R}^n$	Real coordinate space of n-dimension
$C[a,b]$	The space of continuous functions over the interval $[a,b]$
C^n	The space of continuous functions with continuous first n derivatives

List of Figures

List of Tables

Contents

Introduction

Overview of research situation and the necessity of the research

Numerous problems in the fields of mechanics, physics, biology, environment, etc. are reduced to boundary value problems for high order nonlinear ordinary differential equations (ODE), integro-differential equations (IDE) and functional differential equations (FDE). The study of qualitative aspects of these problems such as the existence, uniqueness and properties of solutions, and the methods for finding the solutions always are of interests of mathematicians and engineers. One can find exact solutions of the problems in a very small number of special cases. In general, one needs to seek their approximations by approximate methods, mainly numerical methods. Below we review some important topics in the above field of nonlinear boundary value problems. An important note is that this thesis studies the boundary value problems not at resonance, of which the corresponding homogeneous problems have trivial solution only. Therefore only boundary value problems not at resonance will be mentioned.

a) Existence of solutions and numerical methods for two-point third order nonlinear boundary value problems

High order differential equations, especially third order and fourth order differential equations describe many problems of mechanics, physics and engineering such as bending of beams, heat conduction, underground water flow, thermoelasticity, plasma physics and so on [1–4]. The study of qualitative aspects and solution methods for linear problems, when the equations and boundary conditions are linear, is basically resolved. In recent years, ones draw a great attention to nonlinear differential equations. There are numerous researches on the existence and solution methods for fourth order nonlinear boundary value problems. It is worthy to mention some typical works concerning the existence of solutions and positive solutions, the multiplicity of solutions, and analytical and numerical methods for finding solutions [5–10]. Among the contributions to the study of fourth order nonlinear boundary value problems, there are some results of Vietnamese authors (see [11–14]).

Concerning the not fully or fully third order differential equations

$$u'''(t) = f(t, u(t), u'(t), u''(t)), \quad 0 < t < 1 \tag{0.0.1}$$

there are also many researches. A lot of works studied the existence, uniqueness and positivity of solutions of the problems subject to different boundary conditions. The methods for investigating qualitative aspects of the problems are diverse, among them the monotone technique or method of lower and upper solutions [7, 15–19], the Leray-Schauder continuation principle [20], the fixed point theory on cones [21], etc. It should be said that the above works need an essential assumption that the function $f(t, x, y, z) : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}$ satisfies a Nagumo-type condition [22], or linear growth in

x, y, z at infinity [20], or some complicated conditions, for example, monotone increase in each of x and y [23], or one-sided Lipschitz condition in x for $f = f(t, x)$ [19] and in x, y for $f = f(t, x, y)$ [17]. Sun et al. in [24] investigated the existence of monotone positive solution of the BVP for the simple case $f = f(u(t))$ under difficult to be verified conditions.

Differently from the above approaches to the third order boundary value problems, very recently Kelevedjiev and Todorov [25] using barrier strips type conditions gave sufficient conditions guaranteeing positive or non-negative, monotone, convex or concave solutions.

It should be said that in the mentioned works, no examples of solutions are shown although the sufficient conditions are satisfied and the verification of them is difficult. Therefore, it is desired to overcome the above shortcoming, namely, to construct easily verified sufficient conditions and show examples when these conditions are satisfied and solutions in these examples.

For solving third order linear and nonlinear boundary value problems for the equation (0.0.1) having in mind that the problems under consideration have solutions, there is a great number of methods including analytical and numerical methods. Below we briefly review these methods via some typical works. First we mention some works involving analytical methods. Specifically, in [26] the authors proposed an iterative method based on embedding Green's functions into well-known fixed point iterations, including Picard's and Krasnoselskii–Mann's schemes. The uniform convergence is proved but the method is very difficult to realize because it requires to compute integrals of the product the Green function associated the problem and the function $f(t, u_n(t), u_n'(t), u_n''(t))$ at each iteration. In [27, 28] the Adomian decomposition method and its modification are used. Recently, in 2020, He [29] suggests a simple yet effective way to the third-order ordinary differential equations by the Taylor series technique. In general, for solving the BVPs for nonlinear third order equations numerical methods are widely used. Namely, Al Said et al. [30] solved a third order two point BVP by the use of cubic splines. Noor et al. [31] constructed method of second order accuracy based on quartic splines. Other authors [32, 33] generated finite difference schemes using fourth degree B-spline and quintic polynomial spline for this problem subject to other boundary conditions. El-Danaf [34] constructed a new spline method based on quartic nonpolynomial spline functions that has a polynomial part and a trigonometric part to develop numerical methods for a linear differential equation. Recently, in 2016 Pandey [35] solved the problem for the case $f = f(t, u)$ by the use of quartic polynomial splines. He proved that the convergence of the method is at least $O(h^2)$ for the linear case $f = f(t)$. In the following year, the same author in [36] proposed two difference schemes for the general case $f = f(t, u(t), u'(t), u''(t))$ and also obtained the second order accuracy for the linear case. In 2019, Chaurasia et al. [37] used exponential amalgamation of cubic splines to design a new numerical method of second-order accuracy. *It should be emphasized that all of above mentioned authors only drew attention to the construction of the discrete analogue of the equation* (0.0.1) *associated with some boundary conditions and estimated the error of the obtained solution assuming that the nonlinear system of algebraic equations can be solved by known iterative methods. But the errors arising in the last iterative methods were not taken into account.*

Motivated by the above facts we wish to construct iterative numerical methods of competitive accuracy or more accurate compared with some existing methods, and importantly, to obtain the total error resulting from the error of iterative process at continuous level and the error of discretization of continuous problems at each iteration.

b) Boundary value problems with integral boundary conditions

In recent years, boundary value problems for nonlinear differential equations involving boundary conditions of integral type have attracted attention from many researchers. They constitute an interesting and important class of problems because they arise in many applied fields such as heat conduction, chemical engineering, underground water flow, thermoelasticity, plasma physics and so on. It is worth to mention some works concerning the problems with integral boundary conditions for second order equations such as [38–43]. There are also many papers devoted to the third order and fourth order equations with integral boundary conditions.

Below we mention some works concerning the third order nonlinear equations. The first work we would mention, is of Boucherif et al. [44] in 2009. It is about the problem

$$\begin{aligned} u'''(t) &= f(t, u(t), u'(t), u''(t)), \quad 0 < t < 1, \\ u(0) &= 0, \\ u'(0) - au''(0) &= \int_0^1 h_1(u(s), u'(s))ds, \\ u'(1) + bu''(1) &= \int_0^1 h_2(u(s), u'(s))ds, \end{aligned}$$

where a, b are positive real numbers, f, h_1, h_2 are continuous functions. Based on a priori bounds and a fixed point theorem for a sum of two operators, one a compact operator and the other a contraction, the authors proved the existence of solutions to the problem under complicated conditions on the functions f, h_1, h_2. Independently from the above work, Sun and Li [24] in 2010 considered the problem

$$\begin{aligned} u'''(t) + f(t, u(t), u'(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = 0, \ u'(1) = \int_0^1 g(t)u'(t)dt. \end{aligned}$$

By the use of the Krasnoselskii's fixed point theorem, some sufficient conditions are obtained for the existence and nonexistence of monotone positive solutions to the above mentioned problem.

Next, in 2012 Guo, Liu and Liang [45] studied the boundary value problem with second derivative

$$\begin{aligned} u'''(t) + f(t, u(t), u''(t)) = 0, \quad 0 < t < 1, \\ u(0) = u''(0) = 0, \ u(1) = \int_0^1 g(t)u(t)dt. \end{aligned}$$

They established sufficient conditions for the existence of positive solutions by using the fixed point index theory in a cone and spectral radius of a linear operator. It is a regret that no examples of the functions f and g satisfying the conditions of existence were given.

In another paper, in 2013 Guo and Yang [46] considered a problem with other boundary conditions, namely, the problem

$$\begin{aligned} u'''(t) = f(t, u(t), u'(t)), \quad 0 < t < 1, \\ u(0) = u''(0) = 0, \ u(1) = \int_0^1 g(t)u(t)dt. \end{aligned}$$

Based on the Krasnoselskii fixed-point theorem on cone, the authors obtained the existence of positive solutions of the problem under very complicated and artificial conditions including the growth of the function $f(t,x,y)$.

In 2018, Guendouz et al. [47] studied the problem

$$u'''(t) + f(u(t)) = 0, \quad 0 < t < 1,$$
$$u(0) = u'(0) = 0,\ u(1) = \int_0^1 g(t)u(t)dt.$$

By applying the Krasnoselskii's fixed point theorem on cones they obtained the existence results of positive solutions of the problem. This technique was used also by Benaicha and Haddouchi in [48] for an integral boundary problem for a fourth order nonlinear equation.

Many authors also studied fourth order differential equations with integral boundary conditions (see [48–58]). Below we mention only some typical works. First it is worthy to mention the work of Zhang and Ge [58], where they studied the problem

$$u^{(4)}(t) = w(t)f(t, u(t), u''(t)), \quad 0 < t < 1,$$
$$u(0) = \int_0^1 g(s)u(s)ds,\ u(1) = 0,$$
$$u''(0) = \int_0^1 h(s)u''(s)ds,\ u''(1) = 0,$$

where w may be singular at $t = 0$ and/or $t = 1$, $f \in C([0,1] \times \mathbb{R}^+ \times \mathbb{R}^-, \mathbb{R}^+)$, and $g, h \in L^1[0,1]$ are nonnegative. Using the fixed point theorem of cone expansion and compression of norm type, the authors established the existence and nonexistence of positive solutions.

In 2013, Li et al. [54] studied the fully nonlinear fourth-order problem

$$u^{(4)}(t) = f(t, u(t), u'(t), u''(t), u'''(t)), \quad t \in [0,1],$$
$$u(0) = u'(1) = u'''(1) = 0,\ u''(0) = \int_0^1 h(s, u(s), u'(s), u''(s))ds,$$

where the functions $f \in C([0,1]\times\mathbb{R}^4, \mathbb{R})$, $h \in C([0,1]\times\mathbb{R}^3, \mathbb{R})$ are continuous. By using a fixed point theorem for a sum of two operators, one is completely continuous and the other is a nonlinear contraction, the existence of solutions and monotone positive solutions were established.

Later, in 2015, Lv et al. [55] considered the above problem, which is a simplified form of the problem in [54]

$$u^{(4)}(t) = f(t, u(t), u'(t), u''(t)), \quad t \in [0,1],$$
$$u(0) = u'(1) = u'''(1) = 0,\ u''(0) = \int_0^1 g(s)u''(s)ds,$$

where $f \in C([0,1] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^-, \mathbb{R}^+)$, $g \in C([0,1], \mathbb{R}^+)$. Using the fixed point theorem of cone expansion and compression of norm type, they obtained the existence and nonexistence of concave monotone positive solutions.

It should be emphasized that in all mentioned above works of integral boundary value problems the authors could only show examples of the nonlinear terms satisfying required sufficient conditions, but no exact solutions are shown. Moreover, the known *results are of purely theoretical character concerning the existence of solutions but not methods for finding solutions.*

Therefore, it is needed to give conditions for existence of solutions, to show examples with solutions, and importantly, to construct methods for finding the solutions for integral boundary value problems.

c) Boundary value problems for integro-differential equations

Integro-differential equations describe many phenomena in mechanics, physics, hydromechanics, chemistry, biology, etc. In general, it is impossible to find the exact solutions of the problems involving these equations, especially when they are nonlinear. Hence, a lot of analytical approximation methods and numerical methods have been developed for solving these equations (see [59, 61–69]).

Below, we mention some works concerning the solution methods for integro-differential equations. The first noteworthy one is a recent work of Tahernezhad and Jalilian in 2020 [65], where the authors considered the second order linear problem

$$\begin{aligned} &u''(x) + p(x)u'(x) + q(x)u(x) = f(x) + \int_a^b k(x,t)u(t)dt, \quad a < x < b, \\ &u(a) = \alpha, \ u(b) = \beta, \end{aligned}$$

where $p(x), q(x), k(x,t)$ are sufficiently smooth functions.

Using the exponential spline functions, the authors constructed the numerical solution of the problem and proved that the approximate solution has accuracy $O(h^2)$, where h is the grid size on the computed domain. Before [65] there are interesting works of Chen et al. [60, 69], where the authors used a multiscale Galerkin method for constructing an approximate solution of the above second order problem, for which the convergence rate of the method is two.

Except for the second order integro-differential equations, recently many authors have been interested in integro-differential equations of fourth order appearing in many applications. We first mention the work of Singh and Wazwaz [63]. In this work, the authors developed a technique based on the Adomian decomposition method with the Green's function for designing a series solution of the nonlinear Voltera equation involving the Dirichlet boundary conditions

$$\begin{gathered} u^{(4)}(x) = g(x) + \int_0^x k(x,t)f(u(t))dt, \quad 0 < x < b, \\ u(0) = \alpha_1, \ u'(0) = \alpha_2, \ u(b) = \alpha_3, \ u'(b) = \alpha_4. \end{gathered}$$

Under some conditions the authors proved that the series solution converges as a geometric progression.

For the linear Fredholm integro-differential equation [59]

$$u^{(4)}(x) + \alpha u''(x) + \beta u(x) - \int_a^b K(x,t)u(t)dt = f(x), \quad a < x < b,$$

subject to the above Dirichlet boundary conditions, the difference method and the trapezoidal rule are used to design the corresponding linear system of algebraic equations. A new variant called the Modified Arithmetic Mean iterative method is proposed for solving the latter system, but the error estimate of the method is not obtained.

The boundary value problem for the nonlinear integro-differential equation

$$u^{(4)}(x) - \varepsilon u''(x) - \frac{2}{\pi}\Big(\int_0^\pi |u'(t)|^2 dt\Big) u''(x) = p(x), \quad 0 < x < \pi,$$
$$u(0) = 0,\ u(\pi) = 0,\ u''(0) = 0,\ u''(\pi) = 0$$

was considered in [12, 68], where the authors constructed approximate solutions by the iterative and spectral methods, respectively. Recently, Dang and Nguyen [11] studied the existence and uniqueness of solution and constructed iterative method for finding the solution for the IDE

$$u^{(4)}(x) - M\Big(\int_0^L |u'(t)|^2 dt\Big) u''(x) = f(x, u, u', u'', u'''), \quad 0 < x < L,$$
$$u(0) = 0,\ u(L) = 0,\ u''(0) = 0,\ u''(L) = 0,$$

where $M = M(y) \geq 0$ is a continuous function.

Only three years ago, Wang [66] considered the problem

$$\begin{aligned} u^{(4)}(x) &= f(x, u(x), \int_0^1 k(x,t)u(t)dt),\ 0 < x < 1, \\ u(0) &= 0,\ u(1) = 0,\ u''(0) = 0,\ u''(1) = 0. \end{aligned} \tag{0.0.2}$$

This problem can be seen as a generalization of the linear fourth order problem

$$u^{(4)}(x) + Mu(x) - N\int_0^1 k(x,t)u(t)dt) = p(x),\ 0 < x < 1,$$
$$u(0) = 0,\ u(1) = 0,\ u''(0) = 0,\ u''(1) = 0,$$

where M, N are constants, $p \in C[0,1]$. This linear problem arises from the models for suspension bridges [70, 71], quantum theory [72].

Using the monotone method with a variant of the maximum principle, Wang constructed the sequences of functions, which converge to the extremal solutions of the problem (0.0.2).

From the above reviewed works we see that some integro-differential equations, linear and nonlinear, are studied by different methods. The development of a unified method for investigating both the qualitative and quantitative aspects of extended integro-differential equations is necessary and is of great interest.

d) Boundary value problems for functional differential equations

Functional differential equations have a wide range of applications in sciences and engineering [73]. So, for the last decades they have been extensively studied. There are a lot of works concerning the numerical solution of both initial and boundary value problems for them. There are many solution methods including collocation method [74], iterative methods [75, 76], neural networks [77, 78], and so on. Below we mention some typical results.

First it is worthy to mention the work of Reutskiy in 2015 [74]. In this work, the author considered the linear functional differential equation with proportional delay

$$u^{(n)} = \sum_{j=0}^{J}\sum_{k=0}^{n-1} p^{jk}(x)u^{(k)}(\alpha_j x) + f(x), \quad x \in [0, T]$$

subject to initial or boundary conditions. Here α_j are constants $(0 < \alpha_j < 1)$. The author proposed a method, where the initial equation is replaced by an approximate equation which has an exact analytic solution with a set of free parameters. These free parameters are determined by the use of the collocation procedure. Several examples show the efficiency of the method but theoretically, no error estimates are obtained.

In 2016 Bica et al. [75] considered the problem

$$\begin{aligned} u^{(2p)}(t) &= f(t, u(t), u(\varphi(t))), \quad t \in [a, b], \\ u^{(i)}(a) &= a_i,\ u^{(i)}(b) = b_i, \quad i = \overline{0, p-1} \end{aligned} \tag{0.0.3}$$

where $\varphi : [a, b] \to \mathbb{R},\ a \leq \varphi(t) \leq b, \forall t \in [a, b]$. To solve this problem, the authors constructed successive approximations for the equivalent integral equation by using cubic spline interpolation at each iterative step. For this reason, the authors called the method as the iterated splines method. The authors obtained an error estimate for the approximate solution under very strong conditions and some misunderstanding of smoothness of Green functions. This mistake was corrected in the corrigendum [79] after 5 years from the appearance of [75]. Remark that although in [75] the method was constructed for the general function $\varphi(t)$ but in all numerical examples only the particular case $\varphi(t) = \alpha t$ was considered and the conditions of convergence were not verified. Moreover, it is a regret that in all examples the Lipschitz conditions for the function $f(s, u, v)$ are not satisfied in unbounded domains as required in the conditions (ii) and (iv) [75, page 131].

In 2018, Khuri and Sayfy [76] proposed an iterative method based on Green's function for functional differential equations of arbitrary orders. However, the scope of application of the method is very limited due to the difficulty in calculation of integrals at each iteration.

For solving functional differential equations, beside analytical and numerical methods, in recent years computational intelligence algorithms also are used (see [77, 78]), where feed-forward artificial neural networks are applied. These algorithms in essence are heuristic, therefore no errors estimates are obtained and for achieving the same accuracy as some numerical methods they require large computational efforts .

The further investigation of the existence of solutions for functional differential equations and effective methods for solving them has a great significance. It is why this topic will be one of the tasks of our thesis.

Objectives and contents of the research

The aim of the thesis is to study the existence, uniqueness of solutions and solution methods for some BVPs for high order nonlinear differential, integro-differential and functional differential equations. Specifically, the thesis intends to study the following contents:

Content 1 The existence, uniqueness of solutions and iterative methods for some BVPs for third order nonlinear differential equations.

Content 2 The existence, uniqueness of solutions and iterative methods for some problems for third and fourth order nonlinear differential equations with integral boundary conditions.

Content 3 The existence, uniqueness of solutions and iterative methods for some BVPs for integro-differential and functional differential equations.

Approach and the research method

We shall approach to the above contents from both theoretical and practical points of view, which are the study of qualitative aspects of the existence solutions and construction of numerical methods for finding the solutions. The methodology throughout the thesis is the reduction of BVPs to operator equations in appropriate spaces, the use of fixed point theorems for establishing the existence and uniqueness of solutions and for proving the convergence of iterative methods.

The achievements of the thesis

The thesis achieves the following results:
Result 1 The establishment of theorems on the existence, uniqueness of solutions and positive solutions for third order nonlinear BVPs and the construction of numerical methods for finding the solutions.
These results are published in the two papers [AL1] and [AL2]. Specifically,
- in [AL1] we propose a unified approach to investigate boundary value problems (BVPs) for fully third order differential equations. It is based on the reduction of BVPs to operator equations for the nonlinear terms but not for the functions to be sought as some authors did. By this approach we have established the existence, uniqueness, positivity and monotony of solutions and the convergence of the iterative method for approximating the solutions under some easily verified conditions in bounded domains. These conditions are much simpler and weaker than those of other authors for studying solvability of the problems before by using different methods. Many examples illustrate the obtained theoretical results.
- in [AL2] we establish the existence and uniqueness of solution and propose simple iterative methods on both continuous and discrete levels for a fully third order BVP. We prove that the discrete methods are of second order and third order of accuracy due to the use of appropriate formulas for numerical integration and obtain estimate for total error.
Result 2 The establishment of the existence, uniqueness of solutions and construction

of iterative methods for finding the solutions for nonlinear third and fourth order differential equations with integral boundary conditions. These results are published in the two papers [AL3] and [AL5]. Specifically,
- The work [AL3] is devoted to third order differential equations.
- The work [AL5] concerns fourth order differential equations.

Result 3 The establishment of the existence, uniqueness of solutions and construction of numerical methods for finding the solutions of nonlinear integro-differential equations. The results are published in [AL6].

Result 4 The establishment of the existence, uniqueness of solutions and construction of numerical methods for finding the solutions of nonlinear functional differential equations. The results are published in [AL4].

The obtained results of the thesis are published in the six papers [AL1]-[AL6] (see "List of the works of the author related to the thesis").

Structure of the thesis

Except for "Introduction", "Conclusions" and "References", the thesis contains 4 chapters. In Chapter 1 we recall some auxiliary knowledges. The results of the thesis are presented in Chapters 2, 3 and 4. Namely,

1. Chapter 2 presents the results on the existence, uniqueness of solutions and positive solutions for third order nonlinear BVPs and the construction of numerical methods for finding the solutions.

2. Chapter 3 is devoted to the study of the existence, uniqueness of solutions and construction of iterative methods for finding the solutions for nonlinear third and fourth order differential equations with integral boundary conditions.

3. Chapter 4 presents the results on the existence, uniqueness of solutions and construction of numerical methods for finding the solutions of nonlinear integro-differential equations and functional differential equations.

Chapter 1

Preliminaries

In this chapter we recall some preliminaries on fixed point theorems, Green's functions and quadrature formulas which will be used in the next chapters.

1.1. Some fixed point theorems

1.1.1. Schauder Fixed-Point Theorem

The material of this subsection is taken from [80].

Theorem 1.1.1 (Brouwer Fixed-Point Theorem (1912))**.** Suppose that U is a nonempty, convex, compact subset of $\mathbb{R}^N$, where $N \geq 1$, and that $f : U \to U$ is a continuous mapping. Then f has a fixed point.

A typical example of the Brouwer Fixed-Point Theorem is proof of the existence of solutions of system of nonlinear algebraic equations.

Remark that Brouwer Fixed-Point Theorem is applicable only to continuous mappings in finite dimensional spaces. A generalization of the theorem to infinite dimensional spaces is the Schauder fixed-point theorem.

Definition 1.1.1. Let X and Y be Banach spaces, and $T : D(T) \subseteq X \to Y$ be an operator. T is called compact if and only if:
(i) T is continuous;
(ii) T maps bounded sets into relatively compact sets.

Compact operators play a central role in nonlinear functional analysis. Their importance stems from the fact that many results on continuous operators on $\mathbb{R}^N$ carry over to Banach spaces when "continuous" is replaced by "compact".

Typical examples of compact operators on infinite-dimensional Banach spaces are integral operators with sufficiently regular integrands. Set

$$(Tx)(t) = \int_a^b K(t, s, x(s))ds,$$

$$(Sx)(t) = \int_a^t K(t, s, x(s))ds, \quad \forall t \in [a, b].$$

Suppose

$$K : [a, b] \times [a, b] \times [-R, R] \to \mathbb{K},$$

where $-\infty < a < b < +\infty,\ 0 < R < \infty$ and $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Denote

$$U = \{x \in C([a,b], \mathbb{K}) : \|x\| \le R\},$$

where $\|x\| = \max_{a \le s \le b}$ and $C([a,b], \mathbb{K})$ is the space of continuous maps $x : [a,b] \to \mathbb{K}$. Then the integral operators T and S map U into $C([a,b], \mathbb{K})$ and are compact.

Theorem 1.1.2 (Schauder Fixed-Point Theorem (1930))**.** Let U be a nonempty, closed, bounded, convex subset of a Banach space X, and suppose $T : U \to U$ is a compact operator. Then T has a fixed point.

Corollary 1.1.3 (Alternate Version of the Schauder Fixed-Point Theorem)**.** Let U be a nonempty, compact, convex subset of a Banach space X, and suppose that $T : U \to U$ is a continuous operator. Then T has a fixed point.

The corollary is the direct translation of the Brouwer fixed-point theorem to Banach spaces. The first verison (Theorem 1.1.2) is more frequently used in applications, in which case U is often chosen to be a ball.

1.1.2. Banach Fixed-Point Theorem

We shall determine under which conditions the fixed-point equation

$$x = Tx, \quad x \in M \tag{1.1.1}$$

can be solved using successive approximations

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2, ..., \quad x_0 \in M.$$

The answer is given in the following theorem.

Theorem 1.1.4 (Banach Fixed-Point Theorem (1922) [80])**.** Assume that
(i) we are given an operator $T : M \subset X \to M$;
(ii) M is a closed nonempty set in a complete metric space (X, d);
(iii T is q-contractive, that is,

$$d(Tx, Ty) < qd(x, y) \tag{1.1.2}$$

for all $x, y \in M$ and for a fixed $q, 0 < q < 1$.
Then it follows the conclusions:
(a) Existence and uniqueness: Equation (1.1.1) has exactly one solution, that is, T has exactly one fixed point on M;
(b) Convergence of the iteration: The sequence $x_{n+1} = Tx_n$ of successive approximations converges to the solution, x, for an arbitrary choice of initial point x_0 in M;
(c) Error estimates: For all $n = 0, 1, 2, ...$ we have the a priori error estimate

$$d(x_n, x) \le \frac{q^n}{1-q} d(x_0, x_1).$$

and the a posteriori error estimate

$$d(x_{n+1}, x) \le \frac{q}{1-q} d(x_n, x_{n+1}).$$

(d) Rate of convergence: For all $n = 0, 1, 2, ...$ we have

$$d(x_{n+1}, x) \le qd(x_n, x).$$

Banach Fixed-Point Theorem has many important applications in the qualitative study as well as in approximate solution of nonlinear equations, systems of linear or nonlinear equations, integral equations, differential equations,...

1.2. Green's functions

Green's functions play an important role in the study of existence and uniqueness of boundary value problems for ordinary differential equations.

Consider the linear homogeneous boundary-value problem

$$L[y(x)] \equiv p_0(x)\frac{d^n y}{dx^n} + p_1(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + p_n(x)y = 0, \tag{1.2.1}$$

$$M_i(y(a), y(b)) \equiv \sum_{k=0}^{n-1}\left(\alpha_k^i \frac{d^k y(a)}{dx^k} + \beta_k^i \frac{d^k y(b)}{dx^k}\right) = 0, \quad i = 1, ..., n, \tag{1.2.2}$$

where $p_i(x), i = 0, ..., n$ are continuous functions on (a, b), $p_0(x) \neq 0$ in all points in (a, b) and $(\alpha_k^i)^2 + (\beta_k^i)^2 \neq 0$.

Definition 1.2.1. [83] The function $G(x,t)$ is said to be the Green's function for the boundary value problem (1.2.1)-(1.2.2) if, as a function of its first variable x, it meets the following defining criteria, for any $t \in (a,b)$:
(i) On both intervals $[a,t)$ and $(t,b]$, $G(x,t)$ is a continuous function having continuous derivatives up to n-th order and satisfies the governing equation in (1.2.1) on (a,t) and (t,b), that is:

$$L[G(x,t)] = 0, x \in (a,t); \quad L[G(x,t)] = 0, x \in (t,b).$$

(ii) $G(x,t)$ satisfies the boundary conditions in (1.2.2), that is

$$M_i(G(a,t), G(b,t)) = 0, \quad i = 1, ..., n.$$

(iii) For $x = t$, $G(x,t)$ and all its derivatives up to $(n-2)$ are continuous

$$\lim_{x\to t^+}\frac{\partial^k G(x,t)}{\partial x^k} - \lim_{x\to t^-}\frac{\partial^k G(x,t)}{\partial x^k} = 0, \quad k = 0, ..., n-2.$$

(iv) The $(n-1)$th derivative of $G(x,t)$ is discontinuous when $x = t$, providing

$$\lim_{x\to t^+}\frac{\partial^{n-1} G(x,t)}{\partial x^{n-1}} - \lim_{x\to t^-}\frac{\partial^{n-1} G(x,t)}{\partial x^{n-1}} = -\frac{1}{p_0(t)}.$$

The following theorem specifies the conditions for existence and uniqueness of the Green's function.

Theorem 1.2.1. [83] If the homogeneous boundary-value problem (1.2.1)-(1.2.2) has only a trivial solution, then there exists a unique Green's function associated with the problem.

Consider the linear nonhomogeneous equation

$$L[y(x)] \equiv p_0(x)\frac{d^n y}{dx^n} + p_1(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + p_n(x)y = -f(x), \tag{1.2.3}$$

subject to the homogeneous boundary conditions

$$M_i(y(a), y(b)) \equiv \sum_{k=0}^{n-1}\left(\alpha_k^i \frac{d^k y(a)}{dx^k} + \beta_k^i \frac{d^k y(b)}{dx^k}\right) = 0, \quad i = 1, ...n, \tag{1.2.4}$$

where $p_j(x)$ and the right-hand side term $f(x)$ in (1.2.3) are continous functions, with $p_0(x) \neq 0$ on (a,b) and M_i represent linearly independent forms with constant coefficients.

The following theorem establishes the link between the uniqueness of the solution of (1.2.3)-(1.2.4) and the corresponding homogeneous problem.

Theorem 1.2.2. [83] If the homogeneous boundary-value problem corresponding to (1.2.3)-(1.2.4) has only the trivial solution, then the problem (1.2.3)-(1.2.4) has a unique solution in the form

$$y(x) = \int_a^b G(x,t)f(t)dt,$$

where $G(x,t)$ is the Green's function of the corresponding homogeneous problem.

Let us consider some Green's functions that will later be used in the thesis.

Example 1.2.1. Consider the problem

$$\begin{cases} u'''(x) = \varphi(x), \quad 0 < x < 1, \\ u(0) = u'(0) = u'(1) = 0. \end{cases} \tag{1.2.5}$$

The corresponding Green's function is of the form

$$G(x,t) = \begin{cases} A_1 + A_2 x + A_3 x^2, & 0 \le x \le t \le 1 \\ B_1 + B_2(1-x) + B_3(1-x)^2, & 0 \le t \le x \le 1, \end{cases} \tag{1.2.6}$$

where A_1, A_2, A_3 and B_1, B_2, B_3 are the functions of t. $G(x,t)$ satisfies the condition (i). Because $G(x,t)$ must satisfy the homogeneous boundary conditions in (ii), it follows that

$$A_1 = A_2 = B_2 = 0.$$

Therefore

$$G(x,t) = \begin{cases} A_3 x^2, & 0 \le x \le t \le 1 \\ B_1 + B_3(1-x)^2, & 0 \le t \le x \le 1. \end{cases} \tag{1.2.7}$$

The condition (iii) leads to

$$\begin{cases} B_1 + B_3(1-t)^2 = A_3 t^2 \\ -B_3(1-t) = A_3 t. \end{cases} \tag{1.2.8}$$

From the condition (iv) we have

$$B_3 - A_3 = -1/2. \tag{1.2.9}$$

We can find A_3, B_1, B_3 by solving (1.2.8) and (1.2.9). It follows that

$$A_3 = -\frac{t}{2} + \frac{1}{2}, \quad B_1 = -\frac{t^2}{2} + \frac{t}{2}, \quad B_3 = -\frac{t}{2}.$$

Substitute into (1.2.7) we obtain the Green's function

$$G(x,t) = \begin{cases} x^2(t-1)/2, & 0 \le x \le t \le 1, \\ t(x^2 - 2x + t)/2, & 0 \le t \le x \le 1. \end{cases} \tag{1.2.10}$$

The solution of the problem (1.2.5) can be represented in the form

$$u(x) = -\int_0^1 G(x,t)\varphi(t)dt.$$

Therefore, the solution of the problem with nonhomogeneous boundary conditions

$$\begin{cases} u'''(x) = \varphi(x), \quad 0 < x < 1, \\ u(0) = c_1, u'(0) = c_2, u'(1) = c_3 \end{cases}$$

has the form

$$u(x) = -\int_0^1 G(x,t)\varphi(t)dt + P_2(x)$$

where $P_2(x) = \frac{c_3 - c_2}{2}x^2 + c_2 x + c_1$ is the polynomial of second degree satisfying the boundary conditions $P_2(0) = c_1, P_2'(0) = c_2, P_2'(1) = c_3$.

Example 1.2.2. Consider the problem

$$\begin{cases} u^{(4)}(x) = \varphi(x), \quad 0 < x < 1, \\ u(0) = u''(0) = u(1) = u''(1) = 0. \end{cases} \tag{1.2.11}$$

The corresponding Green's function is of the form

$$G(x,t) = \begin{cases} A_1 + A_2 x + A_3 x^2 + A_4 x^3, & 0 \le x \le t \le 1 \\ B_1 + B_2(1-x) + B_3(1-x)^2 + B_4(1-x)^3, & 0 \le t \le x \le 1, \end{cases} \tag{1.2.12}$$

where A_1, A_2, A_3, A_4 and B_1, B_2, B_3, B_4 are the functions of t. $G(x,t)$ satisfies the condition (i). Because $G(x,t)$ must satisfy the homogeneous boundary conditions in (ii), it follows that

$$A_1 = A_3 = B_1 = B_3 = 0.$$

Therefore

$$G(x,t) = \begin{cases} A_2 x + A_4 x^3, & 0 \le x \le t \le 1 \\ B_2(1-x) + B_4(1-x)^3, & 0 \le t \le x \le 1. \end{cases} \tag{1.2.13}$$

The condition (iii) leads to

$$\begin{cases} B_2(1-t) + B_4(1-t)^3 = A_2 t + A_4 t^3 \\ -B_2 - 3B_4(1-t)^2 = A_2 + 3A_4 t^2 \\ 6B_4(1-t) = 6A_4 t. \end{cases} \tag{1.2.14}$$

From the condition (iv) we have

$$B_4 + A_4 = -1/6. \tag{1.2.15}$$

We can find A_4, B_1, B_2, B_4 by solving (1.2.14) and (1.2.15). It follows that

$$A_2 = \frac{t^3}{6} - \frac{t^2}{2} + \frac{t}{3}, \quad A_4 = \frac{t}{6} - \frac{1}{6},$$

$$B_2 = -\frac{t^3}{6} + \frac{t}{6}, \quad B_4 = -\frac{t}{6}.$$

Substitute into (1.2.13) we obtain the Green's function

$$G(x,t) = \begin{cases} t(x-1)(t^2 - 2x + x^2)/6, & 0 \le t \le x \le 1, \\ x(t-1)(t^2 - 2t + x^2)/6, & 0 \le x \le t \le 1. \end{cases} \tag{1.2.16}$$

The solution of the problem (1.2.11) can be represented in the form

$$u(x) = -\int_0^1 G(x,t)\varphi(t)dt.$$

Therefore, the solution of the problem with nonhomogeneous boundary conditions

$$\begin{cases} u^{(4)}(x) = \varphi(x), \quad 0 < x < 1, \\ u(0) = c_1, u''(0) = c_2, u(1) = c_3, u''(1) = c_4 \end{cases}$$

has the form

$$u(x) = -\int_0^1 G(x,t)\varphi(t)dt + P_3(x)$$

where $P_3(x) = \frac{c_4-c_2}{6}x^3 + \frac{c_2}{2}x^2 + \left(c_3 - c_1 - \frac{c_2}{3} - \frac{c_4}{6}\right)x + c_1$ is the polynomial of third degree satisfying the boundary conditions $P_3(0) = c_1, P_3''(0) = c_2, P_3(1) = c_3, P_3''(1) = c_4$.

1.3. Some quadrature formulas

The material of this section is taken from [84]).

For numerically approximating the definite integral $\int_a^b f(x)dx$, two of the most commonly used quadrature formulas are the Trapezoidal rule and Simpson's rule.

Trapezoidal rule:

Let $f \in C^2[a,b], h = b - a$. Then there exists a point $\xi \in (a,b)$ such that

$$\int_a^b f(x)dx = \frac{h}{2}[f(a) + f(b)] - \frac{1}{12}h^3 f''(\xi).$$

To improve the accuracy of the approximation, divide the interval $[a,b]$ into n subintervals then apply the Trapezoidal rule on each subinterval. This leads to the Composite Trapezoidal rule.

Theorem 1.3.1. Let $f \in C^2[a,b], h = (b-a)/n$, and $x_j = a + jh$, for each $j = 0,1,...,n$. There exists a $\mu \in (a,b)$ for which the Composite Trapezoidal rule for n subintervals can be written with its error term as

$$\int_a^b f(x)dx = \frac{h}{2}\Big[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\Big] - \frac{b-a}{12}h^2 f''(\mu).$$

Briefly,

$$\int_a^b f(x)dx = \frac{h}{2}\Big[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\Big] + O(h^2).$$

Simpson's rule:

Let $f \in C^4[a,b], x_j = a + jh$ for $j = 0,1,2$. Then there exists a $\xi \in (a,b)$ such that

$$\int_a^b f(x)dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] - \frac{h^5}{90}f^{(4)}(\xi).$$

To improve the accuracy of the approximation, divide the interval $[a,b]$ into n subintervals where n is an even number then apply Simpson's rule on each consecutive pair of subintervals. This leads to the Composite Simpson's rule.

Theorem 1.3.2. Let $f \in C^4[a,b]$, n be even, $n = 2m, h = (b-a)/n$, and $x_j = a+jh$, for each $j = 0, 1, ..., n$. There exists a $\mu \in (a,b)$ for which the Composite Simpson's rule for n subintervals can be written with its error term as

$$\int_a^b f(x)dx = \frac{h}{3}\Big[f(a) + 2\sum_{j=1}^{m-1} f(x_{2j}) + 4\sum_{j=1}^{m} f(x_{2j-1}) + f(b)\Big] - \frac{b-a}{180}h^4 f^{(4)}(\mu).$$

Briefly,

$$\int_a^b f(x)dx = \frac{h}{3}\Big[f(a) + 2\sum_{j=1}^{m-1} f(x_{2j}) + 4\sum_{j=1}^{m} f(x_{2j-1}) + f(b)\Big] + O(h^4).$$

Chapter 2

The existence, uniqueness of a solution and an iterative method for two-point third order nonlinear BVPs

2.1. Existence results and a continuous iterative method for third order nonlinear BVPs

2.1.1. Introduction

In this section, we suggest a unified efficient method to study the existence and approximate solutions of BVPs for the nonlinear third order differential equation

$$u'''(t) = f(t, u(t), u'(t), u''(t)), \quad 0 < t < 1 \tag{2.1.1}$$

subject to general boundary conditions

$$\begin{aligned} B_1[u] &= \alpha_1 u(0) + \beta_1 u'(0) + \gamma_1 u''(0) = 0, \\ B_2[u] &= \alpha_2 u(0) + \beta_2 u'(0) + \gamma_2 u''(0) = 0, \\ B_3[u] &= \alpha_3 u(1) + \beta_3 u'(1) + \gamma_3 u''(1) = 0, \end{aligned} \tag{2.1.2}$$

and

$$\begin{aligned} B_1[u] &= \alpha_1 u(0) + \beta_1 u'(0) + \gamma_1 u''(0) = 0, \\ B_2[u] &= \alpha_2 u(1) + \beta_2 u'(1) + \gamma_2 u''(1) = 0, \\ B_3[u] &= \alpha_3 u(1) + \beta_3 u'(1) + \gamma_3 u''(1) = 0, \end{aligned} \tag{2.1.3}$$

such that

$$Rank \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 & 0 & 0 & 0 \\ \alpha_2 & \beta_2 & \gamma_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix} = 3.$$

The boundary conditions (2.1.2) are the general case of those considered in [16, 17, 19, 20, 23], and (2.1.3) are the general case of those in [16, 22]. Notice that the boundary conditions of the form (2.1.3) can be transformed to those of the form (2.1.2) if changing $t = 1 - s$.

To study the problem (2.1.1)-(2.1.2) as well as the problem (2.1.1)-(2.1.3), we use a novel approach based on reducing them to operator equations for the nonlinear terms rather than the functions to be sought. This approach was used to some BVPs for fourth nonlinear equations in previous works [11, 13, 14, 86, 87]. Here, by using this approach we have established the qualitative aspects such as the existence, uniqueness, positivity and monotonicity of solutions and the convergence of the iterative method for finding approximate solutions of the problems (2.1.1)-(2.1.2) under some conditions which are easily verified in bounded domains. These conditions are much simpler and weaker than those in the works of other authors for studying solvability of particular cases of the problems before by using different methods. Many examples illustrate the obtained theoretical results.

2.1.2. Existence results

Since the problem (2.1.1)-(2.1.2) and the problem (2.1.1), (2.1.3) are completely similar, we consider only the first one.

For convenience we rewrite the problem (2.1.1)-(2.1.2) in the form

$$\begin{aligned} &u'''(t) = f(t, u(t), u'(t), u''(t)), \quad 0 < t < 1 \\ &B_1[u] = B_2[u] = B_3[u] = 0, \end{aligned} \tag{2.1.4}$$

where $B_1[u], B_2[u], B_3[u]$ are defined by (2.1.2). We shall associate this problem with an operator equation as follows.
Consider the nonlinear operator A defined on functions $\varphi(x) \in C[0,1]$ by the formula

$$(A\varphi)(t) = f(t, u(t), u'(t), u''(t)), \tag{2.1.5}$$

where $u(t)$ is the solution of the problem

$$\begin{aligned} &u'''(t) = \varphi(t), \quad 0 < t < 1 \\ &B_1[u] = B_2[u] = B_3[u] = 0 \end{aligned} \tag{2.1.6}$$

provided that it has unique solution. It is not hard to verify the following:

Proposition 2.1.1. If $u(t)$ is a solution of the boundary value problem (2.1.4) then the function

$$\varphi(t) = f(t, u(t), u'(t), u''(t))$$

is a fixed point of the operator A defined above by (2.1.5), (2.1.6). Conversely, if the function $\varphi(t)$ is a fixed point of the operator A, i.e., $\varphi(t)$ is a solution of the operator equation

$$A\varphi = \varphi, \tag{2.1.7}$$

then the function $u(t)$ determined from the boundary value problem (2.1.6) solves the problem (2.1.4).

Therefore, the solution of the BVP (2.1.4) is reduced to finding the fixed point of the operator A.

Now return to the problem (2.1.6). Assume that its Green's function is $G(t, s)$. Then the unique solution of the problem may be represented in the form

$$u(t) = \int_0^1 G(t, s)\varphi(s)ds. \tag{2.1.8}$$

By differentiation of both sides of the above formula we obtain

$$u'(t) = \int_0^1 G_1(t, s)\varphi(s)ds, \quad u''(t) = \int_0^1 G_2(t, s)\varphi(s)ds, \tag{2.1.9}$$

where the function $G_1(t,s) = G'_t(t,s)$ is continuous in the domain $Q = [0,1]^2$ and $G_2(t,s) = G'''_{tt}(t,s)$ is continuous in Q except for the line $t = s$.
Next, let

$$\begin{aligned} &\max_{0\le t\le 1}\int_0^1 |G(t,s)|ds = M_0 \\ &\max_{0\le t\le 1}\int_0^1 |G_1(t,s)|ds = M_1, \ \max_{0\le t\le 1}\int_0^1 |G_2(t,s)|ds = M_2. \end{aligned} \tag{2.1.10}$$

Further, for each fixed real number $M > 0$ we define the domain

$$\mathcal{D}_M = \{(t,x,y,z)|\ 0 \le t \le 1,\ |x| \le M_0M,\ |y| \le M_1M,\ |z| \le M_2M\},$$

and by $B[O,M]$ we denote the closed ball of radius M with center 0 in the space of functions continuous in $[0,1]$

$$B[O,M] = \{\varphi \in C[0,1]|\ \|\varphi\| \le M\},$$

where

$$\|\varphi\| = \max_{0\le t\le 1} |\varphi(t)|.$$

Theorem 2.1.2 (Existence of solutions)**.** Assume that there is a number $M > 0$ such that the function $f(t,x,y,z)$ is continuous in the domain $\mathcal{D}_M$ and

$$|f(t,x,y,z)| \le M \tag{2.1.11}$$

$\forall (t,x,y,z) \in \mathcal{D}_M$.

Then, the problem (2.1.4) has a solution $u(t)$ satisfying the estimates

$$|u(t)| \le M_0M,\ |u'(t)| \le M_1M,\ |u''(t)| \le M_2M\ \forall 0 \le t \le 1. \tag{2.1.12}$$

Proof. In view of Proposition 2.1.1, we shall show that the operator A associated with the problem (2.1.4) has a fixed point. To this end, it is easy to verify that $A : B[0,M] \to B[0,M]$. Further, from the compactness of integral operators (2.1.8), (2.1.9) for $\varphi \in C[0,1]$ [88, Sec. 31] and the continuity of the function $f(t,x,y,z)$ it follows that A is a compact operator in the Banach space $C[0,1]$. By the Schauder Fixed Point Theorem [80] A has a fixed point in $B[0,M]$. This fixed point generates the solution of the original problem. The estimates (2.1.12) hold due to the equalities (2.1.8), (2.1.9) and (2.1.10). □

Now assume that the Green's function $G(x,t)$ and its derivative $G_1(x,t)$ have constant signs in the domain $Q = [0,1]^2$. Let us adopt the following convention:
For a function $H(x,t)$ defined and having a constant sign in Q we define

$$\sigma(H) = sign(H(t,s)) = \begin{cases} 1, & \text{if } H(t,s) \ge 0, \\ -1, & \text{if } H(t,s) < 0. \end{cases}$$

To establish the existence of positive solutions of the problem (2.1.1)-(2.1.2) we denote

$$\begin{aligned} \mathcal{D}^+_M = \{(t,x,y,z)|\ &0 \le t \le 1,\ 0 \le x \le M_0M, \\ &0 \le \sigma(G)\sigma(G_1)y \le M_1M,\ |z| \le M_2M\} \end{aligned}$$

and

$$S_M = \{\varphi \in C[0,1]|\ 0 \le \sigma(G)\varphi \le M\}.$$

Theorem 2.1.3 (Existence of positive solution)**.** Assume that there is a number $M > 0$ such that the function $f(t, x, y, z)$ is continuous and

$$0 \leq \sigma(G)f(t, x, y, z) \leq M \quad \forall (t, x, y, z) \in \mathcal{D}_M^+. \tag{2.1.13}$$

Then, the problem (2.1.1),(2.1.2) possesses a monotone nonnegative solution $u(t)$ which satisfies the estimates

$$0 \leq u(t) \leq M_0 M,\ 0 \leq \sigma(G)\sigma(G_1)u'(t) \leq M_1 M,\ |u''(t)| \leq M_2 M. \tag{2.1.14}$$

Furthermore, if $\sigma(G)\sigma(G_1) = 1$ then the problem has a nonnegative and increasing solution, and vice versa, if $\sigma(G)\sigma(G_1) = -1$ then the problem has a nonnegative and decreasing solution.

Moreover, the solution is positive if $f(t, 0, 0, 0) \not\equiv 0$ for $t \in (0, 1)$.

Proof. By replacing $\mathcal{D}_\mathcal{M}$ by $\mathcal{D}_M^+$, $B[0, M]$ by S_M and the condition (2.1.11) by (2.1.13) in the proof of the existence of solution in Theorem 2.1.2, we obtain the existence of monotone nonnegative solution. From the estimates (2.1.14), if $\sigma(G)\sigma(G_1) = 1$ then $u'(t) \geq 0$, which means an increasing solution, and vice versa, if $\sigma(G)\sigma(G_1) = -1$ then the solution is decreasing. Besides, if $f(t, 0, 0, 0) \not\equiv 0$ for $t \in (0, 1)$ then $u = 0$ is not the solution of the problem. Thus, the solution must be positive. □

Theorem 2.1.4 (Existence and uniqueness of solution)**.** Suppose that there exists a number $M \geq 0$ such that

$$|f(t, x, y, z)| \leq M,$$

and the function $f(t, x, y, z)$ satisfies Lipschitz condition with Lipschitz coefficients L_0, L_1, L_2, that is, there exist $L_0, L_1, L_2 \geq 0$ such that

$$|f(t, x_2, y_2, z_2) - f(t, x_1, y_1, z_1)| \leq L_0|x_2 - x_1| + L_1|y_2 - y_1| + L_2|z_2 - z_1| \tag{2.1.15}$$

for any $(t, x, y, z), (t, x_i, y_i, z_i) \in \mathcal{D}_M$ $(i = 1, 2)$ and

$$q := L_0 M_0 + L_1 M_1 + L_2 M_2 < 1. \tag{2.1.16}$$

Then, the problem (2.1.1),(2.1.2) has unique solution $u(t)$ satisfying $|u(t)| \leq M_0 M$, $|u'(t)| \leq M_1 M$, $|u''(t)| \leq M_2 M$ for any $0 \leq t \leq 1$.

Proof. Under the conditions of the theorem, it is easy to verify that the operator A associated with the problem (2.1.1)-(2.1.2) is a contraction operator $A : B[0, M] \to B[0, M]$. By using the contraction principle, A has a unique fixed point in $B[O, M]$, which generates a unique solution $u(t)$ of the problem (2.1.1),(2.1.2).
Similarly as in Theorem 2.1.2, we obtain the estimates for $u(t), u'(t), u''(t)$. Therefore, the theorem is proved. □

Similarly, we obtain the existence and uniqueness of positive solution of the problem (2.1.1)-(2.1.2).

Theorem 2.1.5 (Existence and uniqueness of positive solution)**.** Suppose that the conditions of Theorem 2.1.3 are met in the domain $\mathcal{D}_M^+$. Moreover, suppose that there are numbers $L_0, L_1, L_2 \geq 0$ such that the function $f(t, x, y, z)$ satisfies the Lipschitz conditions (2.1.15), (2.1.16). Then, the problem (2.1.1),(2.1.2) has a unique monotone nonnegative solution $u(t)$ satisfying (2.1.14). Besides, if $f(t, 0, 0, 0) \not\equiv 0$ for $t \in (0, 1)$ then the solution is positive.

Remark 2.1.1. Based on the sign of $G_2(t,s)$, from the representation (2.1.9) for $u''(t)$ we can conclude of the convexity or concavity of solutions of the problem (2.1.4).

2.1.3. Iterative method

Consider the following iterative method for solving the problem (2.1.1), (2.1.2):

1. Given an initial approximation $\varphi_0 \in B[0,M]$, say
$$\varphi_0(t) = 0. \tag{2.1.17}$$

2. Knowing φ_k $(k = 0,1,...)$ compute
$$u_k(t) = \int_0^1 G(t,s)\varphi_k(s)\ ds, \tag{2.1.18}$$
$$y_k(t) = u_k'(t),\ z_k(t) = u_k''(t), \tag{2.1.19}$$
or equivalently,
$$\begin{aligned} y_k(t) &= \int_0^1 G_1(t,s)\varphi_k(s)\ ds, \\ z_k(t) &= \int_0^1 G_2(t,s)\varphi_k(s)\ ds. \end{aligned} \tag{2.1.20}$$

3. Compute the new approximation
$$\varphi_{k+1}(t) = f(t, u_k(t), y_k(t), z_k(t)). \tag{2.1.21}$$

Put
$$p_k = \frac{q^k}{1-q}\|\varphi_1 - \varphi_0\|.$$

Theorem 2.1.6 (Convergence)**.** Suppose that the conditions of Theorem 2.1.4 are met, then the above iterative method converges and there hold the estimates
$$\|u_k - u\| \le M_0 p_k, \quad \|u_k' - u'\| \le M_1 p_k, \quad \|u_k'' - u''\| \le M_2 p_k, \tag{2.1.22}$$
where u is the exact solution of the problem (2.1.1), (2.1.2), and M_0, M_1, M_2 are defined by (2.1.10).

Proof. Indeed, the iterative method above is the successive method for finding the fixed point of the operator A associated with the problem (2.1.1)-(2.1.2). Therefore, it converges and there holds the estimate
$$\|\varphi_k - \varphi\| \le p_k, \tag{2.1.23}$$
where φ is the fixed point of A. Having in mind the representations (2.1.8), (2.1.9), (2.1.18), (2.1.20) and the formulas (2.1.10), from the above estimate we obtain the estimates (2.1.22). Thus, the theorem is proved. □

In many problems when the Green's function and its derivatives have constant sign and the nonlinear term $f(t,x,y,z)$ is monotone in variables x,y,z we can establish the monotonicity of the sequence of approximations $u_k(t)$. We consider a particular case below, which will be met in some examples in the next section.

Theorem 2.1.7 (Monotonicity)**.** Consider the problem (2.1.1)-(2.1.2), where the Green's function $G(t,s)$ and its derivative $G_1(t,s)$ are nonpositive in the domain $Q = [0,1]^2$, the function $f = f(t,x,y) \leq 0$ is decreasing in x,y for $x,y \geq 0$. Then the sequence of iterations $u_k(t)$ computed by the above iterative process is increasing, that is

$$0 = u_0(t) \leq u_1(t) \leq ... \leq u_k(t) \leq ..., \quad t \in [0,1]. \tag{2.1.24}$$

Proof. Indeed, beginning from $\varphi_0 = 0$ by the iterative process (2.1.17)-(2.1.21) we obtain $u_0 = 0,\ y_0 = 0$. Because $f = f(t,x,y) \leq 0$ we have $\varphi_1 = f(t,0,0) \leq 0$. Hence, $u_1(t) = \int_0^1 G(t,s)\varphi_1(s)ds \geq 0$ due to $G(t,s) \leq 0$. Similarly, $y_1(t) \geq 0$. Thus, we have $u_1 \geq u_0,\ y_1 \geq y_0$. Due to the decrease of $f(t,x,y)$ in x,y we have $\varphi_2(t) = f(t,u_1,y_1) \leq f(t,u_0,y_0) = \varphi_1(t)$. Therefore, from the formulas for computing $u_2(t), y_2(t)$ it follows that $u_2 \geq u_1,\ y_2 \geq y_1$. Repeating the above argument we obtain (2.1.24). The theorem is proved. □

2.1.4. Some particular cases and examples

In order to illustrate the theoretical results obtained in the previous section, we consider some particular cases studied by other authors using various methods. In numerical realization of the proposed iterative method, the definite integrals are computed using the trapezoidal rule with second order accuracy. In all examples, computations are carried out on the uniform grid with gridsize $h = 0.01$ on the interval $[0,1]$ until achieving $\|\varphi_k - \varphi_{k-1}\| \leq 10^{-6}$. The number of iterations performed will be indicated.

Through the particular cases together with examples it will be clear of the efficiency of the proposed unified approach to BVPs for nonlinear third order differential equations by the reduction of them to operator equations for the nonlinear terms.

2.1.4.1. Case 1.

Consider the problem

$$\begin{aligned} u^{(3)}(t) &= f(t,u(t),u'(t),u''(t)), \quad 0 < t < 1, \\ u(0) &= u'(0) = u'(1) = 0 \end{aligned}$$

which is the generalization of the problem considered in [19]. There, by using the lower and upper solutions method and the fixed point theorem on cones the authors obtained several results of solution and positive solution. For the case $f = f(t,u(t),u'(t))$ in [17], the authors also established existence results by using the upper and lower solutions method and a new variant of maximum principle. It should be emphasized that the results of these two mentioned works are purely existence and the uniqueness is not established.

The Green's function associated with the considered problem is

$$G(t,s) = \begin{cases} \dfrac{s}{2}(t^2 - 2t + s), & 0 \leq s \leq t \leq 1, \\ \dfrac{t^2}{2}(s-1), & 0 \leq t \leq s \leq 1. \end{cases}$$

Taking derivative of $G(t,s)$ we obtain

$$G_1(t,s) = \begin{cases} s(t-1), & 0 \le s \le t \le 1, \\ t(s-1), & 0 \le t \le s \le 1, \end{cases}$$

$$G_2(t,s) = \begin{cases} s, & 0 \le s \le t \le 1, \\ s-1, & 0 \le t \le s \le 1. \end{cases}$$

It is obvious that

$$G(t,s) \le 0,\ G_1(t,s) \le 0,\ \ 0 \le t, s \le 1$$

and we have

$$M_0 = \max_{0\le t\le 1} \int_0^1 |G(t,s)|\, ds = \frac{1}{12}, \quad M_1 = \max_{0\le t\le 1} \int_0^1 |G_1(t,s)|\, ds = \frac{1}{8},$$
$$M_2 = \max_{0\le t\le 1} \int_0^1 G_2(t,s)\, ds = \frac{1}{2}.$$

Example 2.1.1 (Example 7 in [19]). Consider the problem

$$\begin{aligned} u^{(3)}(t) &= -e^{u(t)}, \quad 0 < t < 1, \\ u(0) &= u'(0) = u'(1) = 0. \end{aligned} \tag{2.1.25}$$

The authors [19] by using the method of lower and upper solutions and the fixed point theorem on cones showed that the above problem has a solution $u(t)$ satisfying $\|u\| \le 1$, $u(t) > 0$ for $t \in (0,1)$ and $u(t)$ is an increasing function. Here, using the theoretical results obtained in the previous section we obtain the results which are stronger than the above results.

Indeed, for the problem (2.1.25) $f = f(t,x) = -e^x$. In the domain

$$\mathcal{D}_M^+ = \Big\{(t,x)|\ 0 \le t \le 1,\ 0 \le x \le \frac{M}{12}\Big\}$$

there hold $-e^{M/12} \le f(t,x) \le 0$. Therefore, selecting $M = 1.1$ we have $-M \le f(t,x) \le 0$. Further, in $\mathcal{D}_M^+$ the function $f(t,x)$ satisfies the Lipschitz condition with $L_0 = e^{M/12} = 1.096$. Thus, $q = L_0/12 = 0.0913$. By Theorem 2.1.5 the problem has a *unique* monotone positive solution $u(t)$ satisfying the estimates

$$0 \le u(t) \le \frac{M}{12} = \frac{1.1}{12} = 0.0917,\ 0 \le u'(t) \le \frac{M}{8} = \frac{1.1}{8} = 0.1357,$$
$$|u''(t)| \le \frac{M}{2} = \frac{1.1}{2} = 0.55.$$

These results are clearly better than those in [19].

The numerical solution of the problem obtained by the iterative method (2.1.17)-(2.1.21) after 5 iterations is depicted in Figure 2.1. From this figure, it is clear that the solution is monotone, positive and is bounded by 0.0917 as shown above by the theory.

Example 2.1.2 (Example 8 in [19]). Consider the problem

$$\begin{aligned} u^{(3)}(t) &= -\frac{5u^3(t) + 4u(t) + 3}{u^2(t) + 1}, \quad 0 < t < 1, \\ u(0) &= u'(0) = u'(1) = 0. \end{aligned} \tag{2.1.26}$$

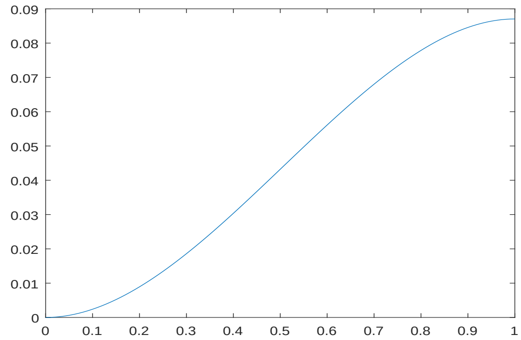

Figure 2.1: Approximate solution in Example 2.1.1

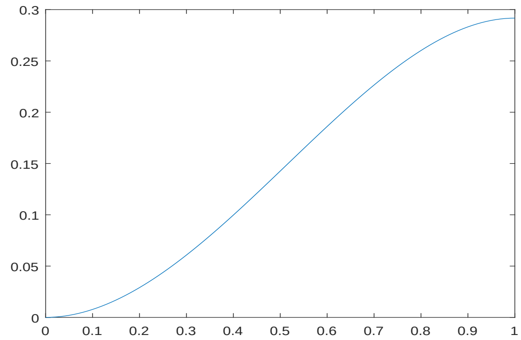

Figure 2.2: Approximate solution in Example 2.1.2

The authors in [19] showed that the problem has an increasing solution $u(t) > 0$ for $t \in (0, 1)$. Analogously as in Example 4.1.1 we obtained that the problem (2.1.26) has a *unique* monotone positive solution $u(t)$ satisfying

$$0 \le u(t) \le 0.3417,\ 0 \le u'(t) \le 0.5125,\ |u''(t)| \le 2.05.$$

The numerical solution of the problem obtained by the iterative method (2.1.17)-(2.1.21) after 8 iterations is depicted in Figure 2.2. From this figure, it is clear that the solution is monotone, positive and is bounded by 0.3417 as shown above by the theory.

Example 2.1.3 (Example 4.2 in [17]). Consider the problem

$$\begin{aligned} u^{(3)}(t) &= -e^{u(t)} - e^{u'(t)}, \quad 0 < t < 1, \\ u(0) &= u'(0) = u'(1) = 0. \end{aligned}$$

By the use of the method of lower and upper solutions and a new variant of maximum principle, the authors in [17] proved that the above problem has a solution $u(t)$ satisfying $\|u\| \le 1$, $u(t) > 0$ for $t \in (0, 1)$ and $u(t)$ is an increasing function. Here, choosing $M = 2.7$, according to Theorem 2.1.5 we conclude that the problem has a *unique* monotone positive solution $u(t)$ satisfying the estimates

$$0 \le u(t) \le 0.2250,\ 0 \le u'(t) \le 0.3375,\ |u''(t)| \le 1.350.$$

The numerical solution of the problem computed by the proposed iterative method (2.1.17)-(2.1.21) after 9 iterations is given in Figure 2.3. From this figure, it is clear that the solution is monotone, positive and is bounded by 0.2250 in agreement with the theory.

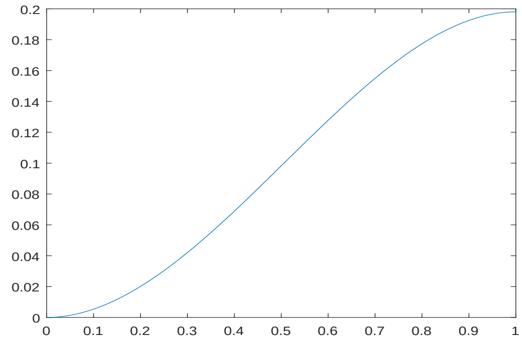

Figure 2.3: Approximate solution in Example 2.1.3

Remark 2.1.2. In the above examples, it can be seen that all the conditions of Theorem 2.1.7 are met. Thus, the sequences of approximations are increasing. Numerical results also confirm this fact.

2.1.4.2. Case 2.

Consider the problem

$$\begin{aligned} u^{(3)}(t) &= f(t, u(t), u'(t), u''(t)), \quad 0 < t < 1, \\ u(0) &= u'(0) = u''(1) = 0. \end{aligned} \tag{2.1.27}$$

In [20] the authors assumed that the function $f(t, x, y, z)$ defined on $[0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is L_p-Caratheodory, and there exist functions $\alpha, \beta, \gamma, \delta \in L_p[0,1]$, $p \geq 1$, such that

$$|f(t, x, y, z) \leq \alpha(t)x + \beta(t)y + \gamma(t)z + \delta(t)|, \quad t \in (0,1)$$

and

$$A_0\|\alpha\|_p + A_1\|\beta\|_p + \|\gamma\|_p < 1,$$

where A_0, A_1 are some constants depending on p. Under these conditions, by using Leray-Schauder continuation principle, the authors proved that the problem has at least one solution. But no examples are given for illustrating the theoretical conclusion.

Here, under the assumption that the function $f(t, x, y, z)$ is continuous, we establish the existence and uniqueness of solution by Theorem 2.1.5. For the problem (2.1.27) the Green's function is

$$G(t,s) = \begin{cases} -st + \dfrac{s^2}{2}, & 0 \leq s \leq t \leq 1, \\ -\dfrac{t^2}{2}, & 0 \leq t \leq s \leq 1. \end{cases}$$

Its first and second derivatives are

$$G_1(t,s) = \begin{cases} -s, & 0 \leq s \leq t \leq 1, \\ -t, & 0 \leq t \leq s \leq 1, \end{cases}$$

$$G_2(t,s) = \begin{cases} 0, & 0 \leq s \leq t \leq 1, \\ -1, & 0 \leq t \leq s \leq 1. \end{cases}$$

It is easily seen that

$$G(t,s) \leq 0, \; G_1(t,s) \leq 0, \; 0 \leq t, s \leq 1$$

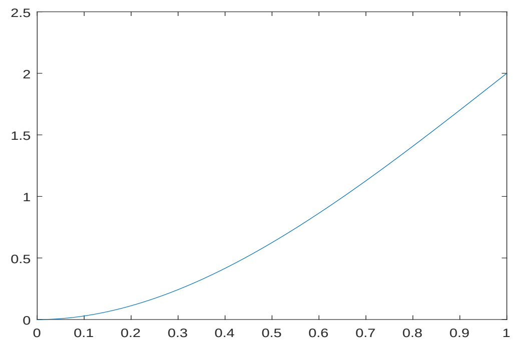

Figure 2.4: Approximate solution in Example 2.1.4

and

$$M_0 = \max_{0\le t\le 1}\int_0^1 |G(t,s)|\,ds = \frac{1}{3},\quad M_1 = \max_{0\le t\le 1}\int_0^1 |G_1(t,s)|\,ds = \frac{1}{2},$$
$$M_2 = \max_{0\le t\le 1}\int_0^1 |G_2(t,s)|\,ds = 1.$$

Example 2.1.4. Consider the following problem

$$\begin{aligned} u'''(t) &= -\frac{1}{36}\big(u'(t)\big)^2 + \frac{1}{24}u(t)u''(t) + \frac{1}{4}t^2 - 6, \quad 0\le t\le 1,\\ u(0) &= u'(0) = u''(1) = 0. \end{aligned} \tag{2.1.28}$$

For the problem

$$f(t,x,y,z) = -\frac{1}{36}y^2 + \frac{1}{24}xz + \frac{1}{4}t^2 - 6.$$

It can be verified that with $M = 7.5,\ L_1 = 0.3125,\ L_2 = 0.2083,\ L_3 = 0.1042$. So, all the conditions of Theorem 2.1.5 are met, and the problem (2.1.28) has a unique positive solution satisfying the estimates $0 \le u(t) \le 2.5,\ 0 \le u'(t) \le 3.75,\ |u''(t)| \le 7.5$.

The numerical solution of the problem computed by the iterative method (2.1.17)-(2.1.21) after 5 iterations is given in Figure 2.4. From this figure it is clear that the solution is bounded by 2.5 in agreement with the theory.

It is interesting that the problem (2.1.28) *has the exact solution* $u(t) = -t^3 + 3t^2$.

This solution satisfies the exact estimates $0 \le u(t) \le 2,\ 0 \le u'(t) \le 3,\ 0 \le u''(t) \le 6$ for $0 \le t \le 1$, which are better than the theoretical estimates above. On the grid with the gridsize $h = 0.01$ the maximal absolute error of the obtained approximate solution compared with the exact solution is $3.7665e-04$.

2.1.4.3. Case 3.

Consider the problem

$$\begin{aligned} u^{(3)}(t) &= f(t,u(t),u'(t),u''(t)), \quad 0 < t < 1,\\ u(0) &= u'(1) = u''(1) = 0. \end{aligned} \tag{2.1.29}$$

Under the conditions similar to those in the previous case, the authors in [20] established the existence of a solution of the problem without illustrative examples. Very recently, in [22], by using the fixed point index theory on cones, the authors studied the existence of positive solutions of the problem (2.1.29) under conditions on the growth of the function $f(t,x,y,z)$ as $|x|+|y|+|z|$ tends to zero and infinity, including a Nagumo-type condition on y and z.

Here, under the assumption that the function $f(t,x,y,z)$ is continuous, we can obtain the existence results by the above theorems. For the problem (2.1.29) the Green's function is

$$G(t,s) = \begin{cases} \dfrac{s^2}{2}, & 0 \le s \le t \le 1, \\ st - \dfrac{t^2}{2}, & 0 \le t \le s \le 1. \end{cases}$$

Its first and second derivatives are

$$G_1(t,s) = \begin{cases} 0, & 0 \le s \le t \le 1, \\ s-t, & 0 \le t \le s \le 1, \end{cases}$$

$$G_2(t,s) = \begin{cases} 0, & 0 \le s \le t \le 1, \\ -1, & 0 \le t \le s \le 1. \end{cases}$$

It is easy to verify that

$$G(t,s) \ge 0,\ G_1(t,s) \ge 0,\ 0 \le t,s \le 1$$

and we obtain

$$M_0 = \frac{1}{6},\ M_1 = \frac{1}{2},\ M_2 = 1.$$

Example 2.1.5. Consider the problem

$$\begin{aligned} &u'''(t) = \frac{1}{18}\big(u'(t)\big)^2 - \frac{1}{12}u(t)u''(t) + \frac{1}{2}t + \frac{11}{2}, \quad 0 \le t \le 1, \\ &u(0) = u'(1) = u''(1) = 0. \end{aligned} \tag{2.1.30}$$

In this example

$$f(t,x,y,z) = \frac{1}{18}y^2 - \frac{1}{12}xz + \frac{1}{2}t + \frac{11}{2}.$$

It can be verified that with $M = 8,\ L_1 = \dfrac{2}{3},\ L_2 = \dfrac{4}{9},\ L_3 = \dfrac{1}{9}$, and the conditions of Theorem 2.1.5 are satisfied. Thus, the problem (2.1.30) has a unique positive, increasing solution that satisfies the estimates $0 \le u(t) \le \dfrac{4}{3},\ 0 \le u'(t) \le 4,\ -8 \le u''(t) \le 0$.

The numerical solution of the problem obtained by the iterative method (2.1.17)-(2.1.21) after 6 iterations is given in Figure 2.5. From this figure, it is clear that the solution is monotone, positive and is bounded by 4/3 in agreement with the above theory.

It can be verified that $u(t) = t^3 - 3t^2 + 3t$ is the exact solution of the problem (2.1.30). This solution is positive, increasing and satisfies the exact estimates $0 \le u(t) \le 1,\ 0 \le u'(t) \le 3,\ -6 \le u''(t) \le 0$ for $0 \le t \le 1$, which are better than the theoretical estimates above. On the grid with the gridsize $h = 0.01$ the maximal error of the obtained approximate solution compared with the exact solution is $3.6256e-04$.

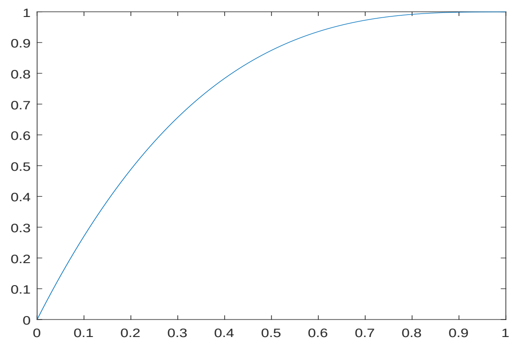

Figure 2.5: Approximate solution in Example 2.1.5

2.1.4.4. Case 4.

Consider the problem

$$\begin{aligned} u^{(3)}(t) &= f(t, u(t), u'(t), u''(t)), \quad 0 < t < 1, \\ u(0) &= u''(0) = u'(1) = 0. \end{aligned} \tag{2.1.31}$$

By the use of the method of lower and upper solutions and Schauder fixed theorem on cones, the author in [23] obtained the existence of a solution under complicated conditions on the nonlinear term.

For the problem (2.1.31) the Green's function is

$$G(t,s) = \begin{cases} \dfrac{t^2}{2} - t + \dfrac{s^2}{2}, & 0 \le s \le t \le 1, \\ t(s-1), & 0 \le t \le s \le 1. \end{cases}$$

Its first and second derivatives are

$$G_1(t,s) = \begin{cases} t-1, & 0 \le s \le t \le 1, \\ s-1, & 0 \le t \le s \le 1, \end{cases}$$

$$G_2(t,s) = \begin{cases} 1, & 0 \le s \le t \le 1, \\ 0, & 0 \le t \le s \le 1. \end{cases}$$

Clearly,

$$G(t,s) \le 0, \; G_1(t,s) \le 0, \; 0 \le t, s \le 1.$$

We have

$$M_0 = \frac{1}{3}, \; M_1 = \frac{1}{2}, \; M_2 = 1.$$

By using theorems in the previous section, we can obtain the results on the existence of solution of the problem (2.1.31).

Example 2.1.6 (Example 3.5 in [23])**.**

$$\begin{aligned} u^{(3)}(t) &= -\frac{1}{4}\big[t + e^{u(t)} + (u'(t))^2 + u''(t)\big], \quad 0 < t < 1, \\ u(0) &= u''(0) = u'(1) = 0. \end{aligned} \tag{2.1.32}$$

Define

$$\mathcal{D}_M^+ = \left\{(t,x,y,z) |\; 0 \le t \le 1, \; 0 \le x \le \frac{M}{3}, \; 0 \le y \le \frac{M}{2}, \; |z| \le M\right\}.$$

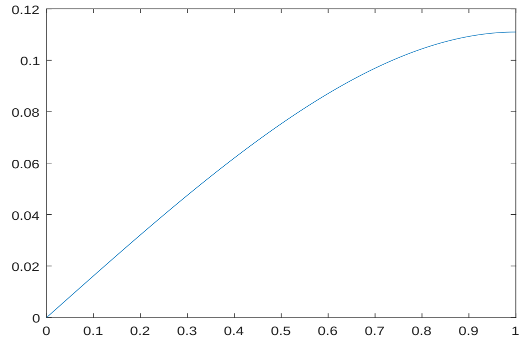

Figure 2.6: Approximate solution in Example 2.1.6

Selecting $M = 0.835$ we have

$$-M \le f(t,x,y,z) = -\frac{1}{4}\big[t + e^x + y^2 + z\big] \le 0.$$

It is easy to calculate the Lipschitz coefficients of $f(t,x,y,z)$:

$$L_0 = \frac{1}{4}e^{M/3} = 0.3302,\ L_1 = \frac{M}{4} = 0.2087,\ L_2 = 1.$$

Thus, $q = L_0/3 + L_1/2 + L_2 = 0.4851 < 1$. By Theorem 2.1.5 the problem has a *unique monotone positive* solution $u(t)$ satisfying

$$0 \le u(t) \le M/3 = 0.2783,\ 0 \le u'(t) \le M/2 = 0.5,\ |u''(t)| \le 1.$$

In [23], the author could only prove that the problem has a positive solution. The numerical solution obtained by the iterative method (2.1.17)-(2.1.21) after 5 iterations is given in Figure 2.6. From this figure, it is clear that the solution is monotone, positive and is bounded by 0.2783 in agreement with the theory.

2.1.4.5. Case 5.

Consider the problem

$$\begin{aligned} u^{(3)}(t) &= f(t,u(t),u'(t),u''(t)), \quad 0 < t < 1,\\ u(0) &= u'(1) = u''(1) = 0. \end{aligned} \tag{2.1.33}$$

In [22], by using the fixed point index theory in cones, the authors obtained the existence of positive solution under conditions which are very complicated and posed on the growth of the function f including a Nagumo-type condition.

For the problem (2.1.33) the Green's function is

$$G(t,s) = \begin{cases} \dfrac{s^2}{2}, & 0 \le s \le t \le 1,\\ st - \dfrac{t^2}{2}, & 0 \le t \le s \le 1. \end{cases}$$

Its first and second derivatives are

$$G_1(t,s) = \begin{cases} 0, & 0 \le s \le t \le 1,\\ s - t, & 0 \le t \le s \le 1, \end{cases}$$

$$G_2(t,s) = \begin{cases} 0, & 0 \le s \le t \le 1, \\ -1, & 0 \le t \le s \le 1. \end{cases}$$

It can be verified that

$$G(t,s) \ge 0,\ G_1(t,s) \ge 0,\ G_2(t,s) \le 0,\ 0 \le t,s \le 1$$

and

$$M_0 = \frac{1}{6},\ M_1 = \frac{1}{2},\ M_2 = 1.$$

Due to the above properties of the Green's function, by the theorems in the previous section we can obtain the results on the existence of solution of the problem (2.1.33).

Example 2.1.7. Consider the following problem

$$\begin{aligned} u'''(t) &= \frac{1}{18}\big(u'(t)\big)^2 - \frac{1}{12}u(t)u''(t) + \frac{1}{2}t + \frac{11}{2}, \quad 0 \le t \le 1, \\ u(0) &= u'(1) = u''(1) = 0. \end{aligned} \tag{2.1.34}$$

In this problem

$$\begin{aligned} f(t,x,y,z) &= \frac{1}{18}y^2 - \frac{1}{12}xz + \frac{1}{2}t + \frac{11}{2}, \\ f(t,0,0,0) &= \frac{1}{2}t + \frac{11}{2} > 0 \quad \forall t \in [0,1]. \end{aligned}$$

It is possible to verify that with $M = 8$ all the conditions of Theorem 2.1.4 are met. Therefore, the problem has a unique positive increasing solution which satisfies $0 \le u(t) \le \frac{4}{3},\ 0 \le u'(t) \le 4,\ |u''(t)| \le 8$.

Notice that the problem has the exact solution $u(t) = t^3 - 3t^2 + 3t$. It is positive, increasing and satisfies the exact estimates $0 \le u(t) \le 1,\ 0 \le u'(t) \le 3,\ -6 \le u''(t) \le 0$ for $0 \le t \le 1$, which are better than the theoretical estimates above.

Example 2.1.8. Consider the problem

$$\begin{aligned} u'''(t) &= u^3(t) + u(t)(u'(t))^2 + u(t)(u''(t))^2,\ 0 \le t \le 1, \\ u(0) &= u'(1) = u''(1) = 0. \end{aligned} \tag{2.1.35}$$

In this problem

$$f(t,x,y,z) = x^3 + xy^2 + xz^2.$$

It is possible to verify that with $0 < M \le \sqrt{\frac{108}{23}}$ Theorem 2.1.5 guarantees that the problem (2.1.35) has a unique nonnegative monotone solution. Since $u(t) \equiv 0$ is a nonnegative solution of the problem, we come to the conclusion that the problem cannot have positive solution, which is contrary to that in [22]. Thus, we believe that there must be some error in their results.

2.1.5. Conclusion

In this section, we have proposed a unified efficient approach to investigate fully nonlinear third order differential equation subject to general two-point linear boundary conditions. The approach is based on the reduction of boundary value problems to fixed point problems of nonlinear operators for the nonlinear terms of the equation but not for the function to be sought. In result, we have obtained the existence, uniqueness, positivity and monotonicity of solution under the conditions which are simpler and easier to verify than those of other authors. The applicability and advantages of the proposed approach are illustrated on some examples taken from the papers of other authors, where our approach yields better results.

The proposed approach is applicable to other boundary value problems for the third order and higher orders nonlinear differential equations. This is the subject of our researches in the future.

2.2. Numerical methods for a third order nonlinear BVP

2.2.1. Introduction

In the previous section, we have established the existence and uniqueness of solutions and the convergence of an iterative method on continuous level for the fully third order differential equations subject to general two-point linear boundary conditions. We also have shown some particular cases and examples for illustrating the obtained theoretical results. In this section, we will discuss numerical realization of the proposed iterative method. The investigation will be done for a case, namely, for Case 1 in the previous section. So, we consider the BVP

$$\begin{aligned} &u^{(3)}(t) = f(t, u(t), u'(t), u''(t)), \quad 0 < t < 1, \\ &u(0) = 0, u'(0) = 0, u'(1) = 0. \end{aligned} \tag{2.2.1}$$

In order to be easily tracked we recall some results concerning the existence of solutions of the above problem. The Green's function of the problem, and its first and second derivatives are

$$G_0(t,s) = \begin{cases} \dfrac{s}{2}(t^2 - 2t + s), & 0 \le s \le t \le 1, \\ \dfrac{t^2}{2}(s-1), & 0 \le t \le s \le 1. \end{cases}$$

$$G_1(t,s) = G'_t(t,s) = \begin{cases} s(t-1), & 0 \le s \le t \le 1, \\ t(s-1), & 0 \le t \le s \le 1, \end{cases}$$

$$G_2(t,s) = G''_{tt}(t,s) \begin{cases} s, & 0 \le s \le t \le 1, \\ s-1, & 0 \le t \le s \le 1. \end{cases} \tag{2.2.2}$$

We have $G_0(t,s) \le 0,\ G_1(t,s) \le 0$ in $Q = [0,1]^2$ and

$$\begin{aligned} &M_0 = \max_{0 \le t \le 1} \int_0^1 |G(t,s)|\, ds = \frac{1}{12}, \quad M_1 = \max_{0 \le t \le 1} \int_0^1 |G_1(t,s)|\, ds = \frac{1}{8}, \\ &M_2 = \max_{0 \le t \le 1} \int_0^1 |G_2(t,s)|\, ds = \frac{1}{2}. \end{aligned} \tag{2.2.3}$$

For each real number $M > 0$ we denote

$$\mathcal{D}_M = \{(t,x,y,z)|\ 0 \le t \le 1,\ |x| \le M_0 M,\ |y| \le M_1 M,\ |z| \le M_2 M\},$$

Theorem 2.2.1 (Existence and uniqueness of solution)**.** Suppose that there exists a number $M \ge 0$ such that

$$|f(t,x,y,z)| \le M,$$

and the function $f(t,x,y,z)$ satisfies Lipschitz condition with Lipschitz coefficients L_0, L_1, L_2, that is, there exist $L_0, L_1, L_2 \geq 0$ such that

$$|f(t,x_2,y_2,z_2) - f(t,x_1,y_1,z_1)| \leq L_0|x_2 - x_1| + L_1|y_2 - y_1| + L_2|z_2 - z_1| \quad (2.2.4)$$

for any $(t,x,y,z),(t,x_i,y_i,z_i) \in \mathcal{D}_M$ $(i = 1,2)$ and

$$q := L_0M_0 + L_1M_1 + L_2M_2 < 1.$$

Then, the problem (2.2.1) has a unique solution $u(t)$ satisfying $|u(t)| \leq M_0M$, $|u'(t)| \leq M_1M$, $|u''(t)| \leq M_2M$ for any $0 \leq t \leq 1$.

Below we recall the iterative method on continuous level for the problem:

1. Given a starting approximation

$$\varphi_0(t) = f(t,0,0,0). \quad (2.2.5)$$

2. Knowing the $k-$th approximation $\varphi_k(t)$ $(k = 0,1,...)$ compute

$$\begin{aligned} u_k(t) &= \int_0^1 G_0(t,s)\varphi_k(s)ds, \\ y_k(t) &= \int_0^1 G_1(t,s)\varphi_k(s)ds, \\ z_k(t) &= \int_0^1 G_2(t,s)\varphi_k(s)ds. \end{aligned} \quad (2.2.6)$$

3. Compute the new approximation

$$\varphi_{k+1}(t) = f(t,u_k(t),y_k(t),z_k(t)). \quad (2.2.7)$$

Set

$$p_k = \frac{q^k}{1-q}, \quad d = \|\varphi_1 - \varphi_0\|. \quad (2.2.8)$$

Theorem 2.2.2 (Convergence)**.** If the conditions of Theorem 2.2.1 are satisfied then the above iterative method converges and there hold the estimates

$$\|u_k - u\| \leq M_0p_kd, \quad \|u_k' - u'\| \leq M_1p_kd, \quad \|u_k'' - u''\| \leq M_2p_kd,$$

where u is the exact solution of the problem (2.2.1) and M_0, M_1, M_2 are defined by (2.2.3).

Remark 2.2.1. Consider the problem with nonhomogeneous boundary conditions

$$\begin{aligned} u^{(3)}(t) &= f(t,u(t),u'(t),u''(t)), \quad 0 < t < 1, \\ u(0) &= c_1, u'(0) = c_2, u'(1) = c_3. \end{aligned} \quad (2.2.9)$$

Let $P_2(t)$ be the polynomial of second degree satisfying the conditions

$$P_2(0) = c_1, P_2'(0) = c_2, P_2'(1) = c_3.$$

It is easy to see that

$$P_2(t) = \frac{c_3 - c_2}{2} t^2 + c_2 t + c_1.$$

Set

$$u(t) = v(t) + P_2(t),$$
$$F(t, v(t), v'(t), v''(t)) = f(t, v(t) + P_2(t), v'(t) + P_2'(t), v''(t) + P_2''(t).$$

Then the problem (2.2.9) is transformed to the problem

$$\begin{aligned} &v^{(3)}(t) = F(t, v(t), v'(t), v''(t)), \quad 0 < t < 1, \\ &v(0) = 0, v'(0) = 0, v'(1) = 0. \end{aligned} \tag{2.2.10}$$

So, we can apply the existence results to this problem. It is worthy to say that the iterative method applied to (2.2.10) becomes the following iterative method

1. Given a starting approximation

$$\varphi_0(t) = f(t, P_2(t), P_2'(t), P_2''(t)). \tag{2.2.11}$$

2. Knowing $\varphi_k(t)$ $(k = 0, 1, ...)$ compute

$$\begin{aligned} u_k(t) &= \int_0^1 G_0(t,s)\varphi_k(s)ds + P_2(t), \\ y_k(t) &= \int_0^1 G_1(t,s)\varphi_k(s)ds + P_2'(t), \\ z_k(t) &= \int_0^1 G_2(t,s)\varphi_k(s)ds + P_2''(t). \end{aligned} \tag{2.2.12}$$

3. Compute the new approximation

$$\varphi_{k+1}(t) = f(t, u_k(t), y_k(t), z_k(t)). \tag{2.2.13}$$

2.2.2. Discrete iterative method 1

To numerically realize the above iterative method we design the corresponding discrete iterative methods. To this end, we consider the uniform grid $\bar{\omega}_h = \{t_i = ih,\ h = 1/N, i = 0, 1, ..., N\}$ on the interval $[0, 1]$ and by $\Phi_k(t), U_k(t), Y_k(t), Z_k(t)$ denote the grid functions defined on this grid and approximate the functions $\varphi_k(t), u_k(t), y_k(t), z_k(t)$ on this grid, respectively.

First, consider the following method, called **Method 1**:

1. Given a starting approximation

$$\Phi_0(t_i) = f(t_i, 0, 0, 0),\ i = 0, ..., N. \tag{2.2.14}$$

2. Knowing the k^{th} approximation $\Phi_k(t_i),\ k = 0, 1, ...;\ i = 0, ..., N$, compute approximately the definite integrals (2.2.6) by the trapezoidal rule

$$\begin{aligned} U_k(t_i) &= \sum_{j=0}^{N} h\rho_j G_0(t_i, t_j)\Phi_k(t_j), \\ Y_k(t_i) &= \sum_{j=0}^{N} h\rho_j G_1(t_i, t_j)\Phi_k(t_j), \\ Z_k(t_i) &= \sum_{j=0}^{N} h\rho_j G_2^*(t_i, t_j)\Phi_k(t_j),\ i = 0, ..., N, \end{aligned} \tag{2.2.15}$$

where ρ_j are the weights

$$\rho_j = \begin{cases} 1/2, \ j = 0, N \\ 1, \ j = 1, 2, ..., N-1 \end{cases}$$

and

$$G_2^*(t,s) = \begin{cases} s, & 0 \le s < t \le 1, \\ s - 1/2, & s = t, \\ s - 1, & 0 \le t < s \le 1. \end{cases} \tag{2.2.16}$$

3. Compute new approximation

$$\Phi_{k+1}(t_i) = f(t_i, U_k(t_i), Y_k(t_i), Z_k(t_i)). \tag{2.2.17}$$

To obtain the error estimates of the method, we need some following auxiliary results.

Proposition 2.2.3. Suppose that the function $f(t,x,y,z)$ and its partial derivatives up to second order are continuous in the domain $\mathcal{D}_M$. Then the functions $u_k(t), y_k(t), z_k(t)$, $k = 0, 1, ...$ generated by the continuous iterative method (2.2.5)-(2.2.7), we have $z_k(t) \in C^3[0,1], \ y_k(t) \in C^4[0,1], \ u_k(t) \in C^5[0,1]$.

Proof. The proposition will be proved by induction. For $k = 0$, by the assumption on the function f we have $\varphi_0(t) \in C^2[0,1]$ because $\varphi_0(t) = f(t, 0, 0, 0)$. In view of the expression (2.2.2) of $G_2(t,s)$ we have

$$z_0(t) = \int_0^1 G_2(t,s)\varphi_0(s)ds = \int_0^t s\varphi_0(s)ds - \int_t^1 (s-1)\varphi_0(s)ds.$$

It is easy to see that $z_0'(t) = \varphi_0(t)$. Hence, $z_0(t) \in C^3[0,1]$. This implies $y_0(t) \in C^4[0,1], \ u_0(t) \in C^5[0,1]$.

Now, assume that $z_k(t) \in C^3[0,1], \ y_k(t) \in C^4[0,1], \ u_k(t) \in C^5[0,1]$. Then, since $\varphi_{k+1}(t) = f(t, u_k(t), y_k(t), z_k(t))$ and the function f by the assumption has continuous derivative in all variables up to second order, it follows that $\varphi_{k+1}(t) \in C^2[0,1]$. Repeating the same argument as for $\varphi_0(t)$ above we obtain that $z_{k+1}(t) \in C^3[0,1], \ y_{k+1}(t) \in C^4[0,1], \ u_{k+1}(t) \in C^5[0,1]$. Thus, the proposition is proved. □

Proposition 2.2.4. For arbitrary function $\varphi(t) \in C^2[0,1]$ we have

$$\int_0^1 G_n(t_i,s)\varphi(s)ds = \sum_{j=0}^{N} h\rho_j G_n(t_i,t_j)\varphi(t_j) + O(h^2), \quad (n = 0, 1) \tag{2.2.18}$$

$$\int_0^1 G_2(t_i,s)\varphi(s)ds = \sum_{j=0}^{N} h\rho_j G_2^*(t_i,t_j)\varphi(t_j) + O(h^2). \tag{2.2.19}$$

Proof. In cases $n = 0$ and $n = 1$, because the functions $G_n(t_i,s)$ are continuous at $s = t_i$ and are polynomials in s in the intervals $[0, t_i]$ and $[t_i, 1]$ we have

$$\int_0^1 G_n(t_i,s)\varphi(s)ds = \int_0^{t_i} G_n(t_i,s)\varphi(s)ds + \int_{t_i}^1 G_n(t_i,s)\varphi(s)ds$$
$$= h\big(\tfrac{1}{2}G_n(t_i,t_0)\varphi(t_0) + G_n(t_i,t_1)\varphi(t_1) + ... + G_n(t_i,t_{i-1})\varphi(t_{i-1}) + \tfrac{1}{2}G_2(t_i,t_i)\varphi(t_i)\big)$$
$$+ h\big(\tfrac{1}{2}G_n(t_i,t_i)\varphi(t_i) + G_n(t_i,t_{i+1})\varphi(t_{i+1}) + ... + G_n(t_i,t_{N-1})\varphi(t_{N-1})$$
$$+ \tfrac{1}{2}G_n(t_i,t_N)\varphi(t_N)\big) + O(h^2)$$
$$= \sum_{j=0}^{N} h\rho_j G_n(t_i,t_j)\varphi(t_j) + O(h^2) \quad (n = 0, 1).$$

Therefore, the estimate (2.2.18) is obtained. The estimate (2.2.19) is established with the help of the following result, which is easy to prove.

Lemma 2.2.1. Let $p(t)$ be a function having continuous derivatives up to second order in the interval $[0,1]$ except for the point $t_i,\ 0<t_i<1$, where it has a jump. Then

$$\int_0^1 p(t)dt = \sum_{j=0}^{N} h\rho_j p(j) + O(h^2), \tag{2.2.20}$$

where $p_j = p(t_j), j \neq i$, $p_i = \frac{1}{2}(p_i^- + p_i^+)$ with $p_i^- = \lim_{t\to t_i - 0} p(t)$, $p_i^+ = \lim_{t\to t_i+0} p(t)$.

□

Proposition 2.2.5. Under the assumption of Proposition 2.2.3, for any $k = 0, 1, ...$ we have

$$\|\Phi_k - \varphi_k\| = O(h^2), \tag{2.2.21}$$

$$\|U_k - u_k\| = O(h^2),\ \|Y_k - y_k\| = O(h^2),\ \|Z_k - z_k\| = O(h^2), \tag{2.2.22}$$

where $\|.\| = \|.\|_{C(\bar{\omega}_h)}$ is the max-norm of the grid functions on $\bar{\omega}_h$.

Proof. The proposition is proved by induction. For $k=0$ we have $\|\Phi_0 - \varphi_0\| = 0$. Next, by the first equation in (2.2.6) and Proposition 2.2.4 we have

$$u_0(t_i) = \int_0^1 G_0(t_i,s)\varphi_0(s)ds = \sum_{j=0}^{N} h\rho_j G_0(t_i,t_j)\varphi_0(t_j) + O(h^2), \quad i = 0, ..., N. \tag{2.2.23}$$

On the other hand, taking into account the first equation in (2.2.15) we have

$$U_0(t_i) = \sum_{j=0}^{N} h\rho_j G_0(t_i,t_j)\varphi_0(t_j). \tag{2.2.24}$$

Therefore, $|U_0(t_i) - u_0(t_i)| = O(h^2)$. It implies that $\|U_0 - u_0\| = O(h^2)$. Analogously, we have

$$\|Y_0 - y_0\| = O(h^2),\ \|Z_0 - z_0\| = O(h^2). \tag{2.2.25}$$

Now assume that (2.2.21) and (2.2.22) hold for $k \geq 0$. We shall prove that they hold for $k+1$.
Indeed, by the Lipschitz condition of the function f and the estimates (2.2.22) it is easy to get the estimate

$$\|\Phi_{k+1} - \varphi_{k+1}\| = O(h^2). \tag{2.2.26}$$

From the first equation in (2.2.6) by Proposition 2.2.4 we obtain

$$u_{k+1}(t_i) = \int_0^1 G_0(t_i,s)\varphi_{k+1}(s)ds = \sum_{j=0}^{N} h\rho_j G_0(t_i,t_j)\varphi_{k+1}(t_j) + O(h^2).$$

On the other hand, by the first formula in (2.2.15) we get

$$U_{k+1}(t_i) = \sum_{j=0}^{N} h\rho_j G_0(t_i,t_j)\Phi_{k+1}(t_j).$$

From the above equalities, we obtain the estimate

$$\|U_{k+1} - u_{k+1}\| = O(h^2).$$

Analogously, we have

$$\|Y_{k+1} - y_{k+1}\| = O(h^2),\ \|Z_{k+1} - z_{k+1}\| = O(h^2).$$

Thus, the proof of the proposition is completed. □

Combining Proposition 2.2.5 and Theorem 2.2.2 yields the following theorem.

Theorem 2.2.6. For the approximate solution of the problem (2.2.1) obtained by the discrete iterative method (2.2.14)-(2.2.17) on the uniform grid $\bar{\omega}_h$ we have the estimates

$$\|U_k - u\| \le M_0 p_k d + O(h^2),\ \|Y_k - u'\| \le M_1 p_k d + O(h^2),\ \|Z_k - u''\| \le M_2 p_k d + O(h^2),$$

where M_0, M_1, M_2 are given by (2.2.3) and p_k, d are given by (2.2.8).

Remark 1. The discrete iterative process (2.2.14)-(2.2.17) is performed until $\|\Phi_{k+1} - \Phi_k\| \le TOL$, where TOL is a given tolerance. From Theorem 2.2.6 it can be seen that the accuracy of the discrete approximate solution depends on both the number q defined in Theorem 2.2.1, which determines the number of iterations of the continuous iterative method and the gridsize h. The number q describes the nature of the BVP, therefore, it is necessary to choose an appropriate h consistent with q as the choice of very small h does not increase the accuracy of the approximate discrete solution.

2.2.3. Discrete iterative method 2

Consider another discrete iterative method, named **Method 2** for realizing the continuous iterative method (2.2.5)-(2.2.7). The steps of this method are the same as of Method 1 with an essential difference in Step 2 and it is now for the even number of grid points, $N = 2n$. Namely,

2'. Knowing $\Phi_k(t_i),\ k = 0, 1, ...;\ i = 0, ..., N$, compute approximately the definite integrals (2.2.6) by the modified Simpson rule

$$\begin{aligned} U_k(t_i) &= F(G_0(t_i, .)\Phi_k(.)), \\ Y_k(t_i) &= F(G_1(t_i, .)\Phi_k(.)), \\ Z_k(t_i) &= F(G_2^*(t_i, .)\Phi_k(.)), \end{aligned}$$

where

$$F(G_l(t_i, .)\Phi_k(.)) = \begin{cases} \sum_{j=0}^N h\rho_j G_l(t_i, t_j)\Phi_k(t_j) \ \text{ if } i \text{ is even} \\ \sum_{j=0}^N h\rho_j G_l(t_i, t_j)\Phi_k(t_j) + \dfrac{h}{6}\Big(G_l(t_i, t_{i-1})\Phi_k(t_{i-1}) - 2G_l(t_i, t_i)\Phi_k(t_i) \\ \quad +G_l(t_i, t_{i+1})\Phi_k(t_{i+1})\Big) \ \text{ if } i \text{ is odd}, \\ l = 0, 1;\ i = 0, 1, 2, ..., N. \end{cases}$$

ρ_j are the weights of the Simpson rule

$$\rho_j = \begin{cases} 1/3,\ j = 0, N \\ 4/3,\ j = 1, 3, ..., N-1 \\ 2/3,\ j = 2, 4, ..., N-2, \end{cases}$$

$F(G_2^*(t_i, .)\Phi_k(.))$ is computed in the same way as $F(G_l(t_i, .)\Phi_k(.))$ above, where G_l is replaced by G_2^* defined by (2.2.16).

Proposition 2.2.7. Suppose that the function $f(t, x, y, z)$ has all continuous partial derivatives up to fourth order in the domain $\mathcal{D}_M$. Then for the functions $u_k(t), y_k(t), z_k(t)$, $\varphi_{k+1}(t)$, $k = 0, 1, ...,$ constructed by the iterative method (2.2.5)-(2.2.7) we have $z_k(t) \in C^5[0,1]$, $y_k(t) \in C^6[0,1]$, $u_k(t) \in C^7[0,1], \varphi_{k+1}(t) \in C^4[0,1]$.

Proposition 2.2.8. For arbitrary $\varphi(t) \in C^4[0,1]$ there hold

$$\int_0^1 G_l(t_i, s)\varphi(s)ds = F(G_l(t_i, .)\varphi(.)) + O(h^3), \quad (l = 0, 1) \tag{2.2.27}$$

$$\int_0^1 G_2(t_i, s)\varphi(s)ds = F(G_2^*(t_i, .)\varphi(.)) + O(h^3). \tag{2.2.28}$$

Proof. Recall that the interval $[0, 1]$ is divided into $N = 2n$ subintervals by the points $t_i = ih, h = 1/N$. In each subinterval $[0, t_i]$ and $[t_i, 1]$ the functions $G_l(t_i, s)$ are polynomials. Hence, if i is even, $i = 2m$ then we represent

$$\int_0^1 G_l(t_i, s)\varphi(s)ds = \int_0^{t_{2m}} G_l(t_i, s)\varphi(s)ds + \int_{t_{2m}}^1 G_l(t_i, s)\varphi(s)ds.$$

Applying the Simpson rule to each integral in the right-hand side we get

$$\int_0^1 G_l(t_i, s)\varphi(s)ds = F(G_l(t_i, .)\varphi(.)) + O(h^4)$$

since $\varphi(t) \in C^4[0, 1]$.
Now consider the case $i = 2m + 1$. We have

$$\begin{aligned} I = \int_0^1 G_l(t_i, s)\varphi(s)ds &= \int_0^{t_{2m}} G_l(t_i, s)\varphi(s)ds + \int_{t_{2m}}^{t_{2m+1}} G_l(t_i, s)\varphi(s)ds \\ &+ \int_{t_{2m+1}}^{t_{2m+2}} G_l(t_i, s)\varphi(s)ds + \int_{t_{2m+2}}^1 G_l(t_i, s)\varphi(s)ds. \end{aligned} \tag{2.2.29}$$

For simplicity we denote

$$f_j = G_l(t_i, s_j)\varphi(s_j)$$

Applying the Simpson rule to the first and the fourth integrals in the right-hand side (2.2.29) and the trapezoidal rule to the second and the third integrals, we obtain

$$\begin{aligned} I &= \frac{h}{3}[f_0 + 4(f_1 + f_3 + ... + f_{2m-1}) + 2(f_2 + f_4 + ... + f_{2m-2}) + f_{2m}] + O(h^4) \\ &+ \frac{h}{2}(f_{2m} + f_{2m+1}) + O(h^3) + \frac{h}{2}(f_{2m+1} + f_{2m+2}) + O(h^3) \\ &+ \frac{h}{3}[f_{2m+2} + 4(f_{2m+3} + f_{2m+5} + ... + f_{2n-1}) + 2(f_{2m+4} + f_{2m+6} + ... + f_{2n-2}) + f_{2n}] + O(h^4) \\ &= \frac{h}{3}[f_0 + 4(f_1 + f_3 + ... + f_{2n-1}) + 2(f_2 + f_4 + ... + f_{2n-2}) + f_{2n}] \\ &+ \frac{h}{6}(f_{2m} - 2f_{2m+1} + f_{2m+2}) + O(h^3) \\ &= F(G_l(t_i, .)\varphi(.)) + O(h^3). \end{aligned}$$

Therefore, in both cases of i, even or odd, we have the estimate (2.2.27).
The estimate (2.2.28) is obtained similarly as (2.2.27) if noticing that

$$2G_2^*(t_i, t_i) = G_2^-(t_i, t_i) + G_2^+(t_i, t_i),$$

where $G_2^{\pm}(t_i, t_i) = \lim_{s \to t_i \pm 0} G_2(t_i, s)$. □

Theorem 2.2.9. Under the assumptions of Proposition 2.2.7, for the approximate solution of the problem (2.2.1) obtained by Method 2 on the uniform grid $\bar{\omega}_h$ we have the error estimates

$$\|U_k - u\| \le M_0 p_k d + O(h^3),\ \|Y_k - u'\| \le M_1 p_k d + O(h^3),$$
$$\|Z_k - u''\| \le M_2 p_k d + O(h^3).$$

Remark 2.2.2. For solving the nonhomogeneous problem (2.2.9) we can construct discrete iterative methods like to those for solving the homogeneous problem (2.2.1).

2.2.4. Examples

To confirm the validity of the obtained theoretical results and the efficiency of the proposed iterative method, we consider some examples. The exact solutions are either known or not known.

Example 2.2.1 (Problem 2 in [35])**.** Consider the problem

$$\begin{aligned} u'''(x) &= x^4 u(x) - u^2(x) + g(x),\ 0 < x < 1,\\ u(0) &= 0,\ u'(0) = -1,\ u'(1) = \sin(1), \end{aligned} \tag{2.2.30}$$

where $g(x) = -3\sin(x) - (x-1)\cos(x) - x^4(x-1)\sin(x) + (x-1)^2\sin^2(x)$. It is easy to verify that the exact solution of the problem is $u^*(x) = (x-1)\sin(x)$.

To apply the above theoretical results for homogeneous problems, we replace $u(x) = v(x) + P(x)$, where $P(x) = \dfrac{1}{2}(1+\sin(1))x^2 - x$ is the polynomial of second degree satisfying the boundary conditions in (2.2.30). Then the original nonhomogeneous problem for $u(x)$ is transformed to the following homogeneous problem for $v(x)$:

$$\begin{aligned} v'''(x) &= x^4 v(x) - v^2(x) - 2P(x)v(x) + x^4 P(x) - P^2(x) + g(x),\ 0 < x < 1,\\ v(0) &= 0,\ v'(0) = 0,\ v'(1) = 0, \end{aligned} \tag{2.2.31}$$

In order to apply Theorem 2.2.1, we need to determine the number M. For the right-hand side function

$$f(x,v) = -v^2(x) + x^4 v(x) - 2P(x)v(x) + x^4 P(x) - P^2(x) + g(x)$$

in the domain $\mathcal{D}_M = \{(x,v)|\ 0 \le t \le 1,\ |v| \le M_0 M\}$, where $M_0 = \frac{1}{12}$ we have

$$\begin{aligned} |f| &\le |v|^2 + |v| + 2|P(x)||v| + |x^4 P(x)| + |P(x)|^2 + |g(x)|\\ &\le (\frac{M}{12})^2 + (1 + 2 * 0.2715)\frac{M}{12} + 0.1 + 0.2715^2 + 4.12\\ &< \frac{M^2}{144} + \frac{1.55M}{12} + 4.3. \end{aligned}$$

Here we use the estimates

$$|P(x)| \le 0.2715,\ |x^4 P(x)| \le 0.1,\ x \in [0,1],$$

that are easily obtained. Besides, for estimating $|g(x)|$ we use the estimates

$$|(x-1)\sin(x)| \le 0.2401,\ |x^4(x-1)\sin(x)| \le 0.0596,\ x \in [0,1].$$

It is easy to prove that for $M = 6$ we have $\frac{M^2}{144} + \frac{1.55M}{12} + 4.3 < M$. It follows that $|f(x,v)| \leq M$ in $\mathcal{D}_M$. Furthermore, in $\mathcal{D}_M$ the function $f(x,v)$ satisfies the Lipschitz condition in v with the Lipschitz coefficient $L_0 = 2.543$. It implies that $q = 0.2119$. Therefore, all assumptions of Theorem 2.2.1 are met, and the problem has a unique solution, and the iterative method converges. The results of the numerical experiments with two different tolerances are reported in Tables 2.1- 2.3.

Table 2.1: The convergence in Example 2.2.1 for $TOL = 10^{-4}$

N	K	$Error_{trap}$	$Order$	$Error_{Simp}$	$Order$
8	3	9.9153e-04		9.7143e-04	
16	3	2.4646e-04	2.0083	1.3101e-04	2.8905
32	3	6.0906e-05	2.0167	1.6020e-05	3.0317
64	3	1.4563e-05	2.0643	1.2587e-06	3.6696
128	3	2.9796e-06	2.2891	8.8553e-07	0.5073
256	3	4.3187e-07	2.7865	8.8165e-07	0.0063

Table 2.2: The convergence in Example 2.2.1 for $TOL = 10^{-6}$

N	K	$Error_{trap}$	$Order$	$Error_{Simp}$	$Order$
8	4	9.99237e-04		9.7223e-04	
16	4	2.4734e-04	2.0044	1.3189e-04	2.8820
32	4	6.1802e-05	2.0008	1.6915e-05	2.9629
64	4	1.5462e-05	1.9989	2.1492e-06	2.9765
128	4	3.8797e-06	1.9947	2.8688e-07	2.9053
256	4	9.8437e-07	1.9787	5.2749e-08	2.4439
512	4	2.6054e-07	1.9177	2.3446e-08	1.1698
1024	4	7.9583e-08	1.7110	1.9786e-08	0.2448

Table 2.3: The convergence in Example 2.2.1 for $TOL = 10^{-10}$

N	K	$Error_{trap}$	$Order$	$Error_{Simp}$	$Order$
8	7	9.9235e-04		9.7222e-04	
16	7	2.4732e-04	2.0045	1.3187e-04	2.8822
32	7	6.1782e-05	2.0011	1.6896e-05	2.9643
64	7	1.5443e-05	2.0003	2.1301e-06	2.9877
128	7	3.8605e-06	2.0001	2.6774e-07	2.9923
256	7	9.6511e-07	2.0000	3.3544e-08	2.9965
512	7	2.4128e-07	2.0000	4.1977e-09	2.9984
1024	7	6.0319e-08	2.0000	5.2483e-10	2.9997

In these tables, N and K are the numbers of grid points and iterations, $Error_{trap}$, $Error_{Simp}$ are the errors $\|U_K - u^*\|$ when using Method 1 and Method 2, respectively, $Order$ is the order of convergence determined by

$$Order = \log_2 \frac{\|U_K^{N/2} - u^*\|}{\|U_K^N - u^*\|}.$$

Here, the superscripts $N/2$ and N of U_K are the number of grid points used to compute U_K on the grid.

From the above tables, it can be seen that for each tolerance the number of iterations is constant and the errors of the approximate solution decrease with the rate (or order) close to 2 for Method 1 and close to 3 for Method 2 until they no longer can be improved. We can explain this as follows. The total error of the actual approximate solution consists of two terms: the error of the iterative method on continuous level and the error of numerical integration at each iteration. When these errors are balanced, the further increase of number of grid points N (or equivalently, the decrease of grid size h) cannot in general improve the accuracy of approximate solution.

Notice that in [35] the author used Newton-Raphson iteration method to solve nonlinear system of equations arising after discretization of the differential problem. The iteration process is continued until the maximum difference between two successive iterations , i.e., $\|U_{k+1} - U_k\|$ is less than 10^{-10}. The number of iterations for achieving this tolerance is not reported. The accuracy for some different N is reported in Table 2.4 (see [35, Table 2]).

Table 2.4: The results in [35] for the problem in Example 2.2.1

N	8	16	32	64
Error	0.11921225e-01	0.33391170e-02	0.87742222e-03	0.23732412e-03

From the above tables, it is clear that our method yields much better accuracy.

Example 2.2.2 (Problem 2 in [36])**.** Consider the problem

$$\begin{aligned} u'''(x) &= -xu''(x) - 6x^2 + 3x - 6,\ 0 < x < 1, \\ u(0) &= 0,\ u'(0) = 0,\ u'(1) = 0. \end{aligned}$$

It is easy to show that with $M = 18, L_0 = L_1 = 0, L_2 = 1, q = 0.5$, the conditions of Theorem 2.2.1 are met, thus the problem has a unique solution. This solution is $u(x) = x^2(\frac{3}{2} - x)$. The results of the numerical experiments with different tolerances are reported in Tables 2.5, 2.6 and 2.7.

Table 2.5: The convergence in Example 2.2.2 for $TOL = 10^{-4}$

N	K	$Error_{trap}$	$Order$	$Error_{Simp}$	$Order$
8	6	0.0078		9.7662e-04	
16	6	0.0020	2.0000	1.2215e-04	2.9991
32	6	4.8837e-04	1.9998	1.5345e-05	2.9929
64	6	1.2216e-04	1.9992	1.9936e-06	2.9443
128	6	3.0604e-05	1.9969	3.2471e-07	2.6181
256	6	7.7157e-06	1.9878	1.1612e-07	1.4835

In [36] the author used Gauss-Seidel iteration method to solve linear system of equations arisen after discretization of the differential problem. The iteration process is continued until the maximum difference between two successive iterations $\|U_{k+1} - U_k\| < 10^{-10}$. The results for some different N are reported in Table 2.8.

From the above tables, it is clear that our method yields better accuracy and requires less computational work.

Table 2.6: The convergence in Example 2.2.2 for $TOL = 10^{-6}$

N	K	$Error_{trap}$	$Order$	$Error_{Simp}$	$Order$
8	8	0.0078		9.7662e-04	
16	6	0.0020	2.0000	1.2215e-04	2.9991
32	6	4.8837e-04	1.9998	1.5345e-05	2.9929
64	6	1.2216e-04	1.9992	1.9936e-06	2.9443
128	6	3.0604e-05	1.9969	3.2471e-07	2.6181
256	6	7.7157e-06	1.9878	1.1612e-07	1.4835
512	6	1.9937e-06	1.9524	9.0051e-08	0.3868
1024	6	5.6316e-07	1.8238	8.6794e-08	0.0532

Table 2.7: The convergence in Example 2.2.2 for $TOL = 10^{-10}$

N	K	$Error_{trap}$	$Error_{Simp}$	N	K	$Error_{trap}$	$Error_{Simp}$
8	11	0.0078	2.0650e-13	64	11	1.2207e-04	2.5890e-13
16	11	0.0020	2.6790e-13	128	11	3.0518e-05	2.5790e-13
32	11	4.8828e-04	2.6279e-13	256	11	7.6294e-06	2.5802e-13

Table 2.8: The results in [36] for the problem in Example 2.2.2

N	128	256	512	1024
Error	0.30696392e-4	0.61094761(-5)	0.14379621e-5	0.41723251e-6
Iter	53	5	3	4

Example 2.2.3. Consider the problem for fully third order differential equation

$$\begin{aligned} &u'''(x) = -e^{u(x)} - e^{u'(x)} - \frac{1}{10}(u''(x))^2,\ 0 < x < 1, \\ &u(0) = 0,\ u'(0) = 0,\ u'(1) = 0. \end{aligned} \tag{2.2.32}$$

The exact solution is not known. It is easy to show that with $M = 3, L_0 = 1.284, L_1 = 1.455, L_2 = 0.3$ and $q = 0.4389$ the conditions of Theorem 2.2.1 are met. Thus, the problem has a unique solution and the iterative process converges.

Table 2.9: The convergence in Example 2.2.3 for $TOL = 10^{-10}$

N	8	16	32	64	128	256
K	15	15	15	15	15	15

The numerical solution of the problem is depicted in Figure 2.7.

In [17] the authors could only obtain the existence but not the uniqueness of a solution to the equation $u'''(x) = -e^{u(x)}$ subject to the boundary conditions as in (2.2.32), and later, in [19] the authors also could only establish a similar result for the equation $u'''(x) = -e^{u(x)} - e^{u'(x)}$.

Remark 2.2.3 (Convergence of the iterative method)**.** It should be emphasized that Theorem 2.2.1 only provides sufficient conditions for the existence and uniqueness of a solution to the problem (2.2.1) and Theorem 2.2.2 gives the convergence rate of the iterative method for finding the solution. When these conditions are not met, the iterative method may or may not converge. To illustrate this remark, we give some examples below.

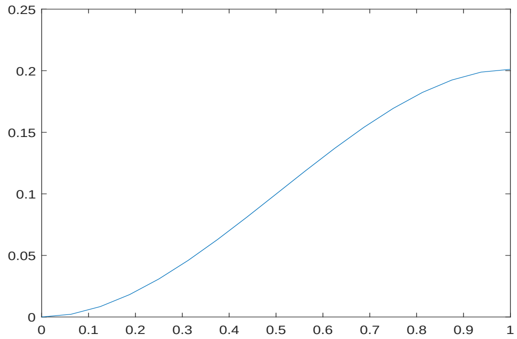

Figure 2.7: Approximate solution in Example 2.2.3.

First, consider the problem

$$u'''(x) = -e^{u(x)} - e^{u'(x)} - (u''(x))^2, \; 0 < x < 1,$$
$$u(0) = 0, \; u'(0) = 0, \; u'(1) = 0.$$

Here, the right-hand side function is $f(x, u, y, z) = -e^u - e^y - z^2$. In the domain

$$\mathcal{D}_M = \left\{ (x, u, y, z) |\ 0 \le x \le 1, \; |u| \le \frac{M}{12}, \; |y| \le \frac{M}{8}, \; |z| \le \frac{M}{2} \right\}$$

we have

$$g(M) := \max_{(x,u,y,z) \in \mathcal{D}_M} |f(x, u, y, z)| = e^{M/12} + e^{M/8} + \left(\frac{M}{2}\right)^2.$$

It is easy to show that $g(M) \ge M + 1.4019 > M$ for any $M > 0$. Hence, there does not exist $M >$ such that $|f(x, u, y, z)| \le M \; \forall (x, u, y, z) \in \mathcal{D}_M$. Therefore, Theorem 2.2.1 cannot guarantee the existence and uniqueness of a solution and the convergence of the iterative method. Nevertheless, for $TOL = 10^{-10}$ the iterative method converges after 23 iterations.

Next, an example when the conditions of Theorem 2.2.1 are not satisfied and the iterative method does not converge is for the equation

$$u'''(x) = -e^{u(x)} - e^{u'(x)} - (u''(x))^2 + 5u''(x) + 10, \; 0 < x < 1.$$

2.2.5. On some extensions of the problem

2.2.5.1. The problem on large intervals

First consider the problem (2.2.1) on the interval $[0, T]$, i.e., the problem

$$\begin{aligned} u^{(3)}(t) &= f(t, u(t), u'(t), u''(t)), \quad 0 < t < T, \\ u(0) &= 0, u'(0) = 0, u'(T) = 0. \end{aligned} \tag{2.2.33}$$

For this problem, it is easy to verify that the Green's function is

$$G_0(t, s) = \begin{cases} \dfrac{s}{2}\left(\dfrac{t^2}{T} - 2t + s\right), & 0 \le s \le t \le T, \\ \dfrac{t^2}{2}\left(\dfrac{s}{T} - 1\right), & 0 \le t \le s \le T. \end{cases}$$

The first and second derivatives of this function with respect to t are

$$G_1(t,s) = \begin{cases} s(\frac{t}{T}-1), & 0 \le s \le t \le T, \\ t(\frac{s}{T}-1), & 0 \le t \le s \le T, \end{cases}$$

$$G_2(t,s) = \begin{cases} \frac{s}{T}, & 0 \le s \le t \le T, \\ \frac{s}{T}-1, & 0 \le t \le s \le T. \end{cases}$$

It is easy to see that $G_0(t,s) \le 0,\ G_1(t,s) \le 0$ in $Q = [0,1]^2$ and

$$\begin{aligned} M_0 &= \max_{0\le t\le T} \int_0^T |G(t,s)|\, ds = \frac{T^3}{12}, \quad M_1 = \max_{0\le t\le T} \int_0^T |G_1(t,s)|\, ds = \frac{T^2}{8}, \\ M_2 &= \max_{0\le t\le T} \int_0^T |G_2(t,s)|\, ds = \frac{T}{2}. \end{aligned} \tag{2.2.34}$$

Clearly, the numbers M_i $(i = 0, 1, 2)$ increase with the increase of T. Therefore, the domain $\mathcal{D}_M$ becomes more extended. This implies that the Lipschitz coefficients L_0, L_1, L_2 of the function $f(t, x, y, z)$ with respect to x, y, z do not decrease, and accordingly, the number $q = L_0M_0 + L_1M_1 + L_2M_2$ increases. This leads to narrowing the scope of applicability of Theorem 2.2.1 on the existence and uniqueness of solution and Theorem 2.2.2 on the convergence of the iterative method.

For demonstrating the above remark we consider some examples.

Example 2.2.4. Consider the problem on $[0, T]$ for the equation of Example 2.2.2, namely, the problem

$$\begin{aligned} u'''(x) &= -xu''(x) - 6x^2 + 3x - 6,\ 0 < x < T, \\ u(0) &= 0,\ u'(0) = 0,\ u'(T) = 0. \end{aligned}$$

Below are the results of convergence for Discrete iterative method 2 with $n = 256$ for some T:

Table 2.10: The convergence in Example 2.2.4 for $TOL = 10^{-6}$

T	1	2	3	4	5
K	8	18	82	2009	no convergence

Here K is the number of iterations for achieving the given tolerance TOL. Notice that from $T = 2$ the conditions of Theorem 2.2.1 are not satisfied but only from $T = 5$ the iterative method diverges. From Table 2.10 clearly that the convergence of the iterative method depends on the width of the interval, where the problem is considered.

Example 2.2.5. Consider the problem

$$\begin{aligned} u'''(x) &= -\frac{1}{6}e^{-u^2} + e^{-(u'')^2},\ 0 < x < T, \\ u(0) &= 0,\ u'(0) = 0,\ u'(T) = 0. \end{aligned}$$

For this example the right-hand side function is $f = f(x, u, y, z) = -\frac{1}{6}e^{-u^2} + e^{-(z)^2}$. In any domain

$$\mathcal{D}_M = \left\{ (x, u, y, z)|\ 0 \le x \le T,\ |u| \le \frac{T^3}{12}M,\ |y| \le \frac{T^2}{8}M,\ |z| \le \frac{T}{2}M \right\},$$

we always have $|f| \leq \frac{7}{6}$. Therefore, in Theorem 2.2.1 we take $M = \frac{7}{6}$. The Lipschitz coefficients of the function f are $L_0 = 0.1430, L_1 = 0, L_2 = 0.8579$. So, $q = 0.1430\ \frac{T^3}{12} + 0.8579\ \frac{T}{2} = 0.0119\ T^3 + 0.4289\ T$. Clearly, for large values of T not all conditions of Theorem 2.2.1 are satisfied, and it is expected that the iterative method will diverge for large T. But it is interesting that this does not occur. Below are the results of the convergence of the iterative method for $n = 200$.

Table 2.11: The convergence in Example 2.2.5 for $TOL = 10^{-6}$

T	1	3	5	10	15	20	40	100
K	6	12	13	16	18	20	27	37

The approximate solution for $T = 100$ is depicted in Figure 2.8.

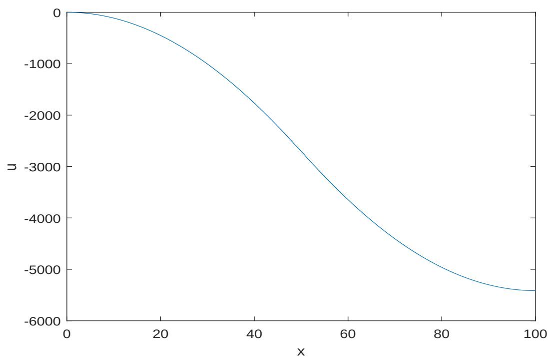

Figure 2.8: Approximate solution in Example 2.2.5.

2.2.5.2. The problem for unbounded nonlinear terms

For the problem with unbounded nonlinear terms (right-hand sides) $f(t, u, y, z)$ caused by singular points, of course, Theorem 2.2.1 cannot work, and Theorem 2.2.2 cannot ensure the convergence of the iterative method. But it is interesting that in some special cases the discrete iterative methods still converge. Below we report some nonlinear terms $f(t, u, y, z)$ for which the iterative method converge:

$$(i)\ \frac{u^2}{\sqrt{|t - \frac{\pi}{4}|}} + e^y + 1, \quad (ii)\ \frac{u^2}{|t - \frac{\pi}{4}|} + e^y + z^2 + 1, \quad (iii)\ \frac{u^2}{|t - \frac{\sqrt{2}}{4}|} + e^y + z^2 + 1.$$

Notice that in the above three functions the singular points are irrational points, therefore, when using the discrete methods on the grids with rational points then the denominators always are not zero. For this reason the computations can be performed.

When we use the uniform grids with the number of grid points $n = 2^k, k = 3, 4, 5, ...,$ the iterative methods also converge for $f = \dfrac{u^2}{\sqrt{|t - \frac{1}{3}|}} + e^y + 1$. This is due to the fact that $i/2^k \neq 1/3$ for any i and k.

Above we only made some remarks on the problem (2.2.1) when the nonlinear term is unbounded. In the future we will study this issue deeply.

2.2.6. Conclusion

In this section, first we proved the existence and uniqueness of a solution for a boundary value problem for fully third order differential equations. Next, to find this solution we proposed iterative methods at both continuous and discrete levels. The numerical implementation of the discrete iterative methods is very simple. It is based on the popular trapezoidal rule of second order accuracy and a modified Simpson rule of third order accuracy for numerical integration. One of the important results in analysis of the proposed numerical methods is that we obtained an estimate for the total error of the actually obtained approximate solution. This total error depends on the number of iterations performed and the discretization parameter, namely, the gridsize. The validity of the theoretical results and the efficiency of the iterative methods are illustrated on examples. In addition, we made some remarks on the iterative method for two extensions of the problem to large intervals and to the case when the nonlinear terms are unbounded due to interior singular points. In the future we will deeply study these issues.

The method for investigating the existence and uniqueness of solution and the iterative schemes for finding solution in this section can be applied to other third order nonlinear boundary value problems, and in general, for higher order nonlinear boundary value problems.

2.3. Chapter conclusion

In this chapter, we have successfully proposed a novel unified method for studying third order nonlinear boundary value problems. It is based on the reduction of boundary value problems to operator equation for nonlinear term (or right-hand side) of the differential equations. This is the essential difference from the existing methods of some other authors. Due to the reduction of boundary value problems to operator equations we have established the existence and uniqueness of solutions of third order differential equation associated with many linear boundary conditions. Besides, we have constructed iterative methods on continuous level and on discrete level for finding the solution. The total error estimate of actual numerical solution was obtained, and many numerical examples supported the theoretical results.

It should be emphasized that by the proposed unified method we have obtained some results which are better than those of other authors, and this method is applicable to other boundary value problems. Besides, the approach for constructing numerical methods for boundary value problems here is completely novel because it is based on the discretization of the corresponding iterative method on continuous level, meanwhile other authors directly approximate differential equations.

The results of this chapter were published in the Scopus paper [AL1] and in the SCIE paper [AL2].

Chapter 3

The existence, uniqueness of a solution and an iterative method for some nonlinear ODEs with integral boundary conditions

3.1. Existence results and an iterative method for fully third order nonlinear integral boundary value problems

3.1.1. Introduction

In this section, we consider the boundary value problem

$$u'''(t) = f(t, u(t), u'(t), u''(t)), \quad 0 < t < 1, \tag{3.1.1}$$

$$u(0) = u'(0) = 0, \quad u(1) = \int_0^1 g(s)u(s)ds, \tag{3.1.2}$$

where $f \in C([0,1] \times \mathbb{R}^3, \mathbb{R}^+), g \in C([0,1], \mathbb{R}^+)$.

A particular case of this problem, namely, the problem

$$\begin{aligned} u'''(t) + f(u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = 0, \ u(1) = \int_0^1 g(t)u(t)dt \end{aligned} \tag{3.1.3}$$

was studied recently by Guendouz et al. in [47]. There, by using Krasnoselskii's fixed point theorem on cones they obtained the existence results of positive solutions of the problem. This technique was used also by Benaicha and Haddouchi in [48] for a fourth order nonlinear boundary problem involving integral conditions.

It should be emphasized that in all of the above-mentioned works the authors only could (even could not) show examples of the nonlinear terms satisfying required sufficient conditions, but *no exact solutions were shown. Moreover, the known results are of purely theoretical characteristics concerning the existence of solutions but not solution methods.*

Here, by the method of reducing BVPs to operator equation for nonlinear terms developed in [13, 14, 86, 89] we obtain the existence, uniqueness and positivity of solution and propose an iterative method for finding the solution. Several examples demonstrate the validity of the theoretical results obtained and the efficiency of the iterative method. Especially, one example of exact solution of the problem is designed so that the functions f and g satisfy all the required conditions.

3.1.2. Existence and uniqueness of solution

To study the problem (3.1.1)-(3.1.2) we reduce it to an operator equation as follows.
First, we introduce the space $\mathcal{B} = C[0,1] \times \mathbb{R}$ of all pairs $w = (\varphi, \alpha)^T$, where $\varphi \in C[0,1], \alpha \in \mathbb{R}$ and equip it with the norm

$$\|w\|_{\mathcal{B}} = \max(\|\varphi\|, r|\alpha|), \tag{3.1.4}$$

where $\|\varphi\| = \max_{0\le t\le 1} |\varphi(t)|$, r is a real number, $r \geq 1$. The constant r will play a significant role in the conditions for the existence and uniqueness of solution and will be selected later in each particular case.

Next we define the operator $A : \mathcal{B} \to \mathcal{B}$ by the equation

$$Aw = \begin{pmatrix} f(t, u(t), u'(t), u''(t) \\ \int_0^1 g(t)u(t)dt \end{pmatrix}, \tag{3.1.5}$$

where $u(t)$ is the solution of the problem

$$u'''(t) = \varphi(t), \quad 0 < t < 1, \tag{3.1.6}$$

$$u(0) = u'(0) = 0, \ u(1) = \alpha. \tag{3.1.7}$$

It is easy to verify the following lemma.

Lemma 3.1.1. If $w = (\varphi, \alpha)^T$ is a fixed point of the operator A in the space $\mathcal{B}$, that is, w is a solution of the operator equation

$$Aw = w \tag{3.1.8}$$

in $\mathcal{B}$, then the function $u(t)$ defined from the problem (3.1.6)-(3.1.7) is a solution of the original problem (3.1.1)-(3.1.2).
Conversely, if $u(t)$ is a solution of (3.1.1)-(3.1.2), then the pair $(\varphi, \alpha)^T$, where

$$\varphi(t) = f(t, u(t), u'(t), u''(t)), \tag{3.1.9}$$

$$\alpha = \int_0^1 g(t)u(t)dt, \tag{3.1.10}$$

is a solution of the operator equation (3.1.8).

Thus, by this lemma, the problem (3.1.1)-(3.1.2) is reduced to the fixed point problem for A.

Remark that the above operator A, which is defined on pairs of functions $\varphi(t)$, $t \in [0,1]$ and boundary values α of $u(t)$ at $t = 1$, is similar to the mixed boundary-domain operator introduced in [90] for studying biharmonic type equation.

Now, we study the properties of A. For this purpose, notice that the problem (3.1.6)-(3.1.7) has a unique solution representable in the form

$$u(t) = \int_0^1 G_0(t,s)\varphi(s)ds + \alpha t^2, \quad 0 < t < 1, \tag{3.1.11}$$

where

$$G_0(t,s) = \begin{cases} -\frac{1}{2}s(1-t)(2t - ts - s), & 0 \le s \le t \le 1 \\ -\frac{1}{2}(1-s)^2 t^2, & 0 \le t \le s \le 1 \end{cases}$$

is the Green's function of the operator $u'''(t)$ associated with the homogeneous boundary conditions $u(0) = u'(0) = u(1) = 0$.

Taking the derivative of both sides of (3.1.11) yields

$$u'(t) = \int_0^1 G_1(t,s)\varphi(s)ds + 2\alpha t, \tag{3.1.12}$$

$$u''(t) = \int_0^1 G_2(t,s)\varphi(s)ds + 2\alpha, \tag{3.1.13}$$

where $G_1(t,s)$ and $G_1(t,s)$ are the first and second derivatives of $G(t,s)$ with respect to t:

$$G_1(t,s) = \begin{cases} -s(st-2t+1), & 0 \le s \le t \le 1, \\ -(1-s)^2 t, & 0 \le t \le s \le 1, \end{cases}$$

$$G_2(t,s) = \begin{cases} -s(s-2), & 0 \le s \le t \le 1, \\ -(1-s)^2, & 0 \le t \le s \le 1. \end{cases}$$

It is easy to see that $G_0(t,s) \le 0$ in $Q = [0,1]^2$, and

$$\begin{aligned} M_0 &= \max_{0\le t\le 1} \int_0^1 |G_0(t,s)|ds = \frac{2}{81}, \\ M_1 &= \max_{0\le t\le 1} \int_0^1 |G_1(t,s)|ds = \frac{1}{18}, \\ M_2 &= \max_{0\le t\le 1} \int_0^1 |G_2(t,s)|ds = \frac{2}{3}. \end{aligned} \tag{3.1.14}$$

Therefore, from (3.1.11), (3.1.12), (3.1.13) and (3.1.14) we obtain

$$\begin{aligned} \|u\| &\le M_0\|\varphi\| + |\alpha|, \\ \|u'\| &\le M_1\|\varphi\| + 2|\alpha|, \\ \|u''\| &\le M_2\|\varphi\| + 2|\alpha|. \end{aligned} \tag{3.1.15}$$

Now for any real number $M > 0$ define the domain

$$\begin{aligned} \mathcal{D}_M = \{(t,x,y,z) \mid 0 \le t \le 1,\ |x| \le (M_0 + \frac{1}{r})M, \\ |y| \le (M_1 + \frac{2}{r})M,\ |z| \le (M_2 + \frac{2}{r})M\}. \end{aligned} \tag{3.1.16}$$

Next, denote

$$C_0 = \int_0^1 g(t)dt,\ C_2 = \int_0^1 t^2 g(t)dt. \tag{3.1.17}$$

Lemma 3.1.2. Suppose that the function $f(t,x,y,z)$ is continuous in $\mathcal{D}_M$, and

$$|f(t,x,y,z)| \le M \quad \text{in } \mathcal{D}_M \tag{3.1.18}$$

and

$$q_1 := rC_0M_0 + C_2 \le 1. \tag{3.1.19}$$

Then the operator A defined by (3.1.5) maps $B[0,M] \in \mathcal{B}$ into itself.

Proof. Take any $w = (\varphi, \alpha)^T \in B[0, M]$. Then $\|\varphi\| \le M$ and $|\alpha| \le M/r$. Let $u(t)$ be the solution of the problem (3.1.6)-(3.1.7). Then from the estimates (3.1.15) for the solution $u(t)$ and its derivatives we obtain

$$\|u\| \le \left(M_0 + \frac{1}{r}\right) M, \quad \|u'\| \le \left(M_1 + \frac{2}{r}\right) M, \quad \|u''\| \le \left(M_2 + \frac{2}{r}\right) M.$$

Therefore, $(t, u(t), u'(t), u''(t)) \in \mathcal{D}_M$. Hence, by the assumption (3.1.18) we have

$$|f(t, u(t), u'(t), u''(t))| \le M.$$

Now estimate $J := k|\int_0^1 g(t)u(t)dt|$. In view of the representation (3.1.11) we obtain

$$\begin{aligned} J \le & r \int_0^1 g(t) \left| \int_0^1 G_0(t, y)\varphi(y)dy \right| dt + r|\alpha| \int_0^1 g(t)t^2 dt \\ & \le rC_0M_0M + C_2M = (rC_0M_0 + C_2)M \le M. \end{aligned} \tag{3.1.20}$$

The above inequalities are valid due to (3.1.14), (3.1.17) and the assumption (3.1.19).

Therefore, by the definition of the norm in the space $\mathcal{B}$ we have

$$\|Aw\|_{\mathcal{B}} \le M,$$

which means that the operator A maps the closed ball $B[0, M]$ in $\mathcal{B}$ into itself. The lemma is proved. □

Lemma 3.1.3. The operator A is a compact operator in $B[0, M]$.

Proof. Indeed, the compactness of A follows from the compactness of the integral operators (3.1.11), (3.1.12), (3.1.13) for the function φ, the continuity of the function $f(t, x, y, z)$ and the compactness of the integral operator $\int_0^1 g(t)u(t)dt$ for the function u. □

Theorem 3.1.1 (Existence of solution)**.** Suppose the conditions of Lemma 3.1.2 are met. Then the problem (3.1.1)-(3.1.2) has a solution.

Proof. By Lemma 3.1.2 and Lemma 3.1.3, the operator A is a compact operator in the Banach space $\mathcal{B}$ mapping the closed ball $B[0, M]$ into itself. Therefore, by the Schauder fixed point theorem, it has a fixed point in $B[0, M]$. This fixed point corresponds to a solution of the problem (3.1.1)-(3.1.2). □

To establish the existence of positive solutions of (3.1.1)-(3.1.2) we introduce the domain

$$\begin{aligned} \mathcal{D}_M^+ = \{(t, x, y, z) \mid 0 \le t \le 1,\ 0 \le x \le (M_0 + \frac{1}{r})M, \\ |y| \le (M_1 + \frac{2}{r})M,\ |z| \le (M_2 + \frac{2}{r})M\}, \end{aligned} \tag{3.1.21}$$

in the space $[0, 1] \times \mathbb{R}^3$ and the domain

$$S_M = \{w = (\varphi, \alpha)^T \mid -M \le \varphi \le 0,\ 0 \le r\alpha \le M\} \tag{3.1.22}$$

in the space $\mathcal{B}$.

Theorem 3.1.2 (Positivity of solution)**.** Assume that the function $f(t,x,y,z)$ is continuous and

$$-M \leq f(t,x,y,z) \leq 0 \text{ in } \mathcal{D}_M^+. \tag{3.1.23}$$

In addition, the condition (3.1.19) is satisfied. Then the problem (3.1.1)-(3.1.2) has a non-negative solution. Moreover, if $f(t,0,0,0) \not\equiv 0$ then this solution is positive.

Proof. First, notice that under the assumptions of the theorem, the operator A maps S_M into itself. Indeed, for any $w = (\varphi, \alpha)^T \in S_M$, $-M \leq \varphi \leq 0, 0 \leq r\alpha \leq M$. Since $G_0(t,s) \leq 0$, from (3.1.11), (3.1.12), (3.1.13) we have

$$0 \leq u(t) \leq (M_0 + \frac{1}{r})M, \ |u'(t)| \leq (M_1 + \frac{2}{r})M, \ |u''(t)| \leq (M_2 + \frac{2}{r})M, \ 0 \leq t \leq 1.$$

So, for the solution $u(t)$ of (3.1.6)-(3.1.7) we have $(t, u(t), u'(t), u''(t)) \in \mathcal{D}_M^+$, and by the assumption (3.1.23) we obtain

$$-M \leq f(t, u(t), u'(t), u''(t)) \leq 0.$$

As in the proof of Theorem 3.1.1 we also have the estimate

$$0 \leq r \int_0^1 g(t)u(t)dt \leq M.$$

Hence, $\big(f(t,u(t),u'(t),u''(t)), \int_0^1 g(t)u(t)dt\big)^T \in S_M$, that is, $A : S_M \to S_M$.
As was shown above, A is a compact operator in S. Therefore, A has a fixed point in S_M, which generates a solution of the problem (3.1.1)-(3.1.2). This solution is nonnegative. Moreover, if $f(t,0,0,0) \not\equiv 0$ then $u(t) \equiv 0$ cannot be the solution. Therefore, the solution is positive. □

Theorem 3.1.3 (Existence and uniqueness)**.** Suppose that there exist numbers $M > 0, L_0, L_1, L_2 \geq 0$ such that

(H1) $|f(t,x,y,z)| \leq M, \ \forall (t,x,y,z) \in \mathcal{D}_M$.

(H2) $|f(t,x_2,y_2,z_2) - f(t,x_1,y_1,z_1)| \leq L_0|x_2 - x_1| + L_1|y_2 - y_1| + L_2|z_2 - z_1|, \ \forall (t,x_i,y_i,z_i) \in \mathcal{D}_M, \ i = 1,2$.

(H3) $q := \max\{q_1, q_2\} < 1$, where $q_1 = rC_0M_0 + C_2$ as was defined by (3.1.19) and

$$q_2 = L_0(M_0 + \frac{1}{r}) + L_1(M_1 + \frac{2}{r}) + L_2(M_2 + \frac{2}{r}). \tag{3.1.24}$$

Then the problem (3.1.1)-(3.1.2) has a unique solution $u \in C^3[0,1]$.

Proof. To prove the theorem, it suffices to show that the operator A defined by (3.1.5) is a contraction map from $B[0,M] \in \mathcal{B}$ into itself. Indeed, under the assumption $(H1)$ and the condition $q_1 < 1$ in the assumption $(H2)$, by Lemma 3.1.2 the operator A maps $B[0,M]$ into itself.

Now, we show that A is a contraction map.

Let $w_i = (\varphi_i, \alpha_i)^T \in B[0,M]$. We have

$$Aw_2 - Aw_1 = \begin{pmatrix} f(t,u_2(t),u_2'(t),u_2''(t) - f(t,u_1(t),u_1'(t),u_1''(t) \\ \int_0^1 g(t)(u_2(t) - u_1(t))ds \end{pmatrix},$$

where $u_i(t)$ $(i = 1, 2)$ is the solution of the problem

$$\begin{cases} u_i'''(t) = \varphi_i(t), & 0 < t < 1 \\ u_i(0) = u_i'(0) = 0, \; u_i(1) = \alpha_i. \end{cases}$$

From the proof of Lemma 3.1.2 it is known that $(t, u_i(t), u_i'(t), u_i''(t)) \in \mathcal{D}_M$. Therefore, by the Lipschitz condition $(H2)$ for f we have

$$\begin{aligned} D_1 :&= |f(t, u_2(t), u_2'(t), u_2''(t) - f(t, u_1(t), u_1'(t), u_1''(t)| \\ &\le L_0|u_2(t) - u_1(t)| + L_1|u_2'(t) - u_1'(t)| + L_2|u_2''(t) - u_1''(t)|. \end{aligned} \tag{3.1.25}$$

Since $u_2(t) - u_1(t)$ is the solution of the problem (3.1.6)-(3.1.7) with the right-hand sides $\varphi_2(t) - \varphi_1(t)$ and $\alpha_2 - \alpha_1$, we have

$$\begin{aligned} \|u_2 - u_1\| &\le M_0\|\varphi_2 - \varphi_1\| + |\alpha_2 - \alpha_1|, \\ \|u_2' - u_1'\| &\le M_1\|\varphi_2 - \varphi_1\| + 2|\alpha_2 - \alpha_1|, \\ \|u_2'' - u_1''\| &\le M_2\|\varphi_2 - \varphi_1\| + 2|\alpha_2 - \alpha_1|. \end{aligned} \tag{3.1.26}$$

As for the element $w = (\varphi, \alpha)^T \in \mathcal{B}$ we use the norm

$$\|w\|_{\mathcal{B}} = \max(\|\varphi\|, r|\alpha|) \quad (r \ge 1),$$

from (3.1.25), (3.1.26) we obtain

$$\begin{aligned} D_1 &\le L_0\left(M_0 + \frac{1}{r}\right)\|w_2 - w_1\|_{\mathcal{B}} + L_1\left(M_1 + \frac{2}{r}\right)\|w_2 - w_1\|_{\mathcal{B}} \\ &+ L_2\left(M_2 + \frac{2}{r}\right)\|w_2 - w_1\|_{\mathcal{B}} \\ &\le \left(L_0\left(M_0 + \frac{1}{r}\right) + L_1\left(M_1 + \frac{2}{r}\right) + L_2\left(M_2 + \frac{2}{r}\right)\right)\|w_2 - w_1\|_{\mathcal{B}} \\ &= q_2\|w_2 - w_1\|_{\mathcal{B}}, \end{aligned} \tag{3.1.27}$$

where q_2 is defined by (3.1.24).
Now consider

$$D_2 := r\Big|\int_0^1 g(t)(u_2(t) - u_1(t))dt\Big|.$$

By analogy with the estimate (3.1.20) it is easy to have

$$D_2 \le (rC_0M_0 + C_2)\|w_2 - w_1\|_{\mathcal{B}} = q_1\|w_2 - w_1\|_{\mathcal{B}}. \tag{3.1.28}$$

From (3.1.27) and (3.1.28) we obtain

$$\|Aw_2 - Aw_1\|_{\mathcal{B}} \le \max\{q_1, q_2\}\|w_2 - w_1\|_{\mathcal{B}}.$$

In view of condition $(H3)$ the operator A is a contraction operator in $B[0, M]$. The theorem is proved. □

Theorem 3.1.4 (Existence and uniqueness of positive solution)**.** If in Theorem 3.1.3 replace $\mathcal{D}_M$ by $\mathcal{D}_M^+$ and the condition $(H1)$ by the condition (3.1.23) then the problem (3.1.1)-(3.1.2) has a unique nonnegative solution $u(t) \in C^3[0, 1]$. Besides, if $f(t, 0, 0, 0) \not\equiv 0$ then this solution is positive.

3.1.3. Iterative method

Assume that all the conditions of Theorem 3.1.3 are met. Then the problem (3.1.1)-(3.1.2) has a unique solution. To find it, consider the following iterative method:

1. Given $w_0 = (\varphi_0, \alpha_0)^T \in B[0, M]$, for example,

$$\varphi_0(t) = f(t, 0, 0, 0), \ \ \alpha_0 = 0. \tag{3.1.29}$$

2. Knowing $\varphi_n(t)$ and α_n $(n = 0, 1, ...)$, compute

$$u_n(t) = \int_0^1 G(t, s)\varphi_n(s)ds + \alpha_n t^2, \tag{3.1.30}$$

$$y_n(t) = \int_0^1 G_1(t, s)\varphi_n(s)ds + 2\alpha_n t, \tag{3.1.31}$$

$$z_n(t) = \int_0^1 G_2(t, s)\varphi_n(s)ds + 2\alpha_n. \tag{3.1.32}$$

3. Compute the new approximations

$$\varphi_{n+1}(t) = f(t, u_n(t), y_n(t), z_n(t)), \tag{3.1.33}$$

$$\alpha_{n+1} = \int_0^1 g(t)u_n(t)dt. \tag{3.1.34}$$

Theorem 3.1.5. Under the assumptions of Theorem 3.1.3 the above iterative method converges, and for the approximate solution $u_n(t)$ and its derivatives $u_n'(t), u_n''(t)$ there hold the estimates

$$\|u_n - u\| \leq \left(M_0 + \frac{1}{r}\right) p_n d, \tag{3.1.35}$$

$$\|u_n' - u'\| \leq \left(M_1 + \frac{2}{r}\right) p_n d, \tag{3.1.36}$$

$$\|u_n'' - u''\| \leq \left(M_2 + \frac{2}{r}\right) p_n d, \tag{3.1.37}$$

where $p_n = \frac{q^n}{1-q}, \ \ d = \|w_1 - w_0\|_{\mathcal{B}}, \ \ w_1 = (\varphi_1, \alpha_1)^T$.

Proof. Notice that the above iterative method is a realization of the successive approximation method for finding the fixed point of operator A. Indeed, let $w_n = (\varphi_n(t), \alpha_n)^T$ be known. Then the next approximation is $w_{n+1} = Aw_n$, where $w_{n+1} = (\varphi_{n+1}(t), \alpha_{n+1})^T$ and

$$Aw_n = \begin{pmatrix} f(t, u_n(t), u_n'(t), u_n''(t) \\ \int_0^1 g(t)u_n(t)dt \end{pmatrix}.$$

In componentwise form we have the formulas (3.1.33) and (3.1.34). In the above formulas $u_n(t)$ is is to be found from the problem

$$u_n'''(t) = \varphi_n(t), \quad 0 < t < 1,$$
$$u_n(0) = u_n'(0) = 0, \ u_n(1) = \alpha_n.$$

Therefore, it is computed by the formula (3.1.30). Its derivatives $u'_{n+1}(t) = y_{n+1}(t), u''_{n+1}(t) = z_{n+1}(t)$ are computed by the formulas (3.1.31), (3.1.32), respectively.
Thus, the iterative method (3.1.30)-(3.1.34) indeed is successive approximation method for finding the fixed point of operator A. Therefore, it converges with the rate of geometric progression and there holds the estimate

$$\|w_n - w\|_{\mathcal{B}} \le \frac{q^n}{1-q}\|w_1 - w_0\|_{\mathcal{B}} = p_n d,$$

where $w_n - w = (\varphi_n - \varphi, \alpha_n - \alpha)^T$.
From the definition of the norm in $\mathcal{B}$ and the above estimate it follows

$$\|\varphi_n - \varphi\| \le \|w_n - w\|_{\mathcal{B}} \le p_n d,$$

$$\|\alpha_n - \alpha\| \le \frac{1}{r}\|w_n - w\|_{\mathcal{B}} \le \frac{1}{r} p_n d.$$

Now, the estimates (3.1.35)-(3.1.37) are easily obtained if taking into account the representations (3.1.11)-(3.1.13), (3.1.30)-(3.1.32), the estimates of the type (3.1.15) and the above estimates. □

To numerically realize the iterative method (3.1.29)-(3.1.34) we cover the interval $[0, 1]$ by the uniform grid $\omega_h = \{t_i = ih,\ h = 1/N, i = 0, 1, ..., N\}$ and use the trapezoidal rule for computing integrals. In all examples in the next section the numerical computations will be performed on the uniform grid with $h = 0.01$ until $\max\left\{\|\varphi_n - \varphi_{n-1}\|, r|\alpha_n - \alpha_{n-1}|\right\} \le 10^{-4}$, where r will be defined for each particular example.

3.1.4. Examples

In order to demonstrate the validity of the obtained theoretical results and the efficiency of the proposed iterative method, in this section we consider some examples.

Example 3.1.1 (Example with exact solution)**.** Consider the problem (3.1.1)-(3.1.2) with

$$f = f(t, u) = -\frac{1}{2} + \frac{1}{3}\left(\frac{1}{6}(t^2 - \frac{t^3}{2})\right)^2 - u^2,$$
$$g(s) = \frac{56}{9}s^4.$$

It is easy to verify that the positive function

$$u(t) = \frac{1}{6}\left(t^2 - \frac{t^3}{2}\right), \quad 0 \le t \le 1$$

is the exact solution of the problem.
For the given $g(s)$, simple calculations give $C_0 = \frac{56}{45}, C_2 = \frac{56}{63}$. Therefore, with $r = 2$ we obtain $q_1 = 0.9503 < 1$. For this r it is possible to choose $M = 0.6$ such that $-M \le f(t, x) \le 0$ for

$$(t, x) \in \mathcal{D}_M^+ = \left\{(t, x) \mid 0 \le t \le 1,\ 0 \le x \le (M_0 + \frac{1}{2})M = 0.5247M\right\}.$$

Indeed,

$$0 \le -f(t,x) = \frac{1}{2} + x^2 - \frac{1}{3}\left(\frac{1}{6}(t^2 - \frac{t^3}{2})\right)^2 \le \frac{1}{2} + x^2 \le \frac{1}{2} + (0.5247M)^2 \le M.$$

Thus, M must satisfy $0.2753M^2 - M + 0.5 \le 0$. The direct calculation of the left side for $M = 0.6$ gives the value $= -0.0670$. So, the choice of M is justified. Further, for $f(t,x)$ we have the Lipshitz coefficient with respect to x in $\mathcal{D}_M^+$, $L_0 = 0.3148$. Consequently, $q_2 = L_0\left(M_0 + \frac{1}{2}\right) = 0.1652$, and $q = 0.9503$. Besides, $f(t,0) \not\equiv 0$. Therefore, by Theorem 3.1.4, the problem has a unique positive solution. It is the above exact solution.

The computation shows that the iterative method (3.1.29)-(3.1.34) converges and the error of the 46th iteration compared with the exact solution is $1.1458e-04$.

Example 3.1.2 (Example 4.1 in [47])**.** Consider the boundary value problem

$$u'''(t) = -u^2e^u, \quad 0 < t < 1,$$
$$u(0) = 0, u'(0) = 0, \ u(1) = \int_0^1 s^4 u(s)ds.$$

In this example

$$f(t,x,y,z) = -x^2e^x, \quad g(s) = s^4.$$

So,

$$C_0 = \int_0^1 g(s)ds = \frac{1}{5}, \quad C_2 = \int_0^1 s^2 g(s)ds = \frac{1}{7}.$$

Choose $r = 2$ in the definition of the norm of the space $\mathcal{B}$ (3.1.4) and in the definition of $\mathcal{D}_M^+$ by (3.1.21). Then $q_1 = rC_0M_0 + C_2 = 0.1527$. For $M = 0.4$ it is possible to verify that $-M \le f(t,x) \le 0$ in $\mathcal{D}_M^+$, $\left|\frac{\partial f}{\partial x}\right| \le 0.5721$ in $\mathcal{D}_M^+$. Therefore,

$$L_0 = 0.5721, \quad q_2 = L_0\left(M_0 + \tfrac{1}{r}\right) = 0.3002.$$

Hence, by Theorem 3.1.4 the problem has a unique nonnegative solution. This solution should be $u(t) \equiv 0$ because $u(t) \equiv 0$ solves the problem. The numerical experiments by the iterative method in Section 3.1.3 confirm this conclusion.

Remark that, in [47] the authors concluded that the problem has at least one positive solution. From our result above, it is clear that their conclusion is not valid.

Example 3.1.3. Consider Example 3.1.2 with the nonlinear term $f = -(1+u^2)$. Clearly, $\frac{f(u)}{u} \to -\infty$ as $u \to +0$ and $u \to +\infty$. Thus, neither Theorem 3.1 nor Theorem 3.2 in [47] are applicable, so the existence of positive solution is not guaranteed.
Now apply our method. Choose $M = 2, r = 3$, then

$$\mathcal{D}_M^+ = \{(t,x) \mid 0 \le t \le 1, \ 0 \le x \le (M_0 + \frac{1}{r})M = 0.7160\}.$$

In $\mathcal{D}_M^+$ we have

$$-M \le f \le 0, \quad |f'_u| \le 1.4321 = L_0,$$
$$q_1 = rC_0M_0 + C_2 = 0.1577, \ q_2 = L_0\left(M_0 + \frac{1}{3}\right) = 0.5127.$$

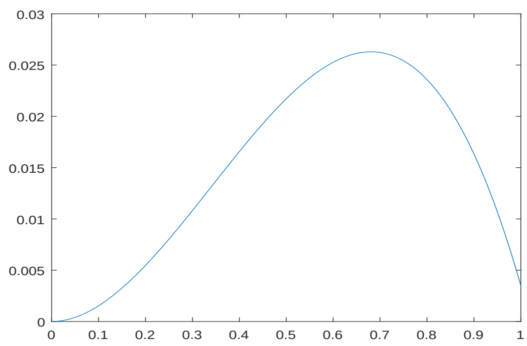

Figure 3.1: Approximate solution in Example 3.1.3.

Hence, by Theorem 3.1.4, the problem has a unique nonnegative solution. Due to $f(t,0) \neq 0$, this solution is positive. The graph of the approximate solution obtained with the given accuracy 10^{-4} after 4 iterations by the iterative method is depicted on Figure 3.1.

Example 3.1.4. Consider Example 3.1.2 with the nonlinear term

$$f = -(u^2 e^u + \frac{1}{5}\sin(u') + \frac{1}{8}\cos(u'') + 1).$$

In this example

$$f(t,x,y,z) = -(x^2 e^x + \frac{1}{5}\sin(y) + \frac{1}{8}\cos(z) + 1).$$

Choose $M = 1.7, r = 4$. It is possible to verify that in $\mathcal{D}_M^+$ we have $-M \le f \le 0$, and the Lipschitz coefficients of f are

$$L_0 = 1.8378, \quad L_1 = \frac{1}{5}, \quad L_2 = \frac{1}{8}.$$

Therefore,

$$q_1 = 0.1626, \; q_2 = 0.7618.$$

Hence, by Theorem 3.1.4, the problem has a unique positive solution. The graph of the approximate solution obtained with the given accuracy 10^{-4} after 6 iterations by the iterative method is depicted on Figure 3.2.

3.1.5. Conclusion

In this section, we have proposed a novel method to study the fully third order differential equation with integral boundary conditions. It is based on the reduction of the boundary value problems to fixed point problem for appropriate operator defined on a space of mixed pairs of functions and numbers. This is the approach successfully used by ourselves before for nonlinear third, fourth and sixth orders two-point boundary value problems. By this approach, we have established the existence, uniqueness and positivity of solution of the problem under the conditions which are easily verified. Moreover, we have proposed an effective solution method and given the convergence analysis for it. The theoretical results have been demonstrated on some examples

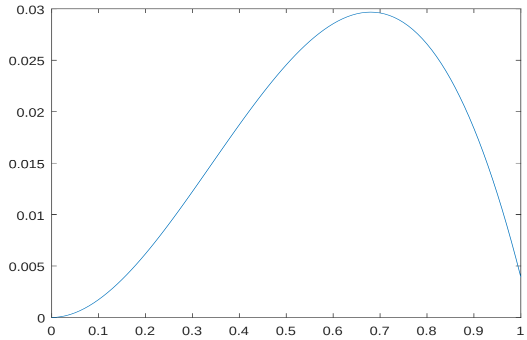

Figure 3.2: Approximate solution in Example 3.1.4.

including an example with exact solution and other examples where the exact solutions are not known. Especially, we have shown that the conclusion on the existence of positive solutions for an example considered before by other authors, is not valid.

The proposed method can be applied to problems with other integral boundary conditions for the third and higher order differential equations. This is the subject of our research in the future.

3.2. Existence results and an iterative method for a fully fourth order nonlinear integral boundary value problem

3.2.1. Introduction

In this section, we consider the boundary value problem

$$u^{(4)}(t) = f(t, u(t), u'(t), u''(t), u'''(t)), \quad 0 < t < 1, \tag{3.2.1}$$

$$u'(0) = u''(0) = u'(1) = 0, \; u(0) = \int_0^1 g(t)u(t)dt, \tag{3.2.2}$$

where $f \in C([0,1] \times \mathbb{R}^4, \mathbb{R}^+)$, $g \in C([0,1], \mathbb{R}^+)$ are given functions.

The simplest particular case of the above problem, when the nonlinear term is $f(u(t))$, was considered recently in [48]. In that paper by employing the Krasnosel'skii's fixed point theorem on cones, the authors proved that the problem has at least one positive solution.

In the paper [AL5], by the method developed in [11, 13, 14, 86, 87, 89, 91, 92] we obtain the results of the existence, uniqueness and positivity of solution and the convergence of an iterative method on both continuous and discrete levels for finding the solution. We also give error analysis of the discrete approximate solution. Five examples, among them an example with exact solution and two examples taken from [48], demonstrate the validity of the obtained theoretical results and the efficiency of the iterative method.

It should be said that for numerical solution of two-point nonlinear BVPs for fourth order differential equations there are many methods, which can be divided into three types. The first type includes methods for constructing discrete systems corresponding to BVPs, for example, [93–96]. In these papers, the authors studied the convergence of the discrete systems without any analysis of errors arising in solving the discrete systems. To the second type of methods there are related the methods of construction of iterative methods on continuous level without attention to how to realize continuous problems at each iteration and error arising at each iteration, see, e.g. [1, 97, 98] and [11, 13, 14, 86, 87, 89, 91, 92]. The third type includes analytical methods such as the Adomian decomposition method [99], the variational iteration method [100], the reproducing kernel method [101], when the solution is sought in series form. Spectral methods also belong to the third type since the exact solution of the problems is expressed in series representation by basis functions. For finding the coefficients of the representation it is needed to solve nonlinear systems of algebraic solutions. At present spectral methods [102] are widely used for solving BVPs for ODE, PDE, integral equations including nonlinear Volterra integral equations [103], [104].

It should be said that in all methods, the estimate of total error of the actually obtained approximate numerical solution has not been addressed. In our opinion, the problem of total error in numerical solution of nonlinear BVPs must be investigated because the total error gives useful information for balancing discretization error and error of iterative process. So, in this section we propose an iterative method at continuous level, its discrete analog and make analysis of the total error of the approximate discrete solution for the BVP with integral boundary condition.

3.2.2. Existence results

To study the problem (3.2.1), (3.2.2) we associate it with an operator equation.

First, we introduce the space $\mathcal{B} = \mathcal{B} = C[0,1] \times \mathbb{R}$ of pairs $w = (\varphi, \mu)^T$, where $\varphi \in C[0,1], \mu \in \mathbb{R}$ and equip it with the norm

$$\|w\|_{\mathcal{B}} = \max(\|\varphi\|, r|\mu|), \tag{3.2.3}$$

where r is a real number, $r \geq 1$ and $\|\varphi\| = \max_{0 \leq t \leq 1} |\varphi(t)|$.

Next, we define the operator A acting on elements $w \in \mathcal{B}$ by the formula

$$Aw = \begin{pmatrix} f(t, u(t), u'(t), u''(t), u'''(t)) \\ \int_0^1 g(s)u(s)ds \end{pmatrix}, \tag{3.2.4}$$

where $u(t)$ is the solution of the problem

$$u^{(4)}(t) = \varphi(t), \quad 0 < t < 1, \tag{3.2.5}$$

$$u'(0) = u''(0) = u'(1) = 0, \ \ u(0) = \mu. \tag{3.2.6}$$

Obviously, due to the continuity of the functions f and g we have $Aw \in \mathcal{B}$. It is easy to verify the following

Lemma 3.2.1. If $w = (\varphi, \mu)^T$ is a fixed point of the operator A in the space $\mathcal{B}$, that is,

$$Aw = w \tag{3.2.7}$$

in $\mathcal{B}$, then the function $u(t)$ found from the problem (3.2.5)-(3.2.6) solves the original problem (3.2.1), (3.2.2).
Conversely, if $u(t)$ is a solution of (3.2.1), (3.2.2), then the pair (φ, μ), where

$$\varphi(t) = f(t, u(t), u'(t), u''(t), u'''(t)), \tag{3.2.8}$$

$$\alpha = \int_0^1 g(s)u(s)ds, \tag{3.2.9}$$

is a solution of the operator equation (3.2.7).

Thus, by this lemma, the solution of problem (3.2.1), (3.2.2) is reduced to finding the fixed point problem for A.

Now, we consider the properties of A. To this end, notice that the problem (3.2.5), (3.2.6) has a unique solution which can be represented in the form

$$u(t) = \int_0^1 G_0(t,s)\varphi(s)ds + \mu, \quad 0 < t < 1, \tag{3.2.10}$$

where $G_0(t,s)$ is the Green's function of the operator $u^{(4)}(t) = 0$ involving the homogeneous boundary conditions $u(0) = u'(0) = u''(0) = u'(1)$. It is not hard to find it in the form

$$G_0(t,s) = \frac{1}{6}\begin{cases} -t^3(1-s)^2 + (t-s)^3, & 0 \le s \le t \le 1 \\ -t^3(1-s)^2, & 0 \le t \le s \le 1. \end{cases} \tag{3.2.11}$$

Differentiating both sides of (3.2.10) gives

$$u'(t) = \int_0^1 G_1(t,s)\varphi(s)ds, \tag{3.2.12}$$

$$u''(t) = \int_0^1 G_2(t,s)\varphi(s)ds, \tag{3.2.13}$$

$$u'''(t) = \int_0^1 G_3(t,s)\varphi(s)ds, \tag{3.2.14}$$

where

$$G_1(t,s) = \frac{1}{2}\begin{cases} -t^2(1-s)^2 + (t-s)^2, & 0 \le s \le t \le 1, \\ -t^2(1-s)^2, & 0 \le t \le s \le 1, \end{cases} \tag{3.2.15}$$

$$G_2(t,s) = \begin{cases} -t(1-s)^2 + (t-s), & 0 \le s \le t \le 1, \\ -t(1-s)^2, & 0 \le t \le s \le 1. \end{cases} \tag{3.2.16}$$

$$G_3(t,s) = \begin{cases} -(1-s)^2 + 1, & 0 \le s < t \le 1, \\ -(1-s)^2, & 0 \le t < s \le 1. \end{cases} \tag{3.2.17}$$

It is easily seen that

$$G_0(t,s) \le 0, \quad G_1(t,s) \le 0,$$

in $Q = [0,1]^2$, and

$$\begin{aligned} M_0 &= \max_{0\le t\le 1} \int_0^1 |G_0(t,s)|ds = 0.0139, \\ M_1 &= \max_{0\le t\le 1} \int_0^1 |G_1(t,s)|ds = 0.0247, \\ M_2 &= \max_{0\le t\le 1} \int_0^1 |G_2(t,s)|ds \le 0.1883, \\ M_3 &= \max_{0\le t\le 1} \int_0^1 |G_3(t,s)|ds = 1.3333. \end{aligned} \tag{3.2.18}$$

Therefore, from (3.2.10), (3.2.12)-(3.2.14) and (3.2.18) we obtain the following estimates for the solution of the problem (3.2.5), (3.2.6):

$$\begin{gathered} \|u\| \le M_0\|\varphi\| + |\mu|, \ \|u'\| \le M_1\|\varphi\|, \\ \|u''\| \le M_2\|\varphi\|, \ \|u'''\| \le M_3\|\varphi\|. \end{gathered} \tag{3.2.19}$$

For any real number $M > 0$, we define the domain

$$\mathcal{D}_M = \{(t, u, y, v, z) \mid 0 \le t \le 1,\ |u| \le (M_0 + \tfrac{1}{r})M,\\ |y| \le M_1 M,\ |v| \le M_2 M,\ |z| \le M_3 M\}. \tag{3.2.20}$$

From now on suppose that the function $f(t, u, y, v, z)$ is continuous in $\mathcal{D}_M$.

Denote

$$C_0 = \int_0^1 g(t)dt > 0. \tag{3.2.21}$$

Lemma 3.2.2. Assume that

$$|f(t, u, y, v, z)| \le M \quad \text{in } \mathcal{D}_M \tag{3.2.22}$$

and

$$q_1 := C_0(rM_0 + 1) \le 1, \tag{3.2.23}$$

where C_0 is defined by (3.2.21). Then, the operator A defined by (3.2.4) maps the closed ball $B[0, M]$ in $\mathcal{B}$ into itself.

Proof. Take $w = (\varphi, \mu)^T \in B[0, M]$. Then $\|\varphi\| \le M$ and $|\mu| \le \frac{M}{r}$.

Return to the problem (3.2.5), (3.2.6). From the estimates (3.2.19) we obtain

$$\|u\| \le \left(M_0 + \frac{1}{r}\right) M, \quad \|u'\| \le M_1 M, \quad \|u''\| \le M_2 M, \quad \|u'''\| \le M_3 M.$$

Hence, $(t, u, u', u'', u''') \in \mathcal{D}_M$ and, due to (3.2.22) we have

$$|f(t, u(t), u'(t), u''(t), u'''(t))| \le M,\ t \in [0, 1].$$

Next, we have the estimates

$$r\left|\int_0^1 g(t)u(t)dt\right| \le r\|u\|C_0 \le rC_0(M_0 + \frac{1}{r}) = C_0(rM_0 + 1) = q_1 M \le M. \tag{3.2.24}$$

Therefore,

$$\|Aw\|_{\mathcal{B}} \le M.$$

□

Lemma 3.2.3. The operator A is a compact operator in $\mathcal{B}[0, M]$.

Proof. The compactness of A follows from the compactness of the integral operators (3.2.10), (3.2.12)-(3.2.14) of $\varphi(s)$, the continuity of the function $f(t, x, y, v, z)$ and the compactness of the integral operator $\int_0^1 g(t)u(t)dt$ of $u(t)$. □

Theorem 3.2.1. Under the conditions of Lemma 3.2.2 the problem (3.2.1), (3.2.2) has a solution.

Proof. By Lemma 3.2.2 and Lemma 3.2.3, the operator A is a compact operator mapping $B[0, M] \subset \mathcal{B}$ into itself. Therefore, by the Schauder fixed point theorem, the operator A has a fixed point in $B[0, M]$. This fixed point corresponds to a solution of the problem (3.2.1), (3.2.2). □

In order to study the positivity of solution of (3.2.1), (3.2.2), we introduce the domain

$$\begin{aligned} \mathcal{D}_M^+ = \{(t,u,y,v,z) \mid 0 \le t \le 1,\ 0 \le u \le (M_0 + \tfrac{1}{r})M, \\ 0 \le y \le M_1 M,\ |v| \le M_2 M,\ |z| \le M_3 M\}, \end{aligned} \tag{3.2.25}$$

in the space $[0,1] \times \mathbb{R}^3$ and the domain

$$S_M = \{w = (\varphi, \mu)^T \mid -M \le \varphi \le 0,\ 0 \le r\mu \le M\} \tag{3.2.26}$$

in the space $\mathcal{B}$.

Theorem 3.2.2 (Positivity of solution)**.** Assume that the function $f(t,u,y,v,z)$ is continuous and

$$-M \le f(t,u,y,v,z) \le 0 \text{ in } \mathcal{D}_M^+, \tag{3.2.27}$$

and there holds the condition (3.2.23). Then the problem (3.2.1), (3.2.2) has a nonnegative solution. In addition, if $f(t,0,0,0,0) \not\equiv 0$ in $(0,1)$ then the solution is positive.

Proof. First, notice that under the assumptions of the theorem, the operator A maps S_M into itself.
Indeed, let $w \in S_M$, $w = (\varphi,\mu)^T, -M \le \varphi \le 0, 0 \le r\mu \le M$. Because $G_i(t,s) \le 0$ for $0 \le t,s \le 1, (i = 0,1)$ from (3.2.10), (3.2.12), (3.2.13) we have

$$0 \le u(t) \le (M_0 + \tfrac{1}{r})M,\ 0 \le u'(t) \le M_1 M,\ |u''(t)| \le M_2 M, |u'''(t)| \le M_3 M\ 0 \le t \le 1.$$

Therefore, for the solution $u(t)$ of (3.2.5), (3.2.6) we have

$$(t, u(t), u'(t), u''(t), u'''(t)) \in \mathcal{D}_M^+,$$

and by the assumption (3.2.27)

$$-M \le f(t, u(t), u'(t), u''(t), u'''(t)) \le 0.$$

In view of (3.2.24) we have

$$0 \le r \int_0^1 g(s)u(s)ds \le C_0(rM_0 + 1)M \le M.$$

Hence, $(f(t,u(t),u'(t),u''(t),u'''(t)), \int_0^1 g(t)u(t)dt)^T \in S_M$, i.e. $A : S_M \to S_M$. Besides, as was shown above, A is a compact operator in S_M. Due to this A has a fixed point in S_M, which generates a solution of the problem (3.2.1), (3.2.2). This solution is nonnegative with its first derivative. Since $f(t,0,0,0,0) \not\equiv 0$ in $(0,1)$ the function $u(t) \equiv 0$ cannot be the solution of the problem. Therefore, this solution should be positive. □

Theorem 3.2.3 (Existence and uniqueness)**.** Assume that there exist numbers $M > 0, L_0, L_1, L_2, L_3 \ge 0$ such that

1. $|f(t,u,y,v,z)| \le M,\ \forall (t,u,y,v,z) \in \mathcal{D}_M$.
2. $|f(t,u_2,y_2,v_2,z_2) - f(t,u_1,y_1,v_1,z_1)| \le L_0|u_2 - u_1| + L_1|y_2 - y_1| + L_2|v_2 - v_1| + L_3|z_2 - z_1|,\ \forall (t,u_i,y_i,v_i,z_i) \in \mathcal{D}_M,\ i = 1,2.$

3. $q := \max\{q_1, q_2\} < 1$, where $q_1 = rC_0M_0 + C_0$ (see (3.2.23)) and

$$q_2 = L_0(M_0 + \tfrac{1}{r}) + L_1M_1 + L_2M_2 + L_3M_3.$$

Then the problem (3.2.1)-(3.2.2) has a unique solution $u \in C^4[0,1]$.

Proof. According to Lemma 3.2.1, the theorem will be proved if we show that the operator A defined by (3.2.4) is a contraction mapping from the closed ball $B[0,M]$ in $\mathcal{B}$ into itself.
In fact, under the assumptions 1) and 3), by Lemma 3.2.2, the operator A maps $B[0,M]$ into itself.

It remains to show that A is a contraction map.

Take $w_i = (\varphi_i, \mu_i)^T \in B[0,M]$, $i = 1, 2$. We have

$$Aw_2 - Aw_1 = \begin{pmatrix} f(t, u_2(t), u_2'(t), u_2''(t), u_2'''(t)) - f(t, u_1(t), u_1'(t), u_1''(t), u_1'''(t)) \\ \int_0^1 g(s)(u_2(s) - u_1(s))ds \end{pmatrix},$$

where $u_i(t),\ (i = 1, 2)$ solves the problem

$$\begin{cases} u_i^{(4)}(t) = \varphi_i(t), \quad 0 < t < 1 \\ u_i'(0) = u_i''(0) = u_i'(1) = 0, \ \ u_i(0) = \mu_i. \end{cases}$$

In the proof of Lemma 3.2.2 it was shown that $(t, u_i(t), u_i'(t), u_i''(t), u_i'''(t)) \in \mathcal{D}_M$. Therefore, by the assumption 2) for f we have

$$\begin{aligned} E_1 :&= |f(t, u_2(t), u_2'(t), u_2''(t), u_2'''(t)) - f(t, u_1(t), u_1'(t), u_1''(t), u_1'''(t))| \\ &\le L_0|u_2(t) - u_1(t)| + L_1|u_2'(t) - u_1'(t)| + L_2|u_2''(t) - u_1''(t)| \\ &+ L_3|u_2'''(t) - u_1'''(t)|. \end{aligned} \tag{3.2.28}$$

Since $u_2(t) - u_1(t)$ is the solution of the problem (3.2.5), (3.2.6) with the right-hand sides $\varphi_2(t) - \varphi_1(t)$ and $\mu_2 - \mu_1$, we have

$$\begin{aligned} \|u_2 - u_1\| &\le M_0\|\varphi_2 - \varphi_1\| + |\mu_2 - \mu_1|, \\ \|u_2' - u_1'\| &\le M_1\|\varphi_2 - \varphi_1\|, \\ \|u_2'' - u_1''\| &\le M_2\|\varphi_2 - \varphi_1\|, \\ \|u_2''' - u_1'''\| &\le M_3\|\varphi_2 - \varphi_1\|. \end{aligned} \tag{3.2.29}$$

From the above estimates and (3.2.28) we obtain

$$\begin{aligned} E_1 &\le \left(L_0\left(M_0 + \frac{1}{r}\right) + L_1M_1 + L_2M_2 + L_3M_3\right)\|w_2 - w_1\|_{\mathcal{B}} \\ &= q_2\|w_2 - w_1\|_{\mathcal{B}} \end{aligned} \tag{3.2.30}$$

if taking into account the definition of the norm in the space $\mathcal{B}$.
Now consider

$$E_2 := \int_0^1 g(s)(u_2(s) - u_1(s))ds.$$

We have

$$|E_2| \le \int_0^1 g(s)|u_2(s) - u_1(s)|ds.$$

In analogy with the estimate (3.2.24) we have

$$|E_2| \le C_0(M_0 + \frac{1}{r})\|w_2 - w_1\|_{\mathcal{B}}.$$

Therefore

$$r|E_2| \le C_0(rM_0 + 1)\|w_2 - w_1\|_{\mathcal{B}} = q_1\|w_2 - w_1\|_{\mathcal{B}}. \tag{3.2.31}$$

From (3.2.30) and (3.2.31) we obtain

$$\|Aw_2 - Aw_1\|_{\mathcal{B}} \le \max\{q_1, q_2\}\|w_2 - w_1\|_{\mathcal{B}}.$$

In view of the assumption 3) the operator A is a contraction operator in $B[0, M]$. This completes the proof of theorem. □

Analogously as the above theorem, it is easy to prove the following result.

Theorem 3.2.4 (Existence and uniqueness of positive solution)**.** If in Theorem 3.2.3 replace $\mathcal{D}_M$ by $\mathcal{D}_M^+$ and the assumption 1) by the assumption (3.2.27) then the problem has a unique nonnegative solution $u(t) \in C^4[0, 1]$. Moreover, if $f(t, 0, 0, 0, 0) \not\equiv 0$ in $(0, 1)$ then the solution is positive.

3.2.3. Iterative method on continuous level

To solve the problem (3.2.1)- (3.2.2) we propose the following iterative method:

1. Given an initial approximation

$$\varphi_0(t) = f(t, 0, 0, 0, 0), \ \mu_0 = 0. \tag{3.2.32}$$

2. Knowing $\varphi_k(t)$ and μ_k $(k = 0, 1, ...)$ compute

$$\begin{aligned} u_k(t) &= \int_0^1 G_0(t, s)\varphi_k(s)ds + \mu_k, \\ y_k(t) &= \int_0^1 G_1(t, s)\varphi_k(s)ds, \\ v_k(t) &= \int_0^1 G_2(t, s)\varphi_k(s)ds, \\ z_k(t) &= \int_0^1 G_3(t, s)\varphi_k(s)ds. \end{aligned} \tag{3.2.33}$$

3. Compute new approximation

$$\begin{aligned} \varphi_{k+1}(t) &= f(t, u_k(t), y_k(t), v_k(t), z_k(t)), \\ \mu_{k+1} &= \int_0^1 g(s)u_k(s)ds. \end{aligned} \tag{3.2.34}$$

As in the previous subsection, it is easy to show that the above iterative method indeed is a realization of the successive approximation method for finding the fixed point of operator A. Therefore, it converges as a geometric progression and we have the estimate

$$\|w_k - w\|_{\mathcal{B}} \le \frac{q^k}{1-q}\|w_1 - w_0\|_{\mathcal{B}} = p_k d,$$

where $w_k - w = (\varphi_k - \varphi, \mu_k - \mu)^T$ and

$$p_k = \tfrac{q^k}{1-q},\ d = \|w_1 - w_0\|_{\mathcal{B}}. \tag{3.2.35}$$

The above estimate can be written in the componentwise form as follows:

$$\|\varphi_k - \varphi\| \le \frac{q^k}{1-q}\|w_1 - w_0\|_{\mathcal{B}} = p_k d,$$

$$|\mu_k - \mu| \le \frac{1}{r}\frac{q^k}{1-q}\|w_1 - w_0\|_{\mathcal{B}} = \frac{1}{r}p_k d.$$

These estimates imply the following result of the convergence of the iterative method (3.2.32)-(3.2.34).

Theorem 3.2.5. The iterative method (3.2.32)-(3.2.34) converges and for the approximate solution $u_k(t)$ there hold error estimates

$$\|u_k - u\| \le \left(M_0 + \frac{1}{r}\right) p_k d,\ \|u_k' - u'\| \le M_1 p_k d,$$
$$\|u_k'' - u''\| \le M_2 p_k d,\ \|u_k''' - u'''\| \le M_3 p_k d$$

where u is the exact solution of the problem (3.2.1)-(3.2.2), p_k and d are defined by (3.2.35), and r is the number available in (3.2.3) for defining the norm of the space $\mathcal{B}$.

3.2.4. Discrete iterative method

To compute numerical solution of the problem (3.2.1)-(3.2.2) we construct discrete iterative method corresponding to the above iterative method on continuous level. To this end we cover the interval $[0, 1]$ by the uniform grid $\bar{\omega}_h = \{t_i = ih,\ h = 1/N, i = 0, 1, ..., N\}$ and denote by $\Phi_k(t), U_k(t), Y_k(t), V_k(t), Z_k(t)$ the grid functions, which are defined on the grid $\bar{\omega}_h$ and approximate the functions $\varphi_k(t), u_k(t), y_k(t), v_k(t), z_k(t)$ on this grid. Also, we denote the approximation of μ_k by $\hat{\mu}_k$.

Consider now the following discrete iterative method.

1. Given a starting approximation

$$\Phi_0(t_i) = f(t_i, 0, 0, 0, 0),\ i = 0, ..., N; \quad \hat{\mu}_0 = 0 \tag{3.2.36}$$

2. Knowing $\Phi_k(t_i),\ i = 0, ..., N$ and $\hat{\mu}_k$ $(k = 0, 1, ...)$ as kth approximation compute approximately the definite integrals (3.2.33) by the trapezoidal rule

$$\begin{aligned}
U_k(t_i) &= \sum_{j=0}^{N} h\rho_j G_0(t_i, t_j)\Phi_k(t_j) + \hat{\mu}_k,\\
Y_k(t_i) &= \sum_{j=0}^{N} h\rho_j G_1(t_i, t_j)\Phi_k(t_j),\\
V_k(t_i) &= \sum_{j=0}^{N} h\rho_j G_2(t_i, t_j)\Phi_k(t_j),\\
Z_k(t_i) &= \sum_{j=0}^{N} h\rho_j G_3^*(t_i, t_j)\Phi_k(t_j),\ i = 0, ..., N,
\end{aligned} \tag{3.2.37}$$

where ρ_j is the weight of the trapezoidal rule

$$\rho_j = \begin{cases} 1/2, \ j = 0, N \\ 1, \ j = 1, 2, ..., N-1 \end{cases}$$

and

$$G_3^*(t,s) = \begin{cases} -(1-s)^2 + 1, & 0 \le s < t \le 1, \\ -(1-s)^2 + 1/2, & s = t, \\ -(1-s)^2, & 0 \le t < s \le 1. \end{cases}$$

3. Compute the new approximation

$$\begin{aligned} \Phi_{k+1}(t_i) &= f(t_i, U_k(t_i), Y_k(t_i), V_k(t_i), Z_k(t_i)), \\ \hat{\mu}_{k+1} &= \sum_{j=0}^{N} h\rho_j g(t_j) U_k(t_j). \end{aligned} \tag{3.2.38}$$

To obtain error estimates for the discrete approximate solution for $u(t)$ and its derivatives we need some following auxiliary results.

Proposition 3.2.6. Assume that the function $f(t,u,y,v,z)$ has all continuous partial derivatives up to order 2 in the domain $\mathcal{D}_M$. Then for the functions $u_k(t), y_k(t), v_k(t), \ z_k(t), k = 0,1,...$ generated by the iterative method (3.2.32)-(3.2.34) there hold $z_k(t) \in C^3[0,1], \ v_k(t) \in C^4[0,1], \ y_k(t) \in C^5[0,1], \ u_k(t) \in C^6[0,1]$.

Proof. The proposition will be proved by induction. For $k = 0$, by the assumption on the function f we have $\varphi_0(t) \in C^2[0,1]$ because $\varphi_0(t) = f(t,0,0,0,0)$. In view of the expression (3.2.17) of $G_3(t,s)$ we have

$$z_0(t) = \int_0^1 G_3(t,s)\varphi_0(s)ds = \int_0^t [-(1-s)^2 + 1]\varphi_0(s)ds - \int_t^1 (1-s)^2\varphi_0(s)ds.$$

The direct differentiation of the integrals in the right-hand side yields $z_0'(t) = \varphi_0(t)$. Hence, $z_0(t) \in C^3[0,1]$. It implies $v_0(t) \in C^4[0,1], \ y_0(t) \in C^5[0,1], \ u_0(t) \in C^6[0,1]$.

Next, assume that $z_k(t) \in C^3[0,1], \ v_k(t) \in C^4[0,1], \ y_k(t) \in C^5[0,1], \ u_k(t) \in C^6[0,1]$. Then, since $\varphi_{k+1}(t) = f(t, u_k(t), y_k(t), v_k(t), z_k(t))$ and by the assumption the function f has continuous derivatives in all variables up to order 2, it follows that $\varphi_{k+1}(t) \in C^2[0,1]$. Repeating the same argument as for $\varphi_0(t)$ above we obtain that $z_{k+1}(t) \in C^3[0,1], \ v_{k+1}(t) \in C^4[0,1], \ y_{k+1}(t) \in C^5[0,1], \ u_{k+1}(t) \in C^6[0,1]$. Thus, the proposition is proved. □

Proposition 3.2.7. For arbitrary function $\varphi(t) \in C^2[0,1]$ we have the estimates

$$\int_0^1 G_n(t_i,s)\varphi(s)ds = \sum_{j=0}^{N} h\rho_j G_n(t_i,t_j)\varphi(t_j) + O(h^2), \quad (n = 0,1,2), \tag{3.2.39}$$

$$\int_0^1 G_3(t_i,s)\varphi(s)ds = \sum_{j=0}^{N} h\rho_j G_3^*(t_i,t_j)\varphi(t_j) + O(h^2). \tag{3.2.40}$$

Proof. In the case $n = 0$ the estimate (3.2.39) is obvious in view of the error estimate of the trapezoidal rule since the function $G_0(t,s)$ defined by (3.2.11) have continuous derivatives up to order 2.

In the cases $n = 1, 2$, although the functions $G_1(t,s), G_2(t,s)$ have not continuous second partial derivatives with respect to t, they are continuous for any $0 \le t, s \le 1$. Due to this continuity, applying the trapezoidal rule to each subinterval $[0, t_i]$ and $[t_i, 1]$ we have

$$\begin{aligned}
&\int_0^1 G_n(t_i,s)\varphi(s)ds = \int_0^{t_i} G_n(t_i,s)\varphi(s)ds + \int_{t_i}^1 G_n(t_i,s)\varphi(s)ds \\
&= h\big(\tfrac{1}{2}G_n(t_i,t_0)\varphi(t_0) + G_n(t_i,t_1)\varphi(t_1) + ... + G_n(t_i,t_{i-1})\varphi(t_{i-1}) + \tfrac{1}{2}G_n(t_i,t_i)\varphi(t_i)\big) \\
&+ O(h^2) + h\big(\tfrac{1}{2}G_n(t_i,t_i)\varphi(t_i) + G_n(t_i,t_{i+1})\varphi(t_{i+1}) + ... + G_n(t_i,t_{N-1})\varphi(t_{N-1}) \\
&+ \tfrac{1}{2}G_n(t_i,t_N)\varphi(t_N)\big) + O(h^2) \\
&= \sum_{j=0}^{N} h\rho_j G_n(t_i,t_j)\varphi(t_j) + O(h^2).
\end{aligned}$$

Thus, the estimate (3.2.39) is proved. The estimate (3.2.40) is obtained by the use of the following result, which is easily proved.

Lemma 3.2.4. Let $p(t)$ be a function having continuous derivatives up to second order in the interval $[0,1]$ except for the point $0 < t_i < 1$, where it has a jump. Denote $\lim_{t\to t_i \pm 0} p(t) = p_i^{\pm}$, and $p_i = \frac{1}{2}(p_i^- + p_i^+)$. Then

$$\int_0^1 p(t)dt = \sum_{j=0}^{N} h\rho_j p_j + O(h^2), \tag{3.2.41}$$

where $p_j = p(t_j)$ for $j \ne i$.

□

Proposition 3.2.8. Let the assumption of Proposition 3.2.6 be satisfied, and additionally assume that the function $g(s) \in C^2[0,1]$. Then for any $k = 0, 1, ...$ we have the estimates

$$\|\Phi_k - \varphi_k\| = O(h^2),\ |\hat{\mu}_k - \mu_k| = O(h^2), \tag{3.2.42}$$

$$\begin{aligned}
\|U_k - u_k\| = O(h^2),\ \|Y_k - y_k\| = O(h^2), \\
\|V_k - v_k\| = O(h^2),\ \|Z_k - z_k\| = O(h^2)
\end{aligned} \tag{3.2.43}$$

where $\|.\| = \|.\|_{C(\bar{\omega}_h)}$ is the max-norm of function on the grid $\bar{\omega}_h$.

Proof. We prove the proposition by induction. For $k = 0$ we have immediately $\|\Phi_k - \varphi_k\| = 0$, $|\hat{\mu}_k - \mu_k| = 0$. Further, by the first equation in (3.2.33) and Proposition 3.2.7 we have

$$u_0(t_i) = \int_0^1 G_0(t_i,s)\varphi_0(s)ds + \mu_0 = \sum_{j=0}^{N} h\rho_j G_0(t_i,t_j)\varphi_0(t_j) + O(h^2),\ i = 0, ..., N \tag{3.2.44}$$

because $\mu_0 = 0$. On the other hand, due to the first equation in (3.2.37) and (3.2.36) we have

$$U_0(t_i) = \sum_{j=0}^{N} h\rho_j G_0(t_i, t_j)\varphi_0(t_j). \tag{3.2.45}$$

Hence, $|U_0(t_i) - u_0(t_i)| = O(h^2)$. Consequently, $\|U_0 - u_0\| = O(h^2)$.
Analogously, we have

$$\|Y_0 - y_0\| = O(h^2),\ \|V_0 - v_0\| = O(h^2),\ \|Z_0 - z_0\| = O(h^2). \tag{3.2.46}$$

Now suppose that (3.2.42) and (3.2.43) are true for $k \geq 0$. We will show that these estimates are true for $k+1$.
Indeed, we have

$$\mu_{k+1} - \hat{\mu}_{k+1} = \sum_{j=0}^{N} h\rho_j g(t_j)\big(u_k(t_j) - U_k(t_j)\big) + O(h^2).$$

In view of the estimate $\|U_k - u_k\| = O(h^2)$ from the above estimate it follows that

$$|\mu_{k+1} - \hat{\mu}_{k+1}| = O(h^2). \tag{3.2.47}$$

Next, by the Lipschitz condition of the function f and the estimates (3.2.42) and (3.2.43) it is easy to obtain the estimate $\|\Phi_{k+1} - \varphi_{k+1}\| = O(h^2)$. Having in mind this estimate and (3.2.47) we obtain the estimate

$$\|U_{k+1} - u_{k+1}\| = O(h^2).$$

Similarly, we obtain

$$\|Y_{k+1} - y_{k+1}\| = O(h^2),\ \|V_{k+1} - v_{k+1}\| = O(h^2),\ \|Z_{k+1} - z_{k+1}\| = O(h^2).$$

Thus, the proposition is proved. □

Combining Proposition 3.2.8 and Theorem 3.2.5 we obtain the following theorem.

Theorem 3.2.9. For the approximate solution of the problem (3.2.1), (3.2.2) obtained by the discrete iterative method (3.2.36)-(3.2.38) on the uniform grid $\bar{\omega}_h$ there hold the estimates

$$\begin{aligned} &\|U_k - u\| \leq \left(M_0 + \frac{1}{r}\right) p_k d + O(h^2),\ \|Y_k - u'\| \leq M_1 p_k d + O(h^2),\\ &\|V_k - u''\| \leq M_2 p_k d + O(h^2),\ \|Z_k - u'''\| \leq M_3 p_k d + O(h^2). \end{aligned} \tag{3.2.48}$$

Proof. The first above estimate is easily obtained if representing

$$U_k(t_i) - u(t_i) = (u_k(t_i) - u(t_i)) + (U_k(t_i) - u_k(t_i))$$

and using the first estimate in Theorem 3.2.5 and the first estimate in (3.2.43). The remaining estimates are obtained in the same way. Thus, the theorem is proved. □

3.2.5. Examples

To confirm the validity of the obtained theoretical results and the efficiency of the proposed discrete iterative method (3.2.36)-(3.2.38) below we consider some examples. In all examples the iterative process will be performed until $\max\{\|\Phi_{k+1} - \Phi_k\|, |\hat{\mu_{k+1}} - \hat{\mu_k}|\} \le TOL$, where TOL is a given tolerance.

Example 3.2.1 (Example with exact solution)**.** Consider the problem with the right-hand side

$$f = f(t,u) = -18 - \frac{1}{5}(\frac{5}{6} + t^3 - \frac{3}{4}t^4)^2 + \frac{1}{5}u^2$$
$$g(t) = 4t^4.$$

It is possible to verify that the positive function

$$u(t) = \frac{5}{6} + t^3 - \frac{3}{4}t^4$$

is the exact solution of the problem.

For the given $g(t)$ we have $C_0 = \int_0^1 g(t)\ dt = \frac{4}{5}$. Taking $r = 4, M = 18.2$ we define

$$\mathcal{D}_M^+ = \{(t,u) \mid 0 \le t \le 1,\ 0 \le u \le (M_0 + \frac{1}{r})M = 4.8030\}.$$

In $\mathcal{D}_M^+$ we have $-M \le f \le 0$, $|f_u'| \le 1.9212 = L_0$. After simple calculations we obtain $q_1 = 0.8445$, $q_2 = 0.5070$. Therefore, $q = 0.8445 < 1$. Hence, by Theorem 3.2.4, the problem has a unique positive solution. It is the above exact solution. Meanwhile, it is easy to verify that neither Theorem 3.1 nor Theorem 3.2 in [48] are applicable, therefore, by this paper the existence of positive solution is not guaranteed. Below we report the results of the numerical experiments for different tolerances (see Tables 3.1-3.3).

Table 3.1: The convergence in Example 3.2.1 for $TOL = 10^{-4}$

N	K	$Error$	N	K	$Error$
30	34	0.0065	500	34	3.9522e-04
50	34	0.0021	1000	34	3.9461e-04
100	34	3.9522e-04	1500	34	3.9413e-04
200	34	3.9522e-04	2000	34	3.9534e-04

Table 3.2: The convergence in Example 3.2.1 for $TOL = 10^{-5}$

N	K	$Error$	N	K	$Error$
30	44	0.0069	300	44	2.8711e-05
50	44	0.0025	500	44	1.6429e-05
100	44	5.8244e-04	1000	44	3.4294e-05
200	44	1.1519e-04	2000	44	3.8906e-05

In the tables N is the number of grid points, K is the number of iterations performed and $Error = \|U_K - u\|$.

Table 3.3: The convergence in Example 3.2.1 for $TOL = 10^{-6}$

N	K	$Error$	N	K	$Error$
50	54	0.0050	1000	54	2.6122e-06
100	54	6.1906e-04	2000	54	3.4403e-06
200	54	3.9533e-04	3000	54	3.4403e-06
500	54	3.9522e-04	4000	54	3.7370e-06

Remark 3.2.1. From the tables we see that for each tolerance the number of iterations is constant and the approximate solution reaches the accuracy $O(h^2)$ ($h = 1/N$) as the tolerance. The further increase of number of grid points does not improve the accuracy of approximate solution.

We can explain this phenomenon as follows:
From Theorem 3.2.9 it is seen that the error of the actual solution, i.e., the discrete solution, consists of two terms. The first term $(M_0 + 1/r)p_k d$ is the error of the iterative method at continuous level (see Theorem 3.2.5) and the second term $O(h^2)$ is the error of discretization at each iteration. The first term depends on the iteration number k by the formula $p_k = q^k/(1-q)$, where q depends on the nature of the boundary value problem (see Theorem 3.2.3). Therefore, it is desired to choose suitable h consistent with q because the choice of very small h does not improve the accuracy of approximate discrete solution. Indeed, suppose h^* is consistent with q in the sense that the quantities $O((h^*)^2)$ and $(M_0 + 1/r)p_K d$ for some K are the same as TOL. Then for any $h < h^*$ the accuracy almost remains the same. Theoretically, the number of iterations K is the minimal natural number k satisfying the inequality $(M_0 + 1/r)p_k d \le TOL$.

Example 3.2.2 (Example 4.1 in [48]). Consider the boundary value problem

$$u^{(4)}(t) = -u^2(e^{-u} + 1), \quad 0 < t < 1,$$
$$u'(0) = u''(0) = u'(1) = 0, \; u(0) = \int_0^1 t^2 u(t)dt.$$

In this example

$$f = f(t, u) = -u^2(e^{-u} + 1), \quad g(t) = t^2.$$

So

$$C_0 = \int_0^1 s^2 ds = \frac{1}{3}.$$

Choosing $M = 0.4$ and $r = 3$, we define

$$\mathcal{D}_M^+ = \{(t, u) \mid 0 \le t \le 1, \; 0 \le u \le (M_0 + 1)M\},$$

where $M_0 = 0.0139$ as was computed in (3.2.18). Then it is easy to verify that

$$-M \le f(t, u) \le 0 \text{ in } \mathcal{D}_M^+$$

and $|\frac{\partial f}{\partial u}| \le 1.622 =: L_0$ in $\mathcal{D}_M^+$. Therefore, $q_1 = rC_0M_0 + C_0 = 0.3472$, $q_2 = L_0(M_0 + \frac{1}{r}) = 0.5633$, and due to this $0 < q < 1$. By Theorem 3.2.4, the problem has a unique nonnegative solution. Since the function $u(t) \equiv 0$ is a solution of the problem, we conclude that the unique solution of the problem is this trivial solution. The computational experiment supports this theoretical conclusion. Remark that in [48] the authors proved that the problem has a positive solution. Hence, we can conclude that their result is not correct.

Example 3.2.3 (Example 4.2 in [48]). Consider the boundary value problem

$$u^{(4)}(t) = -\sqrt{(1+u)} - \sin u, \quad 0 < t < 1,$$
$$u'(0) = u''(0) = u'(1) = 0, \ u(0) = \int_0^1 tu(t)dt.$$

In this example

$$f = f(t,u) = -\sqrt{(1+u)} - \sin u, \quad g(t) = t.$$

So,

$$C_0 = \int_0^1 tdt = \frac{1}{2}.$$

Choosing $r = 3$ and $M = 3$, we have $-M \le f(t,u) \le 0$ in $\mathcal{D}_M^+$, where

$$\mathcal{D}_M^+ = \{(t,u) \mid 0 \le t \le 1, \ 0 \le u \le (M_0 + \frac{1}{r})M\}.$$

In this domain we can take the Lipschitz coefficient $L_0 = 1.5$. Therefore, $q_1 = q_2 = 0.5209$ and then $q = 0.5209 < 1$. Moreover, $f(t,0) = -1 \neq 0$. Hence, by Theorem 3.2.4 the problem has a unique positive solution. Remark that in [48] the authors could only conclude the existence of at least one positive solution.

The numerical computations show that the iterative method described in Section 3.2.3 converges fast. As in Example 3.2.1, the number of iterations for achieving a given tolerance is independent of the grid size. Table 3.4 reports the number of iterations in dependence on TOL.

Table 3.4: The convergence in Example 3.2.3

TOL	10^{-4}	10^{-5}	10^{-6}	10^{-8}
K	12	16	19	26

The graph of the approximate solution for $N = 100$ and $TOL = 10^{-4}$ is depicted in Figure 3.3.

Example 3.2.4. Consider Example 3.2.2 with

$$f = -(1+u^2).$$

Then $\frac{-f(u)}{u} \to +\infty$ as $u \to +0$ and $u \to +\infty$. Thus, neither Theorem 3.1 or Theorem 3.2 in [48] are satisfied, so the existence of positive solution is not guaranteed.
Now apply our theory: Choose $M = 2, r = 3$, then

$$\mathcal{D}_M^+ = \{(t,u) \mid 0 \le t \le 1, \ 0 \le u \le (M_0 + \frac{1}{r})M = 0.6944\}.$$

In $\mathcal{D}_M^+$ we have $-M \le f \le 0$, $|f'_u| \le 1.3888 = L_0$. After simple calculations we obtain $q_1 = 0.3472$, $q_2 = 0.4822$. Hence, by Theorem 3.2.3, the problem has a unique nonnegative solution. Due to $f(t,0) \neq 0, u(t) \not\equiv 0$, it is a positive solution. The performed numerical experiments also show that the number of iterations

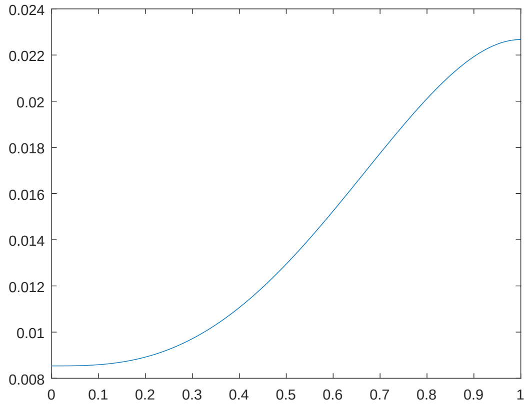

Figure 3.3: Approximate solution in Example 3.2.3.

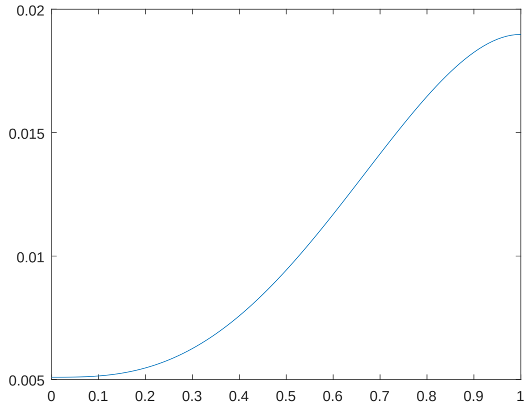

Figure 3.4: Approximate solution in Example 3.2.4.

Table 3.5: The convergence in Example 3.2.4

TOL	10^{-4}	10^{-5}	10^{-6}	10^{-8}
K	7	9	12	16

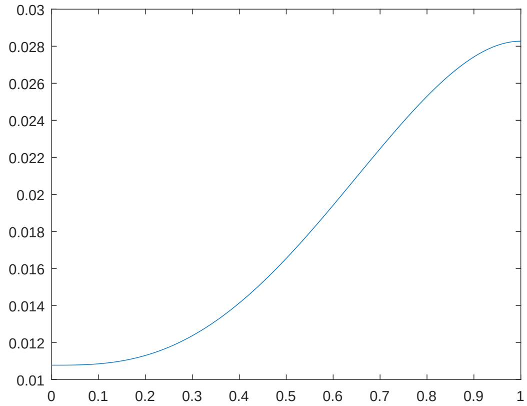

Figure 3.5: Approximate solution in Example 3.2.5.

for achieving a given tolerance is independent of the grid size. Table 3.5 reports the number of iterations in dependence on TOL.

The graph of the approximate solution for $N = 100$ and $TOL = 10^{-4}$ is depicted in Figure 3.4.

Example 3.2.5. Consider the problem (3.2.1)-(3.2.2) with

$$f(t, u, y, v, z) = -(\sqrt{1+u} + \sin y + \frac{1}{3}\cos v + \sin z), \ g(s) = s.$$

It is possible to verify that all the conditions of Theorem 3.2.4 are satisfied. So, the problem has a unique positive solution.

The results on the convergence of the iterative method for this example is presented in Table 3.6.

Table 3.6: The convergence in Example 3.2.5

TOL	10^{-4}	10^{-5}	10^{-6}	10^{-8}
K	11	14	18	25

The approximate solution obtained on the grid with the number of nodes $N = 100$ and $TOL = 10^{-4}$ is depicted on Figure 3.5.

3.2.6. Conclusion

In this section, we have proved the existence, uniqueness and positivity of solution of a fully fourth order nonlinear differential equation involving integral boundary conditions. These results are obtained due to the reduction of the problem to a fixed point problem for an operator acting on pairs of functions and numbers. This idea also was used to study third order nonlinear integral boundary value problem in the previous section. It is a further development of the methodology applied by ourselves before for two-point nonlinear boundary value problems. Ensuring that the integral boundary value problem has a unique solution we proposed an iterative method to find the solution at continuous level. After that we design a discrete scheme as the realization of the continuous iterative method. We also made the analysis of total error of the approximate discrete solution, which results from the error of the continuous iterative method and the error of discretization at each iteration. Several examples confirm the validity of the obtained theoretical results and efficiency of the proposed iterative method.

The methodology used in this paper can be applied to other higher order nonlinear differential equations involving integral boundary conditions including nonlinear boundary conditions. This is the subject of our research in the future.

3.3. Sketch of the method for treating other integral boundary value problems

In the previous sections 3.1 and 3.2 we considered some integral boundary value problems for nonlinear third and fourth order differential equations. For these problems we obtained the results of qualitative aspects such as the existence, uniqueness and positivity of solutions. More importantly, we proposed iterative methods on both continuous and discrete levels. This quantitative aspect of boundary value problems involving integral conditions, to our best knowledge, has not been studied in the literature.

Following the methodology used in the previous sections, in this section we sketch the method for treating some other problems involving boundary conditions of integral type. As in the whole thesis, we denote the Green's function of the problems by $G(t,s)$, its first, second and third derivatives with respect to t by $G_1(t,s)$, $G_2(t,s)$ and $G_3(t,s)$.

Problem 1 (see [24])

$$\begin{aligned} u'''(t) = f(t, u(t), u'(t), u''(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = 0,\ u'(1) = \int_0^1 g(t)u'(t)dt. \end{aligned} \tag{3.3.1}$$

As mentioned in Introduction of the Thesis, some sufficient conditions for the existence and nonexistence of monotone positive solutions are obtained for a particular case of the problem when $f = f(t, u(t), u'(t))$.

Here, similarly as done in Section 3.1 we set

$$\varphi(t) = f(t, u(t), u'(t), u''(t)),$$

$$\alpha = \int_0^1 g(t)u'(t)dt,$$

Then the problem (3.3.1) becomes the problem

$$u'''(t) = \varphi(t), \quad 0 < t < 1, \tag{3.3.2}$$

$$u(0) = u'(0) = 0,\ u'(1) = \alpha \tag{3.3.3}$$

whose solution has the form

$$u(t) = \int_0^1 G(t,s)\varphi(s)ds + \frac{1}{2}\alpha t^2, \quad 0 < t < 1,$$

where

$$G(t,s) = \begin{cases} \frac{s}{2}(t^2 - 2t + s), & 0 \le s \le t \le 1, \\ \frac{t^2}{2}(s-1), & 0 \le t \le s \le 1. \end{cases}$$

is the Green's function of the problem. It suggests us to introduce the operator A in the space $\mathcal{B} = C[0,1] \times \mathbb{R}$ as follows

$$Aw = \begin{pmatrix} f(t, u(t), u'(t), u''(t) \\ \int_0^1 g(t)u'(t)dt \end{pmatrix}, \tag{3.3.4}$$

where $u(t)$ is the solution of the problem (3.3.2).
The solution of the problem (3.3.1) leads to the finding of fixed point of A, whose realization is the following iterative process:

1. Given $w_0 = (\varphi_0, \alpha_0)^T \in B[0,M]$, for example,

$$\varphi_0(t) = f(t,0,0,0), \ \alpha_0 = 0.$$

2. Knowing $\varphi_n(t)$ and α_n $(n = 0,1,...)$, compute

$$\begin{aligned} u_n(t) &= \int_0^1 G(t,s)\varphi_n(s)ds + \frac{1}{2}\alpha_n t^2, \\ y_n(t) &= \int_0^1 G_1(t,s)\varphi_n(s)ds + \alpha_n t, \\ z_n(t) &= \int_0^1 G_2(t,s)\varphi_n(s)ds + \alpha_n. \end{aligned}$$

3. Update

$$\begin{aligned} \varphi_{n+1}(t) &= f(t, u_n(t), y_n(t), z_n(t)), \\ \alpha_{n+1} &= \int_0^1 g(t)y_n(t)dt. \end{aligned}$$

Problem 2 (see [45])

$$\begin{aligned} u'''(t) &= f(t, u(t), u'(t), u''(t)) = 0, \quad 0 < t < 1, \\ u(0) &= u''(0) = 0, \ u(1) = \int_0^1 g(t)u(t)dt. \end{aligned} \tag{3.3.5}$$

For a particular case of the problem, namely, for the case $f(t, u(t), u''(t))$ the authors in [45] established sufficient conditions for the existence of positive solutions.
By the methodology of Section 3.1 the problem (3.3.5) is reduced to the fixed point problem for the operator A defined as

$$Aw = \begin{pmatrix} f(t, u(t), u'(t), u''(t) \\ \int_0^1 g(t)u(t)dt \end{pmatrix}, \tag{3.3.6}$$

where $u(t)$ is the solution of the problem

$$u'''(t) = \varphi(t), \quad 0 < t < 1, \tag{3.3.7}$$
$$u(0) = u''(0) = 0,\ u(1) = \alpha. \tag{3.3.8}$$

This solution is represented in the form

$$u(t) = \int_0^1 G(t,s)\varphi(s)ds + \alpha t, \quad 0 < t < 1,$$

where

$$G(t,s) = \frac{1}{2}\begin{cases} (1-t)(s^2-t), & 0 \le s \le t \le 1, \\ -t(1-s)^2, & 0 \le t \le s \le 1. \end{cases}$$

The solution of the problem (3.3.5) can be found by the iterative method

1. Given $w_0 = (\varphi_0, \alpha_0)^T \in B[0, M]$, for example,

$$\varphi_0(t) = f(t, 0, 0, 0),\ \alpha_0 = 0.$$

2. Knowing $\varphi_n(t)$ and α_n $(n = 0, 1, ...)$, compute

$$u_n(t) = \int_0^1 G(t,s)\varphi_n(s)ds + \alpha_n t,$$
$$y_n(t) = \int_0^1 G_1(t,s)\varphi_n(s)ds + \alpha_n,$$
$$z_n(t) = \int_0^1 G_2(t,s)\varphi_n(s)ds.$$

3. Update

$$\varphi_{n+1}(t) = f(t, u_n(t), y_n(t), z_n(t)),$$
$$\alpha_{n+1} = \int_0^1 g(t)u_n(t)dt.$$

Problem 3

$$\begin{aligned} &u'''(t) = f(t, u(t), u'(t), u''(t)) = 0, \quad 0 < t < 1, \\ &u(0) = 0, u'(0) = \int_0^1 g_1(t)u'(t)dt,\ u'(1) = \int_0^1 g_2(t)u'(t)dt. \end{aligned} \tag{3.3.9}$$

This problem is a simplification of the problem considered in [44], where under very complicated and hard-to-verify conditions Boucherif et al. established the existence of solutions. In order to treat the above problem we also reduce it to the fixed point problem for the operator A acting on the elements $w = (\varphi(t), \alpha, \beta)^T$ of the space $\mathcal{B} = C[0,1] \times \mathbb{R}^2$

$$Aw = \begin{pmatrix} f(t, u(t), u'(t), u''(t) \\ \int_0^1 g_1(t)u'(t)dt \\ \int_0^1 g_2(t)u'(t)dt \end{pmatrix}. \tag{3.3.10}$$

The successive approximation method for finding the fixed point of A is realized by the iterative method:

1. Given $w_0 = (\varphi_0, \alpha_0)^T \in B[0, M]$, for example,

$$\varphi_0(t) = f(t, 0, 0, 0), \ \alpha_0 = 0, \ \beta_0 = 0.$$

2. Knowing $\varphi_n(t)$, α_n and β_n $(n = 0, 1, ...)$, compute

$$\begin{aligned} u_n(t) &= \int_0^1 G(t,s)\varphi_n(s)ds + \frac{1}{2}(\beta_n - \alpha_n)t^2 + \alpha_n t, \\ y_n(t) &= \int_0^1 G_1(t,s)\varphi_n(s)ds + (\beta_n - \alpha_n)t + \alpha_n, \\ z_n(t) &= \int_0^1 G_2(t,s)\varphi_n(s)ds + (\beta_n - \alpha_n). \end{aligned}$$

3. Update

$$\begin{aligned} \varphi_{n+1}(t) &= f(t, u_n(t), y_n(t), z_n(t)), \\ \alpha_{n+1} &= \int_0^1 g_1(t) y_n(t) dt, \\ \beta_{n+1} &= \int_0^1 g_2(t) y_n(t) dt. \end{aligned}$$

Problem 4

$$u^{(4)}(t) = f(t, u(t), u'(t), u''(t)), \quad t \in [0, 1], \tag{3.3.11}$$

$$u(0) = u'(1) = u'''(1) = 0, \ u''(0) = \int_0^1 g(s)u''(s)ds. \tag{3.3.12}$$

In [55] the authors obtained the existence and nonexistence of concave monotone positive solutions.
In order to study the existence, uniqueness and methods for finding the solution of the problem following the methodology in Section 3.2 we also introduce the operator A acting on elements $w = (\varphi(t), \mu)$ by the formula

$$Aw = \begin{pmatrix} f(t, u(t), u'(t), u''(t), u'''(t)) \\ \int_0^1 g(s)u(s)ds \end{pmatrix},$$

where $u(t)$ is the solution of the problem

$$\begin{aligned} &u^{(4)}(t) = \varphi(t), \quad 0 < t < 1, \\ &u(0) = u'(1) = u'''(1) = 0, , \ u''(0) = \mu. \end{aligned}$$

The solution of the problem (3.3.11) will be reduced to the fixed point problem of A and the realization of the successive approximation method for it is the iterative method

1. Given an initial approximation

$$\varphi_0(t) = f(t, 0, 0, 0, 0), \ \mu_0 = 0$$

2. Knowing $\varphi_k(t)$ and μ_k $(k = 0, 1, ...)$ compute

$$u_k(t) = \int_0^1 G_0(t,s)\varphi_k(s)ds + \frac{1}{2}\mu_k t^2 - \mu_k t,$$
$$y_k(t) = \int_0^1 G_1(t,s)\varphi_k(s)ds + \mu_k(t-1),$$
$$v_k(t) = \int_0^1 G_2(t,s)\varphi_k(s)ds + \mu_k,$$
$$z_k(t) = \int_0^1 G_3(t,s)\varphi_k(s)ds.$$

3. Compute new approximation

$$\varphi_{k+1}(t) = f(t, u_k(t), y_k(t), v_k(t), z_k(t)),$$
$$\mu_{k+1} = \int_0^1 g(s)v_k(s)ds.$$

3.4. Chapter conclusion

In this chapter, we study two nonlocal boundary value problems, namely, third order and fourth order nonlinear differential equations with integral boundary conditions. By the reduction of the problems to operator equation for pairs of the right-hand sides of the differential equation and integral boundary condition we have established the existence and uniqueness of solution of the original problems. And more importantly, we were the first to propose iterative methods on continuous level for finding the solution. Especially, in the case of fourth order nonlinear differential equations with integral boundary conditions we have proposed a numerical method based on discretization of the iterative method at continuous level. The total error estimate of the numerical solution was obtained. Many numerical examples confirmed the validity of obtained theoretical results.

The methodology used in this chapter can be applied to nonlinear boundary value problems of any order with integral boundary conditions.

The results of this chapter were published in two papers [AL3] and [AL5] in SCIE journals.

Chapter 4

The existence, uniqueness of a solution and an iterative method for integro-differential and functional differential equations

4.1. Existence results and an iterative method for an integro-differential equation

4.1.1. Introduction

In this section we consider the problem

$$\begin{aligned} u^{(4)}(x) &= f(x, u(x), u'(x), \int_0^1 k(x,t)u(t)dt), \\ u(0) &= 0,\ u(1) = 0,\ u''(0) = 0,\ u''(1) = 0, \end{aligned} \tag{4.1.1}$$

where the function $f(x, u, v, z)$ and $k(x, t)$ are assumed to be continuous. This problem is an extension of the problem

$$\begin{aligned} y^{(4)}(x) &= f(x, y(x), \int_0^1 k(x,t)y(t)dt),\ 0 < x < 1, \\ y(0) &= 0,\ y(1) = 0,\ y''(0) = 0,\ y''(1) = 0 \end{aligned} \tag{4.1.2}$$

considered recently by Wang in [66], where by using the monotone method and a maximum principle, he constructed the sequences of functions, which converge to the extremal solutions of the problem. Remark that the presence of an extra term u' in the right-hand side of (4.1.1) does not allow to use the argument in [66] to study the existence of solutions of the problem. Here, by using the method developed in the previous works [11, 13, 14, 86, 87, 89, 91] we establish the existence and uniqueness of a solution and propose an iterative method at both continuous and discrete levels for finding the solution. The second order convergence of the method is proved. The theoretical results are illustrated by some examples.

4.1.2. Existence results

Using the methodology in [11, 13, 14, 86, 87, 89, 91] we introduce the operator A defined in the space of continuous functions $C[0, 1]$ by

$$(A\varphi)(x) = f(x, u(x), u'(x), \int_0^1 k(x,t)u(t)dt), \tag{4.1.3}$$

where $u(x)$ is the solution of the boundary value problem

$$\begin{aligned} u^{(4)} &= \varphi(x),\ 0 < x < 1, \\ u(0) &= u''(0) = u(1) = u''(1) = 0. \end{aligned} \tag{4.1.4}$$

It is easy to prove the following lemma.

Lemma 4.1.1. If the function $u(x)$ is the solution of the BVP (4.1.1) then the function

$$\varphi(x) = f(x, u(x), u'(x), \int_0^1 k(x,t)u(t)dt)$$

satisfies the operator equation (4.1.5). Conversely, if the function φ is a fixed point of the operator A, that is, φ is the solution of the operator equation

$$A\varphi = \varphi, \tag{4.1.5}$$

where A is defined by (4.1.3)-(4.1.4) then the function $u(x)$ determined from the BVP (4.1.4) is a solution of the BVP (4.1.1).

Because of this, the study of the original BVP (4.1.1) is reduced to the study of the operator equation (4.1.5).

Notice that the BVP (4.1.4) has a unique solution representable in the form

$$u(x) = \int_0^1 G_0(x,s)\varphi(s)ds, \quad 0 < t < 1, \tag{4.1.6}$$

where

$$G_0(x,s) = \frac{1}{6}\begin{cases} s(x-1)(x^2 - x + s^2), & 0 \le s \le x \le 1 \\ x(s-1)(s^2 - s + x^2), & 0 \le x \le s \le 1 \end{cases} \tag{4.1.7}$$

is the Green's function of the operator $u^{(4)}(t) = 0$ associated with the homogeneous boundary conditions $u(0) = u''(0) = u(1) = u''(1) = 0$.
Taking derivative both sides of (4.1.6) yields

$$u'(x) = \int_0^1 G_1(x,s)\varphi(s)ds, \tag{4.1.8}$$

where

$$G_1(x,s) = \frac{1}{6}\begin{cases} s(3x^2 - 6x + s^2 + 2), & 0 \le s \le x \le 1, \\ (s-1)(3x^2 - 2s + s^2), & 0 \le x \le s \le 1. \end{cases} \tag{4.1.9}$$

Set

$$\begin{aligned} M_0 &= \max_{0\le x\le 1} \int_0^1 |G_0(x,s)|ds, \\ M_1 &= \max_{0\le x\le 1} \int_0^1 |G_1(x,s)|ds, \\ M_2 &= \max_{0\le x\le 1} \int_0^1 |k(x,s)|ds. \end{aligned} \tag{4.1.10}$$

It is easy to verify that

$$M_0 = \frac{5}{384}, M_1 = \frac{1}{24}. \tag{4.1.11}$$

Now for any positive number M, we define the domain

$$\begin{aligned} \mathcal{D}_M = \{(x,u,v,z) \mid 0 \le x \le 1,\ &|u| \le M_0 M, \\ &|v| \le M_1 M,\ |z| \le M_0 M_2 M\}, \end{aligned} \tag{4.1.12}$$

and by $B[0, M]$ we denote the closed ball with center 0 and radius M in the space $C[0, 1]$

$$B[0, M] = \{u \in C[0, 1] \mid \|u\| \leq M\},$$

where $\|u\| = \max_{0 \leq x \leq 1} |u(x)|$.

Theorem 4.1.1 (Existence and uniqueness)**.** Assume that $k(x, t)$ is a continuous function in the domain $[0, 1]^2$ and there exist numbers $M > 0$, $L_0, L_1, L_2 \geq 0$ such that:

(i) $f(x, u, v, z)$ is a continuous function in $\mathcal{D}_M$ and $|f(x, u, v, z)| \leq M$, $\forall (x, u, v, z) \in \mathcal{D}_M$.

(ii) $|f(x_2, u_2, v_2, z_2) - f(x_1, u_1, v_1, z_1)| \leq L_0|u_2 - u_1| + L_1|v_2 - v_1| + L_2|z_2 - z_1|$, $\forall (x_i, u_i, v_i, z_i) \in \mathcal{D}_M,\ i = 1, 2$.

(iii) $q = L_0 M_0 + L_1 M_1 + L_2 M_0 M_2 < 1$.

Then the problem (4.1.1) has a unique solution $u \in C^4[0, 1]$ satisfying the estimates $|u(x)| \leq M_0 M$, $|u'(x)| \leq M_1 M$ for any $0 \leq x \leq 1$.

Proof. Under the conditions of the theorem, we shall show that A is a contraction operator in $B[0, M]$. Then the operator equation (4.1.5) has a unique solution $u \in C^{(4)}[0, 1]$ which leads to the existence and uniqueness of solution of the BVP (4.1.1).

Indeed, take $\varphi \in B[0, M]$. Then the problem (4.1.4) has a unique solution of the form (4.1.6). Combining this with (4.1.10) we have $|u(x)| \leq M_0\|\varphi\|$ for all $x \in [0, 1]$. Similarly, we obtain $\|u'(x)\| \leq M_1\|\varphi\| \forall x \in [0, 1]$. Denote by K the integral operator defined by

$$(Ku)(x) = \int_0^1 k(x, t)u(t)dt.$$

Then from the last equation in (4.1.10) we obtain $|(Ku)(x)| \leq M_0 M_2\|\varphi\|$, $x \in [0, 1]$. Therefore, if $\varphi \in B[0, M]$, that is, $\|\varphi\| \leq M$ then for any $x \in [0, 1]$ we have

$$|u(x)| \leq M_0 M,\ |u'(x)| \leq M_1 M,\ |(Ku)(x)| \leq M_0 M_2 M.$$

Thus, $(x, u(x), u'(x), (Ku)(x)) \in \mathcal{D}_M$. The condition (i) leads to

$$|f(x, u(x), u'(x), (Ku)(x))| \leq M \quad \forall x \in [0, 1].$$

Therefore, $|(A\varphi)(x)| \leq M$, $\forall x \in [0, 1]$ and $\|A\varphi\| \leq M$ which means $A : B[0, M] \to B[0, M]$.

Next, take $\varphi_1, \varphi_2 \in B[0, M]$. From the conditions (ii) and (iii) we have

$$\|A\varphi_2 - A\varphi_1\| \leq (L_0 M_0 + L_1 M_1 + L_2 M_0 M_2)\|\varphi_2 - \varphi_1\| = q\|\varphi_2 - \varphi_1\|.$$

Because $q < 1$, A is a contraction in $B[O, M]$. Thus the proof of the theorem is completed. □

For studying positive solutions of the BVP (4.1.1) we denote

$$\begin{aligned} \mathcal{D}_M^+ = \{(x, u, v, z) \mid 0 \leq x \leq 1,\ 0 \leq u \leq M_0 M, \\ |v| \leq M_1 M,\ |z| \leq M_0 M_2 M\}, \end{aligned} \tag{4.1.13}$$

and

$$S_M = \{\varphi \in C[0, 1], 0 \leq \varphi(x) \leq M\}.$$

Theorem 4.1.2 (Positivity of solution)**.** Assume that $k(x,t)$ is a continuous function in the domain $[0,1]^2$ and there exist numbers $M > 0,\ L_0, L_1, L_2 \geq 0$ satisfying:

(i) $f(x,u,v,z)$ is a continuous function in $\mathcal{D}_M^+$ and $0 \leq f(x,u,v,z) \leq M,\ \forall (x,u,v,z) \in \mathcal{D}_M^+$ and $f(x,0,0,0) \not\equiv 0$.

(ii) $|f(x_2,u_2,v_2,z_2) - f(x_1,u_1,v_1,z_1)| \leq L_0|u_2 - u_1| + L_1|v_2 - v_1| + L_2|z_2 - z_1|$, $\forall (x_i,u_i,v_i,z_i) \in \mathcal{D}_M^+,\ i = 1,2$.

(iii) $q = L_0M_0 + L_1M_1 + L_2M_0M_2 < 1$.

Then the problem (4.1.1) has a unique positive solution $u \in C^4[0,1]$ satisfying the estimates $0 \leq u(x) \leq M_0M$, $|u'(x)| \leq M_1M$ for any $0 \leq x \leq 1$.

Proof. By replacing $\mathcal{D}_M$ by $\mathcal{D}_M^+$, $B[0,M]$ by S_M in the proof of Theorem 4.1.1, we obtain the existence of a nonnegative solution. Because $f(x,0,0,0) \not\equiv 0$, this solution must be positive. □

4.1.3. Numerical method

Assume that all the assumptions of Theorem 4.1.1 are met. Then the problem (4.1.1) has a unique solution. To find it, consider the following iterative method:

1. Given an initial approximation

$$\varphi_0(x) = f(x,0,0,0). \tag{4.1.14}$$

2. Knowing the $m-$th approximation $\varphi_m(x)$ $(m = 0,1,...)$ compute

$$\begin{aligned} u_m(x) &= \int_0^1 G_0(x,t)\varphi_m(t)dt, \\ v_m(x) &= \int_0^1 G_1(x,t)\varphi_m(t)dt, \\ z_m(x) &= \int_0^1 k(x,t)u_m(t)dt. \end{aligned} \tag{4.1.15}$$

3. Compute the new approximation

$$\varphi_{m+1}(x) = f(x,u_m(x),v_m(x),z_m(x)). \tag{4.1.16}$$

This iterative method is in fact the successive approximation method for finding the fixed point of operator A defined by (4.1.3)-(4.1.4). As shown in the proof of Theorem 4.1.1 the operator A is a contraction mapping in the closed ball $B[0,M]$. Therefore, the iterative method converges and there holds the estimate

$$\|\varphi_m - \varphi\| \leq \frac{q^m}{1-q}\|\varphi_1 - \varphi_0\| = p_m d,$$

where φ is the fixed point of A and

$$p_m = \tfrac{q^m}{1-q},\ d = \|\varphi_1 - \varphi_0\|. \tag{4.1.17}$$

This fact leads to the following result.

Theorem 4.1.3 (Convergence)**.** If the conditions of Theorem 4.1.1 are satisfied then the iterative method (4.1.14)-(4.1.16) converges and there hold the estimates for the approximate solution $u_k(t)$

$$\|u_m - u\| \leq M_0 p_m d, \ \|u'_m - u'\| \leq M_1 p_m d,$$

where u is the exact solution of the problem (4.1.1), d and p_m are determined by (4.1.17).

For the numerical realization of the iterative method above, we design a corresponding discrete iterative method. In order to do this, construct the uniform grid $\bar{\omega}_h = \{x_i = ih, \ h = 1/N, i = 0, 1, ..., N\}$ the interval $[0, 1]$, and by $\Phi_m(x), U_m(x), V_m(x), Z_m(x)$ denote the grid functions defined on this grid and approximating the functions $\varphi_m(x)$, $u_m(x), v_m(x), z_m(x)$.

Consider the following discrete iterative method:

1. Given an initial approximation

$$\Phi_0(x_i) = f(x_i, 0, 0, 0), \ i = 0, ..., N. \tag{4.1.18}$$

2. Knowing the $m-$th approximation $\Phi_m(x_i), \ m = 0, 1, ...; \ i = 0, ..., N,$, compute approximately the definite integrals (4.1.15) by the trapezoidal rule

$$\begin{aligned} U_m(x_i) &= \sum_{j=0}^{N} h\rho_j G_0(x_i, x_j)\Phi_m(x_j), \\ V_m(x_i) &= \sum_{j=0}^{N} h\rho_j G_1(x_i, x_j)\Phi_m(x_j), \\ Z_m(x_i) &= \sum_{j=0}^{N} h\rho_j k(x_i, x_j)U_m(x_j), \ i = 0, ..., N, \end{aligned} \tag{4.1.19}$$

where ρ_j is the weight of the trapezoidal rule, that is

$$\rho_j = \begin{cases} 1/2, \ j = 0, N \\ 1, \ j = 1, 2, ..., N-1. \end{cases}$$

3. Compute the new approximation

$$\Phi_{m+1}(x_i) = f(x_i, U_m(x_i), V_m(x_i), Z_m(x_i)). \tag{4.1.20}$$

To obtain the error estimates for the approximate solution for $u(t)$ and its derivatives on the grid, we need the following auxiliary results.

Proposition 4.1.4. Suppose that the function $f(t, u, v, z)$ has all continuous partial derivatives up to second order in the domain $\mathcal{D}_M$ and the kernel function $k(x, t)$ also has all continuous partial derivatives up to second order in $[0, 1]^2$. Then for the functions $\varphi_m(x), u_m(x), v_m(x), z_m(x), m = 0, 1, ...$, constructed by the iterative method (4.1.14)-(4.1.16) we have $\varphi_m(x) \in C^2[0, 1], \ u_m(x) \in C^6[0, 1]$, $v_m(x) \in C^5[0, 1], \ z_m(x) \in C^2[0, 1]$.

Proof. The proposition will be proved by induction. For $k = 0$, due to the condition on the function f we have $\varphi_0(t) \in C^2[0,1]$ because $\varphi_0(x) = f(x,0,0,0)$. Since

$$u_0(x) = \int_0^1 G_0(x,t)\varphi_0(t)dt$$

it implies that $u_0(x)$ is the solution of the problem

$$u_0^{(4)}(x) = \varphi_0(x), \quad x \in (0,1),$$
$$u_0(0) = u_0(1) = u_0''(0) = u_0''(1) = 0.$$

Thus, $u_0(x) \in C^6[0,1]$, which leads to $v_0(x) \in C^5[0,1]$ since $v_0(x) = u_0'(x)$. Due to the condition $k(x,t)$ has all continuous derivatives up to second order, the function $z_0(x) = \int_0^1 k(x,t)u_0(t)dt$ belongs to $C^2[0,1]$.

Now assume $\varphi_m(x) \in C^2[0,1]$, $u_m(x) \in C^6[0,1]$, $v_m(x) \in C^5[0,1]$, $z_m(x) \in C^2[0,1]$. Then, since $\varphi_{m+1}(x) = f(x, u_m(x), v_m(x), z_m(x))$ and because the function f is assumed to have continuous derivative in all variables up to second order, it follows that $\varphi_{m+1}(x) \in C^2[0,1]$. Repeating the same argument as for $\varphi_0(x)$ above we have $u_{m+1}(x) \in C^6[0,1]$, $v_{m+1}(x) \in C^5[0,1]$, $z_{m+1}(x) \in C^2[0,1]$ Therefore, the proof is completed. □

Proposition 4.1.5. For any function $\varphi(x) \in C^2[0,1]$ there holds the estimate

$$\int_0^1 G_n(x_i,t)\varphi(t)dt = \sum_{j=0}^{N} h\rho_j G_n(x_i,t_j)\varphi(t_j) + O(h^2) \quad (n = 0,1). \tag{4.1.21}$$

Proof. This estimate is obvious due to the error estimate of the trapezoidal rule since $G_n(x_i,t)$ $(n = 0,1)$ are continuous at t_j and are polynomials in the intervals $[0,t_j]$ and $[t_j,1]$. □

Proposition 4.1.6. Under the conditions of Proposition 4.1.4, for any $m = 0,1,...$ there hold the estimates

$$\|\Phi_m - \varphi_m\| = O(h^2), \quad \|U_m - u_m\| = O(h^2), \tag{4.1.22}$$

$$\|V_m - v_m\| = O(h^2), \ \|Z_m - z_m\| = O(h^2) \tag{4.1.23}$$

where $\|.\| = \|.\|_{\bar{\omega}_h}$ is the max-norm of function on the grid $\bar{\omega}_h$.

Proof. The proposition will be proved by induction. For $m = 0$ we have immediately $\|\Phi_0 - \varphi_0\| = 0$. Next, by the first equation in (4.1.15) and Proposition 4.1.5 we obtain

$$u_0(x_i) = \int_0^1 G_0(x_i,t)\varphi_0(t)dt = \sum_{j=0}^{N} h\rho_j G_0(x_i,t_j)\varphi_0(t_j) + O(h^2) \tag{4.1.24}$$

for any $i = 0,...,N$. On the other hand, taking into account the first equation in (4.1.19) we obtain

$$U_0(x_i) = \sum_{j=0}^{N} h\rho_j G_0(x_i,t_j)\Phi_0(t_j). \tag{4.1.25}$$

Thus, $|U_0(t_i) - u_0(t_i)| = O(h^2)$ since $\Phi_0(t_j) = \varphi_0(t_j) = f(t_j, 0, 0, 0)$. Therefore, $\|U_0 - u_0\| = O(h^2)$.
Analogeously, we have

$$\|V_0 - v_0\| = O(h^2). \tag{4.1.26}$$

Next, due to the trapezoidal rule we have

$$z_0(x_i) = \int_0^1 k(x_i, t)u_0(t)dt = \sum_{j=0}^{N} h\rho_j k(x_i, t_j)u_0(t_j) + O(h^2),$$

while by the third equation in (4.1.19) we obtain

$$Z_0(x_i) = \sum_{j=0}^{N} h\rho_j k(x_i, t_j)U_0(t_j),\ i = 0, ..., N.$$

Thus,

$$\begin{aligned}
\Big|Z_0(x_i) - z_0(x_i)\Big| &= \Big|\sum_{j=0}^{N} h\rho_j k(x_i, t_j)(U_0(t_j) - u_0(t_j))\Big| + O(h^2) \\
&\le \sum_{j=0}^{N} h\rho_j |k(x_i, t_j)||U_0(t_j) - u_0(t_j)| + O(h^2) \\
&\le Ch^2 \sum_{j=0}^{N} h\rho_j |k(x_i, t_j)| + O(h^2) \\
&\le CC_1 h^2 \sum_{j=0}^{N} h\rho_j + O(h^2) = O(h^2)
\end{aligned}$$

since $|U_0(t_j) - u_0(t_j)| \le Ch^2$, $|k(x_i, t_j)| \le C_1$, where C, C_1 are some constants.

Now assume that (4.1.22) and (4.1.23) hold for $m \ge 0$. We shall prove that these estimates hold for $m+1$. By the Lipschitz condition of the function f and the estimates (4.1.22) and (4.1.23), it follows the estimate

$$\|\Phi_{m+1} - \varphi_{m+1}\| = O(h^2).$$

Now from the first equation in (4.1.15) by Proposition 4.1.5 we obtain

$$u_{m+1}(x_i) = \int_0^1 G_0(x_i, t)\varphi_{m+1}(t)dt = \sum_{j=0}^{N} h\rho_j G_0(x_i, x_j)\varphi_{m+1}(x_j) + O(h^2).$$

On the other hand, by the first formula in (4.1.19) we obtain

$$U_{m+1}(x_i) = \sum_{j=0}^{N} h\rho_j G_0(x_i, x_j)\Phi_{m+1}(x_j).$$

From this equality and the above estimates we have the estimate

$$\|U_{m+1} - u_{m+1}\| = O(h^2).$$

Similarly, we have

$$\|V_{m+1} - v_{m+1}\| = O(h^2),\ \|Z_{m+1} - z_{k+1}\| = O(h^2).$$

Therefore, the proof is completed. □

Combining Proposition 4.1.6 and Theorem 4.1.3 leads to the following result.

Theorem 4.1.7. Suppose that all the assumptions of Theorem 4.1.1 and Proposition 4.1.4 are met. Then, for the approximate solution of the problem (4.1.1) obtained by the discrete iterative method on the grid $\bar{\omega}_h$ there hold the estimates

$$\|U_m - u\| \le M_0 p_m d + O(h^2),\ \|V_m - u'\| \le M_2 p_m d + O(h^2). \tag{4.1.27}$$

Proof. The first estimate is easily obtained if writing

$$U_m(t_i) - u(t_i) = (u_m(t_i) - u(t_i)) + (U_m(t_i) - u_m(t_i))$$

and using the first estimate in Theorem 4.1.3 and the second estimate in (4.1.22). The second estimate is similarly proved. □

4.1.4. Examples

Example 4.1.1. Consider the problem (4.1.1) with

$$\begin{aligned}
&k(x,t) = e^x \sin(\pi t), \quad (x,t) \in [0,1] \times [0,1],\\
&f(x, u(x), u'(x), \int_0^1 k(x,t)u(t)dt) = u^2(x) \int_0^1 k(x,t)u(t)dt + u(x)u'(x)\\
&-\frac{1}{2}e^x \sin^2(\pi x) + \pi^4 \sin(\pi x) - \frac{\pi}{2}\sin(2\pi x).
\end{aligned}$$

In this case

$$f(x,u,v,z) = u^2 z + uv - \frac{1}{2}e^x \sin^2(\pi x) + \pi^4 \sin(\pi x) - \frac{\pi}{2}\sin(2\pi x)$$

and $M_2 = \dfrac{2e}{\pi}$. It is easy to show that $u = \sin(\pi x)$ is the exact solution of the problem. In the domain

$$\mathcal{D}_M = \{(x,u,v,z) \mid 0 \le x \le 1,\ |u| \le M_0 M, |u'| \le M_1 M, |z| \le M_0 M_2 M\}$$

we have

$$|f(x,u,v,z)| \le M_0^3 M_2 M^3 + M_0 M_1 M^2 + \pi^4 + \frac{\pi}{2} + \frac{e}{2}.$$

It can be verified that if $M = 113$ then the conditions of Theorem 4.1.1 are met with $L_0 = 12.2010, L_1 = 1.4714, L_2 = 2.1649, q = 0.2690$. Thus, the problem has a unique solution $u(x)$ satisfying the estimates $|u(x)| \le 1.4714, |u'(x)| \le 4.7083$. These theoretical estimates are somewhat larger than the exact estimates $|u(x)| \le 1, |u'(x)| \le \pi$.

The numerical results obtained by the discrete iterative method (4.1.18)-(4.1.20) are reported in Tables 4.1 and 4.2. Here, $Error = \|U_m - u\|$.

Notice that if the stopping criterion is $\|\Phi_m - \Phi_{m-1}\| \le 10^{-10}$ instead of $\|U_m - u\| \le h^2$ then better accuracy of the approximate solution are obtained with more iterations (see Table 4.2).

From Table 4.2 we observe that the accuracy of the approximation is close to $O(h^4)$ even though by the above theory it is $O(h^2)$ only.

Table 4.1: The convergence in Example 4.1.1 for stopping criterion $\|U_m - u\| \le h^2$

N	h^2	m	$Error$
50	4.0000e-04	2	1.4305e-04
100	1.0000e-04	3	2.8588e-06
150	4.4444e-05	3	2.8599e-06
200	2.5000e-05	3	2.8602e-06
300	1.1111e-05	3	2.8603e-06
400	6.2500e-06	3	2.8603e-06
500	4.0000e-06	3	2.8603e-06
800	1.5625e-06	4	5.7485e-08
1000	1.0000e-06	4	5.7486e-08

Table 4.2: The convergence in Example 4.1.1 for stopping criterion $\|\Phi_m - \Phi_{m-1}\| \le 10^{-10}$

N	h^2	m	$Error$
50	4.0000e-04	7	2.2152e-08
100	1.0000e-04	7	1.3831e-09
150	4.4444e-05	7	2.7279e-10
200	2.5000e-05	7	8.5995e-11
300	1.1111e-05	7	1.6618e-11
400	6.2500e-06	7	4.9447e-12
500	4.0000e-06	7	1.7567e-12
800	1.5625e-06	7	1.4588e-13
1000	1.0000e-06	7	3.3318e-13

Example 4.1.2 (Example 4.2 in [66]). Consider the problem

$$\begin{aligned} u^{(4)}(x) &= \sin(\pi x)[(2 - u^2(x)) \int_0^1 tu(t)dt + 1], x \in (0,1), \\ u(0) &= 0,\ u(1) = 0,\ u''(0) = 0,\ u''(1) = 0. \end{aligned} \tag{4.1.28}$$

This is the problem (4.1.1) with

$$\begin{aligned} &k(x,t) = \sin(\pi x)t, \quad (x,t) \in [0,1] \times [0,1], \\ &f(x, u(x), u'(x), \int_0^1 k(x,t)u(t)dt) = (2 - u^2(x)) \int_0^1 \sin(\pi x)tu(t)dt + \sin(\pi x). \end{aligned}$$

Here, $f(x,u,v,z) = (2 - u^2)z + \sin(\pi x)$.
It is easy to verify that $M_2 = \max_{0 \le x \le 1} \int_0^1 |k(x,t)|dt = \frac{1}{2}$. With M_0 and M_1 in (4.1.11) we denote

$$\mathcal{D}_M = \Big\{(x,u,v,z) \mid 0 \le x \le 1,\ |u| \le \frac{5}{384}M,\ |v| \le \frac{1}{24}M, |z| \le \frac{5}{768}M\Big\}. \tag{4.1.29}$$

It can be verified that if $M = 1.1$ then the conditions of Theorem 4.1.1 are met with $L_0 = 2.0515e-04, L_1 = 0, L_2 = 2, q = 0.0130$. Thus, the problem (4.1.28) has a unique solution satisfying the estimates $|u(x)| \le 0.0143$, $|u'(x)| \le 0.0458$.

Notice that in [66] by the monotone method the author could only prove the convergence of the iterative sequences to extremal solutions of the problem but not the existence and uniqueness of solution.

Using the discrete iterative method (4.1.18)-(4.1.20) on the grid with gridsize $h = 0.01$ and the stopping criterion $\|\Phi_m - \Phi_{m-1}\| \le 10^{-10}$, an approximate solution is found after 7 iterations. Figure 4.1 depicts the graph of this approximate solution.

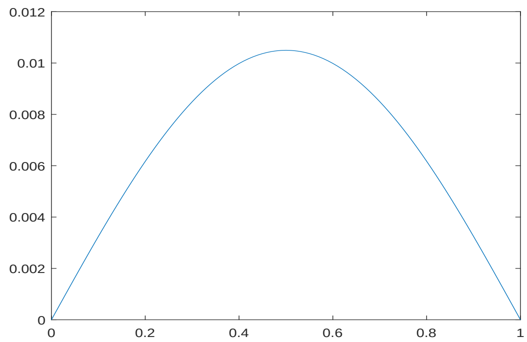

Figure 4.1: Approximate solution in Example 4.1.2.

4.1.5. Conclusion

In this section, we have established the existence and uniqueness of a solution for a fourth order nonlinear integro-differential equation with the Navier boundary conditions and proposed an iterative method at both continuous and discrete levels for finding the solution. The second order of accuracy of the discrete method has been proved. The validity of the obtained theoretical results and the efficiency of the iterative method are demonstrated in some examples where the exact solution is either known or unknown. It must be emphasized that for the example considered in [66], we have established the existence and uniqueness of solution and found it numerically but the author could only show the convergence of the iterative sequences constructed by the monotone method to extremal solutions.

The method used in this section with appropriate modifications can be applied to nonlinear integro-differential equations of any order with other boundary conditions and more complicated nonlinear terms. This is the direction of our research in the future.

4.2. Existence results and an iterative method for functional differential equations

4.2.1. Introduction

In this section, we propose a novel approach to functional differential equations (FDE), which differs from that of Bica et al. [75] for FDEs of even orders which used iterated cubic splines. This approach of ours can be applied to FDEs of any orders with nonlinear terms containing derivatives. For simplicity, we consider the FDE of the form

$$u''' = f(t, u(t), u(\varphi(t))), \quad t \in [0, a] \qquad (4.2.1)$$

subject to the general boundary conditions

$$\begin{aligned} B_1[u] &= \alpha_1 u(0) + \beta_1 u'(0) + \gamma_1 u''(0) = b_1, \\ B_2[u] &= \alpha_2 u(0) + \beta_2 u'(0) + \gamma_2 u''(0) = b_2, \\ B_3[u] &= \alpha_3 u(1) + \beta_3 u'(1) + \gamma_3 u''(1) = b_3, \end{aligned} \tag{4.2.2}$$

or

$$\begin{aligned} B_1[u] &= \alpha_1 u(0) + \beta_1 u'(0) + \gamma_1 u''(0) = b_1, \\ B_2[u] &= \alpha_2 u(1) + \beta_2 u'(1) + \gamma_2 u''(1) = b_2, \\ B_3[u] &= \alpha_3 u(1) + \beta_3 u'(1) + \gamma_3 u''(1) = b_3, \end{aligned} \tag{4.2.3}$$

such that

$$rank \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 & 0 & 0 & 0 \\ \alpha_2 & \beta_2 & \gamma_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix} = 3.$$

Assume $\varphi(t) : [0, a] \to [0, a]$ is a continuous function.

As a development of the unified approach for fully third order nonlinear differential equation

$$u''' = f(t, u(t), u'(t), u''(t))$$

in the previous works [13, 14], here we establish the existence and uniqueness of solution of the problem (4.2.1)-(4.2.2) and propose an iterative method for finding the solution at both continuous and discrete levels. The validity of obtained theoretical results and the efficiency of the proposed numerical method will be demonstrated in some examples.

4.2.2. Existence and uniqueness of a solution

Following the approach in [13, 14] (see also [11, 86]) for investigating the problem (4.2.1)-(4.2.2) we define the nonlinear operator A the space of continuous functions $C[0, a]$ by

$$(A\psi)(t) = f(t, u(t), u(\varphi(t))), \tag{4.2.4}$$

where $u(t)$ is the solution of the problem

$$\begin{aligned} &u'''(t) = \psi(t), \quad 0 < t < a \\ &B_1[u] = b_1, B_2[u] = b_2, B_3[u] = b_3, \end{aligned} \tag{4.2.5}$$

with $B_1[u], B_2[u], B_3[u]$ defined by (4.2.2). It is possible to prove the following proposition.

Proposition 4.2.1. If the function ψ is a fixed point of the operator A, that is, ψ is the solution of the operator equation

$$A\psi = \psi, \tag{4.2.6}$$

where A is defined by (4.2.4)-(4.2.5) then the function $u(t)$ determined from the BVP (4.2.5) is a solution of the BVP (4.2.1)-(4.2.2). Conversely, if the function $u(x)$ is the solution of the BVP (4.2.1)-(4.2.2) then the function

$$\psi(t) = f(t, u(t), u(\varphi(t)))$$

satisfies the operator equation (4.2.6).

Now, let $G(t,s)$ be the Green's function associated to the problem (4.2.5). Then the solution of the problem can be written as

$$u(t) = g(t) + \int_0^a G(t,s)\psi(s)ds, \tag{4.2.7}$$

where $g(t)$ is the polynomial of second degree satisfying the boundary conditions

$$B_1[g] = b_1, B_2[g] = b_2, B_3[g] = b_3. \tag{4.2.8}$$

Denote

$$M_0 = \max_{0\le t\le a} \int_0^a |G(t,s)|ds. \tag{4.2.9}$$

For any number $M > 0$, introduce the domain

$$\mathcal{D}_M = \Big\{(t,u,v) \mid 0 \le t \le a; |u| \le \|g\| + M_0M; |v| \le \|g\| + M_0M\Big\}, \tag{4.2.10}$$

where $\|g\| = \max_{0\le t\le a} |g(t)|$.
Denote by $B[0,M]$ the closed ball with center 0 and radius M in the space $C[0,a]$.

Theorem 4.2.2 (Existence and uniqueness)**.** Suppose that:

(i) $\varphi(t)$ is a continuous function that maps $[0,a]$ to $[0,a]$.

(ii) $f(t,u,v)$ is a continuous function and bounded by M in the domain $\mathcal{D}_M$, that is,

$$|f(t,u,v)| \le M \quad \forall (t,u,v) \in \mathcal{D}_M. \tag{4.2.11}$$

(iii) $f(t,u,v)$ satisfies the Lipschitz conditions in the variables u, v with the Lipschitz coefficients $L_1, L_2 \ge 0$ in $\mathcal{D}_M$, that is,

$$\begin{aligned} |f(t,u_2,v_2) - f(t,u_1,v_1)| \le L_1|u_2 - u_1| + L_2|v_2 - v_1| \\ \forall (t,u_i,v_i) \in \mathcal{D}_M \ (i = 1,2). \end{aligned} \tag{4.2.12}$$

(iv)

$$q := (L_1 + L_2)M_0 < 1. \tag{4.2.13}$$

Then the problem (4.2.1)-(4.2.2) has a unique solution $u(t) \in C^3[0,a]$ satisfying the estimate

$$|u(t)| \le \|g\| + M_0M \quad \forall t \in [0,a]. \tag{4.2.14}$$

Proof. First we prove that the operator A maps $B[0,M] \to B[0,M]$. Indeed, for any $\psi \in B[0,M]$ we have $\|\psi\| \le M$. Let $u(t)$ be the solution of the problem (4.2.5). From (4.2.7) we have

$$|u(t)| \le \|g\| + M_0M \quad \forall t \in [0,a]. \tag{4.2.15}$$

Because $0 \le \varphi(t) \le a$ we obtain

$$|u(\varphi(t)| \le \|g\| + M_0M \quad \forall t \in [0,a].$$

Thus, if $t \in [0,a]$ then $(t, u(t), u(\varphi(t))) \in \mathcal{D}_M$. The condition (4.2.11) follows that $|f(t,u(t),u(\varphi(t)))| \le M \quad \forall t \in [0,a]$. Due to (4.2.4) we obtain $|(A\psi)(t)| \le M \quad \forall t \in [0,a]$, which means $\|A\psi\| \le M$ or $A\psi \in B[0,M]$.

Next, we show that A is a contraction mappping in $B[0, M]$. Let $\psi_1, \psi_2 \in B[0, M]$ and $u_1(t), u_2(t)$ be the solutions of the problem (4.2.5), respectively. Due to the condition (4.2.12) we have

$$|A\psi_2 - A\psi_1| \le L_1|u_2(t) - u_1(t)| + L_2|u_2(\varphi(t)) - u_1(\varphi(t))|. \tag{4.2.16}$$

Using the representations

$$u_i(t) = g(t) + \int_0^a G(t, s)\psi_i(s)ds, \quad (i = 1, 2)$$

and (4.2.9) we obtain

$$|u_2(t) - u_1(t)| \le M_0\|\psi_2 - \psi_1\|,$$
$$|u_2(\varphi(t)) - u_1(\varphi(t))| \le M_0\|\psi_2 - \psi_1\|.$$

Combining the above estimates and (4.2.16), due to the condition (4.2.13) it follows that

$$\|A\psi_2 - A\psi_1\| \le q\|\psi_2 - \psi_1\|, \quad q < 1.$$

Therefore, A is a contraction in $B[0, M]$.

Hence, the operator equation (4.2.6) has a unique solution $\psi \in B[0, M]$. By Proposition 4.2.1 the solution of the problem (4.2.5) for this right-hand side $\psi(t)$ is the solution of the original problem (4.2.1)-(4.2.2). □

Remark 4.2.1. Theorem 4.2.2 still holds if replacing the third order equation (4.2.1) by the higher order equation (0.0.3). Moreover, the assumptions of this theorem are weaker than the assumptions (i)-(iii) in [75, page 131] since in our theorem the Lipschitz conditions should be satisfied only in a bounded domain $\mathcal{D}_M$ instead of the unbounded one $[a, b] \times \mathbb{R} \times \mathbb{R}$ as in [75] and there always is $(L_1 + L_2)M_0 \le (L_1 + L_2)(b - a)M_G$ since $M_0 \le (b - a)M_G$.

4.2.3. Solution method and its convergence

Consider the following iterative method:

1. Given $\psi_0 \in B[0, M]$, say,
$$\psi_0(t) = f(t, 0, 0). \tag{4.2.17}$$

2. Knowing $\psi_k(t)$ $(k = 0, 1, ...)$ compute
$$\begin{aligned} u_k(t) &= g(t) + \int_0^a G(t, s)\psi_k(s)ds, \\ v_k(t) &= g(\varphi(t)) + \int_0^a G(\varphi(t), s)\psi_k(s)ds. \end{aligned} \tag{4.2.18}$$

3. Compute the new approximation
$$\psi_{k+1}(t) = f(t, u_k(t), v_k(t)). \tag{4.2.19}$$

Denote

$$p_k = \frac{q^k}{1 - q}, \ d = \|\psi_1 - \psi_0\|. \tag{4.2.20}$$

Theorem 4.2.3 (Convergence)**.** If the conditions of Theorem 4.2.2 are satisfied then the above iterative method converges and there holds the estimate

$$\|u_k - u\| \leq M_0 p_k d,$$

where u is the exact solution of the problem (4.2.1)-(4.2.2) and M_0 is given by (4.2.9).

This theorem directly follows from the convergence of the successive approximation method for finding fixed point of the operator A, the representations (4.2.7) and the first equation in (4.2.18).

For the numerical realization of this iterative method, we design the corresponding discrete iterative method. To do this, we construct the uniform grid $\bar{\omega}_h = \{t_i = ih,\ h = a/N, i = 0, 1, ..., N\}$ on the interval $[0, a]$ and denote by $\Phi_k(t), U_k(t), V_k(t)$ the grid functions defined on this grid and approximating the functions $\psi_k(t), u_k(t), v_k(t)$, respectively.

The discrete iterative method is as follows:

1. Given
$$\Psi_0(t_i) = f(t_i, 0, 0),\ \ i = 0, ..., N. \tag{4.2.21}$$

2. Knowing $\Psi_k(t_i),\ k = 0, 1, ...;\ i = 0, ..., N$, compute approximately the definite integrals (4.2.18) by the trapezoidal rule
$$\begin{aligned} U_k(t_i) &= g(t_i) + \sum_{j=0}^{N} h\rho_j G(t_i, t_j)\Psi_k(t_j), \\ V_k(t_i) &= g(\xi_i) + \sum_{j=0}^{N} h\rho_j G(\xi_i, t_j)\Psi_k(t_j), i = 0, ..., N, \end{aligned} \tag{4.2.22}$$
where ρ_j are the weights of the trapezoidal rule
$$\rho_j = \begin{cases} 1/2,\ j = 0, N \\ 1,\ j = 1, 2, ..., N-1 \end{cases}$$
and $\xi_i = \varphi(t_i)$.

3. Compute the new approximation
$$\Psi_{k+1}(t_i) = f(t_i, U_k(t_i), V_k(t_i)). \tag{4.2.23}$$

For investigating the convergence of this discrete iterative method, the following auxiliary results are needed.

Proposition 4.2.4. If the function $f(t, u, v)$ has all partial derivatives continuous up to second order and the function $\varphi(t)$ also has continuous derivatives up to second order then the functions $\psi_k(t), u_k(t), v_k(t)$ constructed by the iterative method (4.2.17)-(4.2.19) also have continuous derivatives up to second order.

This proposition is obvious.

Proposition 4.2.5. For any function $\psi(t) \in C^2[0,a]$ there hold the estimates

$$\int_0^a G(t_i,s)\psi(s)ds = \sum_{j=0}^{N} h\rho_j G(t_i,s_j)\psi(s_j) + O(h^2), \tag{4.2.24}$$

$$\int_0^a G(\xi_i,s)\psi(s)ds = \sum_{j=0}^{N} h\rho_j G(\xi_i,s_j)\psi(s_j) + O(h^2), \tag{4.2.25}$$

where in order to avoid possible confusion we denote $s_j = t_j$.

Proof. The validity of (4.2.24) is guaranteed by [14, Proposition 3]. Here we notice that (4.2.24) is not automatically deduced from the estimate for the composite trapezoidal rule because the function $\frac{\partial^2 G(t_i,s)}{\partial s^2}$ has discontinuity at $s = t_i$.

Now we prove the estimate (4.2.25). Since $0 \le \xi_i = \varphi(t_i) \le a$, there are possible two cases:
Case 1: ξ_i coincides with one node s_j of the grid $\bar{\omega}_h$, that is, there exists $s_j \in \bar{\omega}_h$ such that $\xi_i = s_j$. Because the Green's function $G(t,s)$ as a function of s is continuous at $s = \xi_i$ and is a polynomial of s in the intervals $[0,\xi_i]$ and $[\xi_i,a]$, therefore

$$\begin{aligned}
&\int_0^a G(\xi_i,s)\psi(s)ds = \int_0^{\xi_i} G(\xi_i,s)\psi(s)ds + \int_{\xi_i}^a G(\xi_i,s)\psi(s)ds \\
&= h\Big(\frac{1}{2}G(\xi_i,s_0)\psi(s_0) + \sum_{m=1}^{j-1} G(\xi_i,s_m)\psi(s_m) + \frac{1}{2}G(\xi_i,s_j)\psi(s_j)\Big) + O(h^2) \\
&+ h\Big(\frac{1}{2}G(\xi_i,s_j)\psi(s_j) + \sum_{m=j+1}^{N-1} G(\xi_i,s_m)\psi(s_m) + \frac{1}{2}G(\xi_i,s_N)\psi(s_N)\Big) + O(h^2) \\
&= \sum_{j=0}^{N} h\rho_j G(t_i,s_j)\psi(s_j) + O(h^2).
\end{aligned}$$

Hence, (4.2.25) is proved for Case 1.
Case 2: ξ_i lies between s_l and s_{l+1}, that is, $s_l < \xi_i < s_{l+1}$ for some $l = \overline{0, N-1}$. In this case, we represent

$$\int_0^a G(\xi_i,s)\psi(s)ds = \int_0^{s_l} F(s)ds + \int_{s_l}^{\xi_i} F(s)ds + \int_{\xi_i}^{s_{l+1}} F(s)ds + \int_{s_{l+1}}^a F(s)ds \tag{4.2.26}$$

where $F(s) = G(\xi_i,s)\psi(s)$. Note that $F(s) \in C^2$ in $[s_l,\xi_i]$ and $[\xi_i,s_{l+1}]$. Applying the composite trapezoidal rule to the first and the last integrals in the right-hand side of (4.2.26) we obtain

$$\begin{aligned}
T_1 :&= \int_0^{s_l} F(s)ds + \int_{s_{l+1}}^a F(s)ds \\
&= \sum_{j=0}^{l} \rho_j^{(l-)} F(s_j) + \sum_{j=l+1}^{N} \rho_j^{(l+)} F(s_j) + O(h^2),
\end{aligned} \tag{4.2.27}$$

where

$$\rho_j^{(l-)} = \begin{cases} \frac{1}{2}, & j = 0, l \\ 1, & 1 < j < l \end{cases}, \quad \rho_j^{(l+)} = \begin{cases} \frac{1}{2}, & j = l+1, N \\ 1, & l+1 < j < N. \end{cases}$$

To compute the second and the third integrals, we apply the trapezoidal rule

$$\begin{aligned} T_2 := & \int_{s_l}^{\xi_i} F(s)ds + \int_{\xi_i}^{s_{l+1}} F(s)ds \\ = & \frac{1}{2}\big[(F(s_l) + F(\xi_i))(\xi_i - s_l) + (F(\xi_i) + F(s_{l+1}))(s_{l+1} - \xi_i)\big] + O(h^2). \end{aligned} \tag{4.2.28}$$

Using the points s_l and s_{l+1} for linearly interpolating $F(s)$ in the point ξ_i we obtain

$$F(\xi_i) = F(s_l)\frac{\xi_i - s_{l+1}}{s_l - s_{l+1}} + F(s_{l+1})\frac{\xi_i - s_l}{s_{l+1} - s_l} + O(h^2).$$

It follows that

$$F(\xi_i)(s_{l+1} - s_l) = F(s_l)(s_{l+1} - \xi_i) + F(s_{l+1})(\xi_i - s_l) + O(h^3). \tag{4.2.29}$$

T_2 can be transformed to

$$T_2 = \frac{1}{2}\big[F(s_l)(\xi_i - s_l) + F(s_{l+1})(s_{l+1} - \xi_i)\big] + F(\xi_i)(s_{l+1} - s_l) + O(h^2)$$

Coupled this with (4.2.29) we have

$$T_2 = \frac{1}{2}h(F(s_l) + F(s_{l+1})) + O(h^3).$$

Combining this estimate with (4.2.27) and (4.2.26) we obtain

$$\int_0^a G(\xi_i, s)\psi(s)ds = \sum_{j=0}^{N} h\rho_j G(\xi_i, s_j)\psi(s_j) + O(h^2).$$

Therefore, (4.2.25) is proved for Case 2.

The proof of Proposition 4.2.5 is completed. □

Remark 4.2.2. If in Proposition 4.2.5 we replace $G(t_i, s)$ and $G(\xi_i, s)$ by $|G(t_i, s)|$ and $|G(\xi_i, s)|$, respectively then we also have the similar estimates

$$\int_0^a |G(t_i, s)|\psi(s)ds = \sum_{j=0}^{N} h\rho_j |G(t_i, s_j)|\psi(s_j) + O(h^2), \tag{4.2.30}$$

$$\int_0^a |G(\xi_i, s)|\psi(s)ds = \sum_{j=0}^{N} h\rho_j |G(\xi_i, s_j)|\psi(s_j) + O(h^2), \tag{4.2.31}$$

Proposition 4.2.6. If the conditions of Theorem 4.2.2 are satisfied then there hold the following estimates

$$\|\Psi_k - \psi_k\|_{\bar{\omega_h}} = O(h^2), \tag{4.2.32}$$

$$\|U_k - u_k\|_{\bar{\omega_h}} = O(h^2), \tag{4.2.33}$$

where $\|.\|_{\bar{\omega_h}}$ is the max-norm of grid function defined on the grid $\bar{\omega_h}$.

Proof. The proof is done by induction. For $k=0$ we have at once $\|\Psi_0 - \psi_0\|_{\bar{\omega_h}}$ because $\Psi_0(t_i) = f(t_i, 0, 0)$ and also $\psi_0(t_i) = f(t_i, 0, 0)$, $i = \overline{0, N}$. Next, combining (4.2.18) and Proposition 4.2.5 we obtain

$$\begin{aligned} u_0(t_i) &= g(t_i) + \int_0^a G(t_i, s)\psi_0(s)ds \\ &= g(t_i) + \sum_{j=0}^{N} h\rho_j G(t_i, s_j)\psi_0(s_j) + O(h^2). \end{aligned}$$

By (4.2.22) we also have

$$U_0(t_i) = g(t_i) + \sum_{j=0}^{N} h\rho_j G(t_i, s_j)\Psi_0(s_j).$$

Hence,

$$|U_0(t_i) - u_0(t_i)| = O(h^2).$$

This means $\|U_0 - u_0\|_{\bar{\omega_h}} = O(h^2)$. Therefore, the estimates (4.2.32) and (4.2.33) hold for $k = 0$.

Now, assume that these estimates hold for $k \geq 0$. We must prove that they also hold for $k+1$. Indeed, from (4.2.19), (4.2.23) and the Lipschitz conditions for the function $f(t, u, v)$ we obtain

$$\begin{aligned} |\Psi_{k+1}(t_i) - \psi_{k+1}(t_i)| &= |f(t_i, U_k(t_i), V_k(t_i)) - f(t_i, u_k(t_i), v_k(t_i))| \\ &\leq L_1|U_k(t_i) - u_k(t_i)| + L_2|V_k(t_i) - v_k(t_i)|. \end{aligned} \tag{4.2.34}$$

From Proposition 4.2.5 we have

$$\begin{aligned} v_k(t_i) &= g(\varphi(t_i)) + \int_0^a G(\varphi(t_i), s)\psi_k(s)ds \\ &= g(\xi_i) + \sum_{j=0}^{N} h\rho_j G(\xi_i, s_j)\psi_k(s_j) + O(h^2). \end{aligned}$$

Due to (4.2.22) we obtain

$$\begin{aligned} |V_k(t_i) - v_k(t_i)| &= |\sum_{j=0}^{N} h\rho_j G(\xi_i, s_j)(\Psi_k(s_j) - \psi_k(s_j))| + O(h^2) \\ &\leq \sum_{j=0}^{N} h\rho_j |G(\xi_i, s_j)| \|\Psi_k - \psi_k\|_{\bar{\omega_h}} + O(h^2). \end{aligned} \tag{4.2.35}$$

For $\psi(s) = 1$, the estimate (4.2.31) is equivalent to

$$\int_0^a |G(\xi_i, s)|ds = \sum_{j=0}^{N} h\rho_j |G(\xi_i, s_j)| + O(h^2).$$

Therefore

$$\begin{aligned} \sum_{j=0}^{N} h\rho_j |G(\xi_i, s_j)| &= \int_0^a |G(\xi_i, s)|ds + O(h^2) \\ &\leq \max_{0 \leq t \leq a} \int_0^1 |G(t, s)|ds + O(h^2) = M_0 + O(h^2). \end{aligned}$$

Coupling this with (4.2.35) leads to

$$|V_k(t_i) - v_k(t_i)| \le \|\Psi_k - \psi_k\|_{\bar{\omega_h}} + O(h^2).$$

Because of the induction hypothesis, we have

$$\|V_k - v_k\|_{\bar{\omega_h}} = O(h^2). \tag{4.2.36}$$

Combining the induction hypothesis $\|U_k - u_k\|_{\bar{\omega_h}} = O(h^2)$ and (4.2.36), from (4.2.34) it follows that

$$\|\Psi_{k+1} - \psi_{k+1}\|_{\bar{\omega_h}} = O(h^2). \tag{4.2.37}$$

To show that

$$\|U_{k+1} - u_{k+1}\|_{\bar{\omega_h}} = O(h^2), \tag{4.2.38}$$

we take into account that

$$|U_{k+1}(t_i) - u_{k+1}(t_i)| \le \sum_{j=0}^{N} h\rho_j |G(t_i, s_j)||\Psi_{k+1}(s_j) - \psi_{k+1}(s_j)| + O(h^2).$$

Similarly as above, we shall obtain

$$|U_{k+1}(t_i) - u_{k+1}(t_i)| = O(h^2),$$

which implies (4.2.38).
Therefore, the proposition is proved. □

Coupling Proposition 4.2.6 with Theorem 4.2.3 leads to the following result.

Theorem 4.2.7. Under the conditions of Theorem 4.2.2 for the approximate solution of the problem (4.2.1)-(4.2.2) obtained by the discrete iterative method (4.2.21)-(4.2.23) there holds the estimate

$$\|U_k - u\|_{\bar{\omega_h}} \le M_0 p_k d + O(h^2),$$

where p_k and d are defined by (4.2.20).

Remark 4.2.3. If applying Bica's method, which uses a cubic spline interpolation procedure at each iteration, to the third order problem then $O(h^4)$ convergence cannot be ensured since Corollary 1 in [105, p. 50] is not applicable because of the properties of the Green's function for the third order equation.

Remark 4.2.4. For the discrete iterative method (4.2.17) -(4.2.19), $O(h^2)$ convergence is obtained. Naturally, one may think of applying Gaussian quadratures to compute the integrals in (4.2.18). However, this cannot be done due to the nodes of Gaussian quadratures not coinciding with the grid nodes where the computation of the solution is done.

Remark 4.2.5. The results in Section 4.2.2 and 4.2.3 are obtained for the nonlinear third order FDE with nonlinear term $f = f(t, u(t), u(\varphi(t)))$. In similar fashion, we can obtain similar results of existence and convergence of the iterative method at continuous level for the general case

$$f = f(t, u(t), u(\varphi(t)), u'(\varphi_1(t)), u''(\varphi_2(t))),$$

where the functions $\varphi(t), \varphi_1(t), \varphi_2(t)$ are continuous from $[0, a]$ to $[0, a]$. However, in order to numerically realize the iterative method, we must notice that the second derivative $\frac{\partial^2 G(t,s)}{\partial t^2}$ of the Green's function has discontinuity at $s = t$. In this case, to compute integrals containing $\frac{\partial G(t,s)}{\partial t}$ and $\frac{\partial^2 G(t,s)}{\partial t^2}$ we must use the formulas constructed in the previous work [14].

Remark 4.2.6. The iterative method developed in this section for the third order nonlinear FDE can be applied to nonlinear FDE of any order.

4.2.4. Examples

In the following examples, the iterative method (4.2.21)-(4.2.23) is performed until $\|\Psi_k - \Psi_{k-1}\|_{\bar{\omega_h}} \leq 10^{-10}$. In the tables of results for the convergence of the iterative method $Error = \|U_K - u\|_{\bar{\omega_h}}$, K is the number of iterations performed.

Example 4.2.1. Consider the problem

$$\begin{aligned} &u'''(t) = e^t - \frac{1}{4}u(t) + \frac{1}{4}u^2(\frac{t}{2}), \quad 0 < t < 1, \\ &u(0) = 1,\ u'(0) = 1,\ u'(1) = e. \end{aligned} \tag{4.2.39}$$

For this problem, the exact solution is $u(t) = e^t$ and the corresponding Green's function is

$$G(t,s) = \begin{cases} \frac{s}{2}(t^2 - 2t + s), & 0 \leq s \leq t \leq 1, \\ \frac{t^2}{2}(s-1), & 0 \leq t \leq s \leq 1. \end{cases}$$

It can be verified that

$$M_0 = \max_{0 \leq t \leq a} \int_0^1 |G(t,s)| ds = \frac{1}{2}.$$

The polynomial of second degree satisfying the boundary conditions of the problem is

$$g(t) = 1 + t + \frac{e-1}{2}t^2.$$

Thus, $\|g\| = 2 + \dfrac{e-1}{2} = 2.7183$. In this example $f(t,u,v) = e^t - \frac{1}{4}u + \frac{1}{4}v^2$. It is possible to show that if $M = 6.5$ then $|f(t,u,v)| \leq M$ in the domain $\mathcal{D}_M$ defined by (4.2.10). Besides, in $\mathcal{D}_M$, $f(t,u,v)$ satisfies the Lipschitz conditions in u and v with Lipschitz coefficients $L_1 = \frac{1}{4}, L_2 = 1.7004$. Hence, $q := (L_1 + L_2)M_0 = 0.16$. Therefore, all the conditions of Theorem 4.2.2 are met, implying that (4.2.39) has a unique solution. This is the above exact solution.

The results of convergence of the iterative method (4.2.21)-(4.2.23) are reported in Table 4.3. These results support the conclusion that the iterative

Table 4.3: The convergence in Example 4.2.1.

N	h^2	K	$Error$
50	4.0000e-04	3	6.1899e-05
100	1.0000e-04	3	1.5475e-05
150	4.4444e-05	3	6.877 -06
200	2.5000e-05	3	3.8688e-06
300	1.1111e-05	3	1.7195e-06
400	6.2500e-06	3	9.6721e-07
500	4.0000e-06	3	6.1901e-07
800	1.5625e-06	3	2.4180e-07
1000	1.0000e-06	3	1.5475e-07

method is of $O(h^2)$ accuracy.

Remark 4.2.7. Theorem 4.2.7 gives sufficient conditions for convergence of the iterative method (4.2.21)-(4.2.23). In the cases when these conditions are not met, the iterative can still converge to some solution. For example, if $f(t,u,v) = e^t + u^2 + v^2 + 1$ with the same boundary conditions as in (4.2.39) the iterative method converges after 15 iterations. And if $f(t,u,v) = e^{2t} - u^3 + v^2 + 5$, the iterative process reaches $TOL = 10^{-10}$ after 16 iterations. Notice that the number of iterations does not depend on the grid size as in Example 4.2.1.

Example 4.2.2. Consider the problem

$$\begin{aligned} &u'''(t) = \sin(u^2(t)) + \cos(u^2(t^2)), \quad 0 < t < 1, \\ &u(0) = 0,\ u'(0) = \pi,\ u'(1) = -\pi. \end{aligned} \tag{4.2.40}$$

For this problem $f(t,u,v) = \sin(u^2) + \cos(v^2)$ and $\varphi(t) = t^2$. It is possible to show that all the conditions of Theorem 4.2.3 are met, hence the problem has a unique solution. Also, by Theorem 4.2.7 the iterative method (4.2.21)-(4.2.23) converges. The computation results show that the iterative method stops after 8 iterations for any number of grid points. The graph of the approximate solution is depicted in Figure 4.2.

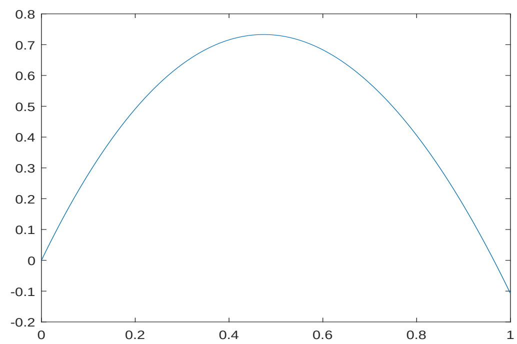

Figure 4.2: Approximate solution in Example 4.2.2.

Example 4.2.3 (Example 5 in [76]). Consider the problem

$$\begin{aligned} &u'''(t) = -1 + 2u^2(t/2), \quad 0 < t < \pi, \\ &u(0) = 0,\ u'(0) = 1,\ u(\pi) = 0. \end{aligned} \tag{4.2.41}$$

For this problem, the exact solution is $u(t) = \sin(t)$, and the corresponding Green's function is

$$G(t,s) = \begin{cases} -\dfrac{t^2(\pi - s)^2}{2\pi^2} + \dfrac{(t-s)^2}{2}, & 0 \le s \le t \le \pi, \\ -\dfrac{t^2(\pi - s)^2}{2\pi^2}, & 0 \le t \le s \le \pi \end{cases}$$

and $f(t,u,v) = -1 + 2v^2$.

The results of convergence of the iterative method (4.2.21)-(4.2.23) are reported in Table 4.4. These results also confirm the accuracy $O(h^2)$ of the iterative method.

Table 4.4: The convergence in Example 4.2.3.

N	h^2	K	$Error$
50	4.0000e-04	25	1.4455e-04
100	1.0000e-04	25	3.6142e-05
150	4.4444e-05	25	1.6063e-05
200	2.5000e-05	25	9.0345e-06
300	1.1111e-05	25	4.0155e-06
400	6.2500e-06	25	2.2587e-06
500	4.0000e-06	25	1.4456e-06
800	1.5625e-06	25	5.6467e-07
1000	1.0000e-06	25	3.6139e-07

4.2.5. Conclusion

In this section, we have proposed a unified approach to nonlinear functional differential equations via boundary value problems for nonlinear third order functional differential equations as a particular case. The existence and uniqueness of solution have been established and the convergence of second order of the discrete iterative method for finding the solution has been proved. The validity of the theoretical results and the efficiency of the numerical method have been demonstrated in various examples.

Our approach is applicable to boundary value problems for nonlinear functional differential equations of any order associated with general linear boundary conditions. It is also applicable to integro-differential equations.

4.3. Chapter conclusion

This chapter concerns integro-differential and functional differential equations. As in the previous chapters, we reduce the boundary value problems for these types of differential equations to suitable operator equations. Due to this we have established the existence and uniqueness of solutions and proved the convergence of the iterative method as the successive approximation method for the fixed point of the corresponding operator equation. Moreover, we have constructed the discrete iterative method which gives the approximate numerical solutions of the two above mentioned problems with second-order accuracy. Many numerical examples supported the theoretical results.

We would like to emphasize that our method for boundary value problems of functional differential equations is much simpler than the method of iterated cubic splines of Bica et al. [75] in the construction as well as in the proof of the convergence. By investigating the reduced implicit operator equation we have simultaneously established the existence and uniqueness of solution and the convergence of the iterative method. This methodology can be applied to boundary value problems for functional differential equations of any order.

The results of this chapter were published in two papers [AL4] and [AL6] in SCIE journals.

General Conclusions

In this thesis, we have successfully studied the existence, uniqueness of solutions and the iterative methods for solving some nonlinear boundary value problems for some high order differential equations including integro-differential and functional differential equations. The main achievements of the thesis include:

1. The establishment of the existence, uniqueness of solutions and positive solutions for third order nonlinear BVPs and the construction of numerical methods for finding the solutions; The proposal of discrete iterative methods of second and third order accuracy for solving third order nonlinear differential equations.

2. The establishment of the existence, uniqueness of solutions and construction of iterative methods for finding the solutions for nonlinear third and fourth order differential equations with integral boundary conditions.

3. The establishment of the existence, uniqueness of solutions and construction of numerical methods for finding the solutions of nonlinear integro-differential and functional differential equations.

The validity and applicability of the theoretical results and the effectiveness of the constructed iterative methods have been confirmed by many experimental examples.

The methodology throughout the thesis has been shown to be superior to those of many other authors due to its simplicity and coherence and can be applied to a wide range of boundary value problems for differential equations.

A weakness of this methodology is that it is only applicable to problems for differential equations with non-singular right-hand sides. Therefore, the future goals of the thesis are:

1. The further development of the above results for the case of singular right-hand sides and the case of unbounded domains.

2. The construction of iterative methods of higher order accuracy.

3. The study of the problems with nonlinear boundary conditions.

List of works of the author related to the thesis

[AL1] Q. A Dang, Q. L. Dang, A unified approach to fully third order nonlinear boundary value problems, J. Nonlinear Funct. Anal. 2020 (2020), Article ID 9 (Scopus, Q3).

[AL2] Q. A Dang, Q. L. Dang, Simple numerical methods of second- and third-order convergence for solving a fully third-order nonlinear boundary value problem, Numerical Algorithms, (2021) 87:1479-1499 (SCIE, Q1).

[AL3] Q. A Dang, Q. L. Dang, Existence results and iterative method for fully third order nonlinear integral boundary value problems, Applications of Mathematics, 66 (2021) 657-672 (SCIE, Q3).

[AL4] Q. A Dang, Q. L. Dang, A unified approach to study the existence and numerical solution of functional differential equation, Applied Numerical Mathematics 170 (2021) 208–218 (SCI, Q1).

[AL5] Q. A Dang, Q. L. Dang, Existence results and iterative method for a fully fourth-order nonlinear integral boundary value problem, Numerical Algorithms, 85 (2020) 887-907 (SCIE, Q1).

[AL6] Q. L. Dang, Q. A Dang, Existence results and numerical method for solving a fourth-order nonlinear integro-differential equation, Numerical Algorithms, 90, 563-576 (2022) (SCIE, Q1).

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Appendix: MATLAB codes for some examples

I. Numerical computation in Example 2.2.1:

```
clear all
for k=3:10
        n=2^k   % number of grid points
        c1=0; c2=-1; c3=sin(1); % u(0)=c1, u'(0)=c2, u'(1)=
           c3
        % Compute solution of the BVP using modified
           Simpson's rule:
        [iter,t,u,y,z]=bvp3_simp(n,10^(-10),c1,c2,c3);
        % Compute solution of the BVP using trapezoidal
           rule:
        % [iter,t,u,y,z]=bvp3_trap(n,10^(-10),c1,c2,c3);
        for i=1:n+1
                err(i)=u(i)-ucx(t(i));
        end
        if k==3
                error=chuan(err)
        else
                error2=chuan(err);
                order=log2(error/error2)
                error=error2
        end
end

function [iter,t,u,y,z]=bvp3_trap(n,TOL,c1,c2,c3)
% function for computing solution of the BVP using
   trapezoidal rule
h=1/n;
iter=0;
for i=1:n+1
    t(i)=(i-1)*h;
    s(i)=(i-1)*h;
    P_2(i)=(c3-c2)/2*t(i)^2 + c2*t(i) +c1;
    dP_2(i)=(c3-c2)*t(i)+c2;
    ddP_2(i)=(c3-c2);
    p0(i)=f(t(i),P_2(i),dP_2(i),ddP_2(i));
    ud(i)=ucx(t(i));
end
ssp=1;
while ssp > TOL
    iter=iter+1;
```

```
    for i=1:n+1
        for j=1:n+1
            Gp(j)= G(t(i),s(j))*p0(j);
            G1p(j)= G1(t(i),s(j))*p0(j);
            G2p(j)= G2(t(i),s(j))*p0(j);
        end
        v(i)=trap(0,1,n,Gp);
        y(i)=trap(0,1,n,G1p);
        z(i)=trap(0,1,n,G2p);
        p(i)=f(t(i),v(i)+P_2(i),y(i)+dP_2(i),z(i)+ddP_2(i))
           ;
    end
    ssp=chuan(p-p0);
    p0=p;
end
for i=1:n+1
    u(i)=v(i)+P_2(i);
end
end

function trap=trap(a,b,n,f)
% numerical integration by trapezoidal rule over [a,b] with
    n subintervals
h=(b-a)/n;
S=0;
for j=2:n
    S=S+f(j);
end
S=S+(f(1)+f(n+1))/2;
trap=S*h;
end

function [iter,t,u,y,z]=bvp3_simp(n,TOL,c1,c2,c3)
% function for computing solution of the BVP using modified
    Simpson's rule
h=1/n;
iter=0;
for i=1:n+1
    t(i)=(i-1)*h;
    s(i)=(i-1)*h;
    P_2(i)=(c3-c2)/2*t(i)^2 + c2*t(i) +c1;
    dP_2(i)=(c3-c2)*t(i)+c2;
    ddP_2(i)=(c3-c2);
    p0(i)=f(t(i),P_2(i),dP_2(i),ddP_2(i));
    ud(i)=ucx(t(i));
end
ssp=1;
while ssp > TOL
  iter=iter+1;
  for i=1:n+1
```

```
    if rem(i-1,2)==0
      for j=1:n+1
        Gp(j)= G(t(i),s(j))*p0(j);
        G1p(j)= G1(t(i),s(j))*p0(j);
        G2p(j)= G2(t(i),s(j))*p0(j);
      end
      v(i)=simpson(0,1,n,Gp);
      y(i)=simpson(0,1,n,G1p);
      z(i)=simpson(0,1,n,G2p);
      p(i)=f(t(i),v(i)+P_2(i),y(i)+dP_2(i),z(i)+ddP_2(i));
    else
      for j=1:n+1
        Gp(j)= G(t(i),s(j))*p0(j);
        G1p(j)= G1(t(i),s(j))*p0(j);
        G2p(j)= G2(t(i),s(j))*p0(j);
      end
      v(i)=simpson(0,1,n,Gp)+h*(Gp(i-1)-2*Gp(i)+Gp(i+1))/6;
      y(i)=simpson(0,1,n,G1p)+h*(G1p(i-1)-2*G1p(i)+G1p(i+1)
         )/6;
      z(i)=simpson(0,1,n,G2p)+h*(G2p(i-1)-2*G2p(i)+G2p(i+1)
         )/6;
      p(i)=f(t(i),v(i)+P_2(i),y(i)+dP_2(i),z(i)+ddP_2(i));
    end
  end
  ssp=chuan(p-p0);
  p0=p;
end
for i=1:n+1
  u(i)=v(i)+P_2(i);
end
end

function S=simpson(a,b,n,f)
% numerical integration by Simpson's rule over [a,b] with n
   subintervals
h=(b-a)/n;
S=0;
for k=0:(n/2-1)
    S=S+f(2*k+1)+4*f(2*k+2)+f(2*k+3);
end
S=h/3*S;
end

function ucx=ucx(x) % exact solution
ucx=(x-1)*sin(x);
end

function hvp = f(t,u,y,z)   % right-hand side function f
hvp= t^4*u-u^2+sin(t)^2*(t-1)^2-cos(t)*(t-1)-3*sin(t)-t^4*(
   t-1)*sin(t);
```

```
end

function green = G(t,s) % Green's function
if s <= t
    green= s/2*(t^2-2*t+s);
else
    green= t^2/2*(s-1);
end
end

function green1 = G1(t,s) % first derivative of the Green's
    function
if s <= t
    green1= s*(t-1);
else
    green1= t*(s-1);
end
end

function green2 = G2(t,s) % second derivative of the Green'
   s function with the change at jump point t=s
if s < t
    green2= s;
elseif s==t
    green2= s-1/2;
else
    green2= s-1;
end
end

function chuan=chuan(y)
chuan=norm(y,inf);
end
```

II. Numerical computation in Example 3.2.1:

```
clear all
n=30            % number of grid points
TOL=10^-4;
[iter,t,u,y,v,z]=bvp4_intBC(n,TOL);
K=iter  % number of iterations
h=1/n;
for i=1:n+1
    x=(i-1)*h;
    ud(i)=5/6+x^3-3/4*x^4;      % exact solution
    err(i)=u(i)-ud(i);
end
error=chuan(err)

function [iter,t,u,y,v,z]=bvp4_intBC(n,TOL)
```

```
% function for computing the solution of 4th order BVP with
    integral boundary condition
h=1/n;
iter=0;
for i=1:n+1
    t(i)=(i-1)*h;
    s(i)=(i-1)*h;
    p0(i)=f(t(i),0.1,0.1,0,0);
end
alpha0=0.1;
ssp=1; ssa=1;
while ssp > TOL || ssa > TOL
    iter=iter+1;
    for i=1:n+1
        for j=1:n+1
            Gp(j)= G(t(i),s(j))*p0(j);
            G1p(j)= G1(t(i),s(j))*p0(j);
            G2p(j)= G2(t(i),s(j))*p0(j);
            G3p(j)= G3(t(i),s(j))*p0(j) ;
        end
        u(i)=trap(0,1,n,Gp)+alpha0;
        y(i)=trap(0,1,n,G1p);
        v(i)=trap(0,1,n,G2p);
        z(i)=trap(0,1,n,G3p);
        p(i)=f(t(i),u(i),y(i),v(i),z(i));
        gu(i)=gs(t(i))*u(i);
    end
    alpha=trap(0,1,n,gu);
    ssp=chuan(p-p0);
    ssa=abs(alpha-alpha0);
    p0=p;
    alpha0=alpha;
end
end

function hvp = f(t,u,y,v,z) % right-hand side function f
hvp=-18+u^2/5-(5/6+t^3-3/4 *t^4)^2/5;
end

function g_s=gs(s)  % function g(s)
g_s = 4*s^4;
end

function green = G(t,s) % Green's function
if s <= t
    green= -(1/6)*t^3*(1-s)^2+(t-s)^3/6;
else
     green= -(1-s)^2*t^3/6;
end
end
```

```
function green1 = G1(t,s)
if s <= t
    green1= (-t^2*(1-s)^2+(t-s)^2)/2;
else
    green1= -t^2*(1-s)^2/2;
end
end

function green2 = G2(t,s)
if s <= t
    green2= -t*(1-s)^2+(t-s);
else
    green2=-t*(1-s)^2;
end
end

function green3 = G3(t,s)
if s < t
    green3= -(1-s)^2+1;
elseif s>t
    green3=-(1-s)^2;
else
    green3=-(1-s)^2+1/2;
end
end

function trap=trap(a,b,n,f)
h=(b-a)/n;
S=0;
for j=2:n
    S=S+f(j);
end
S=S+(f(1)+f(n+1))/2;
trap=S*h;
end

function chuan=chuan(y)
chuan=norm(y,inf);
end
```

III. Numerical computation in Example 4.1.1:

```
clear all
for n = [50 100 150 200 300 400 500 800 1000]
    n   % number of grid points
    h=1/n;
    h_square = h^2
    % [iter,x,u]=bvp4_2(n,h^2);
    [iter,x,u]=ide4(n,10^(-10));
```

```
    m=iter  % number of iterations
    for i=1:n+1
        err(i)=u(i)-ucx(x(i));
    end
    error=chuan(err)
end

function [iter,x,u]=ide4(n,TOL)
% function for computing the solution of 4th order IDE
h=1/n;
iter=0;
for i=1:n+1
    t(i)=(i-1)*h;
    x(i)=(i-1)*h;
    p0(i)=f(x(i),0,0,0);
    ud(i)=ucx(x(i));
end
for i=1:n+1
    for j=1:n+1
        k(i,j)=ker(x(i),t(j));
    end
end
ssp=1;
ssu=1;
while and(ssp > TOL,iter <1000)
    iter=iter+1;
    for i=1:n+1
        for j=1:n+1
            Gp(j)= G(x(i),t(j))*p0(j);
            G1p(j)= G1(x(i),t(j))*p0(j);
        end
        u(i)=trap(0,1,n,Gp);
        y(i)=trap(0,1,n,G1p);
        err(i)=u(i)-ud(i);
    end
    ssu=chuan(err);
    for i=1:n+1
        for j=1:n+1
            ku(j)= k(i,j)*u(j);
        end
        z(i)=trap(0,1,n,ku);
    end
    for i=1:n+1
            p(i)=f(x(i),u(i),y(i),z(i));
    end
    ssp=chuan(p-p0);
    p0=p;
end
for i=1:n+1
err(i)=u(i)-ud(i);
```

```
end
ssu=chuan(err);
end

function ucx=ucx(x)      % exact solution
ucx=sin(pi*x);
end

function ker=ker(x,t)    % kernel function
ker = exp(x)* sin(pi*t);
end

function hvp=f(x,u,y,z) % right-hand side function
hvp=u^2*z + 2*u*y/2- exp(x)/2*(sin(pi*x))^2 + pi^4*sin(pi*x
   ) - pi*sin(2*pi*x)/2;
end

function green = G(x,s) % Green's function
if x <= s
    green = x*(s-1)*(s^2-2*s+x^2)/6;
else
    green = s*(x-1)*(x^2-2*x+s^2)/6;
end
end

function green1 = G1(x,s)
if x <= s
        green1 = (s-1)*(3*x^2-2*s+s^2)/6;
else
        green1 = s*(3*x^2-6*x+s^2+2)/6;
end
end

function trap=trap(a,b,n,f)
h=(b-a)/n;
S=0;
for j=2:n
    S=S+f(j);
end
S=S+(f(1)+f(n+1))/2;
trap=S*h;
end

function chuan=chuan(y)
chuan=norm(y,inf);
end
```

IV. Numerical computation in Example 4.2.1:

```
clear all
```

```
for n = [50 100 150 200 300 400 500 800 1000]
    n   % number of grid points
    c1=1; c2=1; c3=exp(1);
    [iter,t,u,v]=FDE3(n,10^(-10),c1,c2,c3);
    h=1/n;
    h_square=h*h
    iter    % number of iterations
    for i=1:n+1
        err(i)=u(i)-ucx(t(i));
    end
    error=chuan(err)
end

function [iter,t,u,v]=FDE3(n,TOL,c1,c2,c3)
% function for computing the solution of 3rd order FDE
h=1/n;
iter=0;
for i=1:n+1
    t(i)=(i-1)*h;
    s(i)=(i-1)*h;
    xi(i)=phi(t(i));
    g(i)=(c3-c2)/2*t(i)^2 + c2*t(i) +c1;
    gp(i)=(c3-c2)/2*xi(i)^2 + c2*xi(i) +c1;
    psi0(i)=f(t(i),0,0);
end
ssp=1;
while and(ssp > TOL, iter<100)
    iter=iter+1;
    for i=1:n+1
        for j=1:n+1
            Gp(j)= G(t(i),s(j))*psi0(j);
            G1p(j)= G(xi(i),s(j))*psi0(j);
        end
        u(i)=trap(0,1,n,Gp)+g(i);
        v(i)=trap(0,1,n,G1p)+gp(i);
        psi(i)=f(t(i),u(i),v(i));
    end
    ssp=chuan(psi-psi0);
    psi0=psi;
end
end

function ucx=ucx(x) % exact solution
ucx=exp(x);
end

function phi=phi(x)
phi=x/2;
end
```

```
function hvp=f(t,u,v)
hvp=exp(t)-u/4+v^2/4;
end

function green = G(t,s)
if s <= t
    green= s/2*(t^2-2*t+s);
else
    green= t^2/2*(s-1);
end
end

function trap=trap(a,b,n,f)
h=(b-a)/n;
S=0;
for j=2:n
    S=S+f(j);
end
S=S+(f(1)+f(n+1))/2;
trap=S*h;
end

function chuan=chuan(y)
chuan=norm(y,inf);
end
```