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THE EXISTENCE, UNIQUENESS AND ITERATIVE METHODS FOR SOME NONLINEAR BOUNDARY VALUE PROBLEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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Introduction

1. Overview of research situation and the necessity of the research

Numerous problems in the fields of mechanics, physics, biology, environment, etc. are reduced to boundary value problems for high order nonlinear ordinary differential equations (ODE), integro-differential equations (IDE) and functional differential equations (FDE). The study of qualitative aspects of these problems such as the existence, uniqueness and properties of solutions, and the methods for finding the solutions always are of interests of mathematicians and engineers. One can find exact solutions of the problems in a very small number of special cases. In general, one needs to seek their approximations by approximate methods, mainly numerical methods.

Among higher order equations, fourth order ones have been widely studied on both qualitative and quantitative aspects because of their various applications. Some doctoral theses on nonlinear fourth order boundary value problems have been successfully defended in Vietnam recently, such as those of Ngo Thi Kim Quy (2017) and Nguyen Thanh Huong (2019).

Besides the fourth order equations, third order ones have also received attention from researchers because they are the mathematical models of numerous problems in chemical engineering, heat conduction, astrophysics, etc... Concerning the not fully or fully third order differential equations

$$u'''(t) = f(t, u(t), u'(t), u''(t)), \quad 0 < t < 1 \tag{1}$$

with different boundary conditions, there have been many studies on qualitative aspects such as those of Li & Li (2017), Yao & Feng (2002), Feng (2008), Hopkin & Kosmatov (2007), Bai (2008), Sun et al. (2014),... By different methods like the lower and upper solutions method, Schauder's and Krasnoselskii's fixed point theorems, etc... they have established the existence, positivity and monotony of solutions under complicated conditions that are hard to verify. Moreover, no examples of solutions are shown although the sufficient conditions are satisfied and the verification of them is difficult. Some other authors such as Pandey (2016, 2017), Al-Said & Noor (2007), Danaf (2008), Khan & Sultana (2012), Lv & Gao (2017), He (2020) under the assumptions that the problems have unique solution have proposed solution methods like the use of difference method for the derivatives, polynomial or non-polynomial splines, method of series,...

Therefore, it is of great necessity to study sufficient conditions that are easy to verify for the existence and uniqueness of solutions of boundary value problems for nonlinear third order differential equations. Also, it is no less important to construct efficient numerical methods for finding the solutions of these problems.

Recently, third and fourth order nonlinear equations with integral boundary conditions have gathered plenty of interest among researchers. Some results have been achieved on the existence of solutions like those of Boucherif et al. (2009), Guo et al. (2012), Wang (2015), Benaicha et al. (2016), Li et al. (2013),etc... Integro-differential equations and functional differential equations have also received increasing attention. Fascinating results on the existence and methods for finding solutions have been obtained by Aruchnan et al. (2015), Chen et al. (2015), Lakestania et al. (2010), Tahernezhad (2020), Wang (2020), Bica et al. (2016), Khuri & Sayfy (2018), Hou (2021),... Sufficient conditions for these results were often complicated and difficult to verify. Therefore, the proposal of a unified approach to these problems on both qualitative and quantitative aspects under easy-to-verify conditions is of great need.

Motivated by the above facts, in this thesis we shall study the topic: *"The existence, uniqueness and iterative methods for some nonlinear boundary value problems of ordinary differential equations"*.

2. Objectives of the research

The aim of the thesis is to study the existence, uniqueness of solutions and solution methods for some BVPs for high order nonlinear differential, integro-differential and functional differential equations.

3. Contents and approach of the research

The thesis intends to study the following contents:

Content 1 The existence, uniqueness of solutions and iterative methods for some BVPs for third order nonlinear differential equations.

Content 2 The existence, uniqueness of solutions and iterative methods for some problems for third and fourth order nonlinear differential equations with integral boundary conditions.

Content 3 The existence, uniqueness of solutions and iterative methods for some BVPs for integro-differential and functional differential equations.

We shall approach to the above contents from both theoretical and practical points of view, which are the study of qualitative aspects of the existence solutions and construction of numerical methods for finding the solutions. The methodology through all the thesis is to the reduction of BVPs to operator equations in appropriate spaces, use Banach fixed point theorem for establishing the existence and uniqueness of solutions and for proving the convergence of continuous iterative methods, then construct discrete realizations of these methods.

4. Structure of the thesis

Except for "Introduction", "Conclusions" and "References", the thesis contains 4 chapters. In Chapter 1 we recall some auxiliary knowledges. The results of the thesis are presented in Chapters 2, 3 and 4. Namely,

1. Chapter 2 presents the results on the existence, uniqueness of solutions and positive solutions for third order nonlinear BVPs and the construction of numerical methods for finding the solutions.
2. Chapter 3 is devoted to the study of the existence, uniqueness of solutions and construction of iterative methods for finding the solutions for nonlinear third and fourth order differential equations with integral boundary conditions.
3. Chapter 4 presents the results on the existence, uniqueness of solutions and construction of numerical methods for finding the solutions of nonlinear integro-differential equations and functional differential equations.

5. The achievements of the thesis

The thesis achieves the following results:

(i) The establishment of theorems on the existence, uniqueness of solutions and positive solutions for third order nonlinear boundary value problems and the construction of numerical methods for finding the solutions.

(ii) The establishment of the existence, uniqueness of solutions and construction of iterative methods for finding the solutions for nonlinear third and fourth order differential equations with integral boundary conditions.

(iii) The establishment of the existence, uniqueness of solutions and construction of numerical methods for finding the solutions of fourth order integro-differential equations and of third order functional differential equations.

The obtained results of the thesis are published in the six papers [AL1]-[AL6] (see "List of the works of the author related to the thesis").

Chapter 1

Preliminaries

This chapter contains essential preliminary knowledges for the next chapters, taken from the books of Zeidler (1986), Melnikov et al. (2012), Burden and Faires (2011). The chapter includes:

1. Schauder's and Banach's fixed-point theorems.
2. Green's functions.
3. Some quadrature formulas.

Chapter 2

Existence results and an iterative method for two-point third order nonlinear BVPs

In this chapter, we investigate the existence, uniqueness of solution and the iterative methods on continuous level as well as discrete level for solving some two-point boundary value problems for nonlinear fully third-order differential equations.

2.1 Existence results and a continuous iterative method for third order nonlinear BVPs

Consider the boundary value problem

$$\begin{aligned} &u'''(t) = f(t, u(t), u'(t), u''(t)), \quad 0 < t < 1 \\ &B_1[u] = B_2[u] = B_3[u] = 0, \end{aligned} \tag{2.1}$$

where $B_1[u], B_2[u], B_3[u]$ are the boundary condition operators

$$\begin{aligned} B_1[u] &= \alpha_1 u(0) + \beta_1 u'(0) + \gamma_1 u''(0), \\ B_2[u] &= \alpha_2 u(0) + \beta_2 u'(0) + \gamma_2 u''(0), \\ B_3[u] &= \alpha_3 u(1) + \beta_3 u'(1) + \gamma_3 u''(1), \end{aligned} \tag{2.2}$$

satisfying

$$rank \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 & 0 & 0 & 0 \\ \alpha_2 & \beta_2 & \gamma_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix} = 3.$$

Denote by $G(t,s)$ the Green's function of the corresponding homogeneous problem of (2.1), by $G_1(t,s), G_2(t,s)$ the first and second derivatives with respect to t of $G(t,s)$, $G_0(t,s) = G(t,s)$ and

$$M_i = \max_{0 \le t \le 1} \int_0^1 |G_i(t,s)| ds, \ i = 0, 1, 2. \tag{2.3}$$

For each $M > 0$, introduce the domain

$$\mathcal{D}_M = \{(t, x, y, z) | \ 0 \le t \le 1, \ |x| \le M_0 M, \ |y| \le M_1 M, \ |z| \le M_2 M\}.$$

Theorem 2.1.2 (Existence of solutions). *Suppose that there exists a number $M > 0$ such that the function $f(t, x, y, z)$ is continuous and bounded by M in the domain $\mathcal{D}_M$, i.e.,*

$$|f(t, x, y, z)| \le M \tag{2.4}$$

for any $(t, x, y, z) \in \mathcal{D}_M$.

Then, the problem (2.1) has a solution $u(t)$ satisfying

$$|u(t)| \le M_0 M, \ |u'(t)| \le M_1 M, \ |u''(t)| \le M_2 M \text{ for any } 0 \le t \le 1. \tag{2.5}$$

This theorem can be proved by reducing the problem (2.1) to the operator equation $A\varphi = \varphi$, where the operator A is defined by

$$(A\varphi)(t) = f(t, u(t), u'(t), u''(t)), \tag{2.6}$$

where $u(t)$ is a solution of the problem

$$\begin{aligned} u'''(t) &= \varphi(t), \quad 0 < t < 1 \\ B_1[u] &= B_2[u] = B_3[u] = 0. \end{aligned} \tag{2.7}$$

Suppose that $G(x,t)$ and $G_1(x,t)$ are of constant signs in the square $Q = [0,1]^2$. For a function $H(x,t)$ defined and having a constant sign in Q we define

$$\sigma(H) = sign(H(t,s)) = \begin{cases} 1, & \text{if } H(t,s) \ge 0, \\ -1, & \text{if } H(t,s) < 0. \end{cases}$$

In order to investigate the existence of positive solutions of the problem (2.1) we introduce the notations

$$\begin{aligned} \mathcal{D}_M^+ = \{(t, x, y, z) |\ & 0 \le t \le 1, \ 0 \le x \le M_0 M, \\ & 0 \le \sigma(G)\sigma(G_1) y \le M_1 M, \ |z| \le M_2 M\}, \end{aligned}$$

$$S_M = \{\varphi \in C[0,1] |\ 0 \le \sigma(G)\varphi \le M\}.$$

Theorem 2.1.3 (Existence of positive solution). *Suppose that there exists a number $M > 0$ such that the function $f(t, x, y, z)$ is continuous and*

$$0 \le \sigma(G) f(t, x, y, z) \le M \tag{2.8}$$

for any $(t, x, y, z) \in \mathcal{D}_M^+$. Then, the problem (2.1) has a monotone nonnegative solution $u(t)$ satisfying

$$0 \le u(t) \le M_0 M, \ 0 \le \sigma(G)\sigma(G_1) u'(t) \le M_1 M, \ |u''(t)| \le M_2 M. \tag{2.9}$$

Theorem 2.1.4 (Existence and uniqueness of solution). *Assume that there exist numbers $M, L_0, L_1, L_2 \ge 0$ such that*

$$|f(t, x, y, z)| \le M,$$

$$|f(t, x_2, y_2, z_2) - f(t, x_1, y_1, z_1)| \le L_0 |x_2 - x_1| + L_1 |y_2 - y_1| + L_2 |z_2 - z_1| \tag{2.10}$$

for any $(t, x, y, z), (t, x_i, y_i, z_i) \in \mathcal{D}_M$ $(i = 1, 2)$ and

$$q := L_0 M_0 + L_1 M_1 + L_2 M_2 < 1. \tag{2.11}$$

Then, the problem (2.1) has a unique solution $u(t)$ such that $|u(t)| \le M_0 M$, $|u'(t)| \le M_1 M$, $|u''(t)| \le M_2 M$ for any $0 \le t \le 1$.

Consider the following iterative method for solving the problem (2.1):

1. Given an initial approximation $\varphi_0 \in B[0, M]$, say
$$\varphi_0(t) = 0. \tag{2.12}$$

2. Knowing φ_k $(k = 0, 1, ...)$, compute
$$\begin{aligned} u_k(t) &= \int_0^1 G(t,s)\varphi_k(s)\, ds, \quad y_k(t) = \int_0^1 G_1(t,s)\varphi_k(s)\, ds, \\ z_k(t) &= \int_0^1 G_2(t,s)\varphi_k(s)\, ds. \end{aligned} \tag{2.13}$$

3. Update the new approximation
$$\varphi_{k+1}(t) = f(t, u_k(t), y_k(t), z_k(t)). \tag{2.14}$$

Set
$$p_k = \frac{q^k}{1-q}\|\varphi_1 - \varphi_0\|. \tag{2.15}$$

Theorem 2.1.6 (Convergence). *Under the assumptions of Theorem 2.1.4 the above iterative method converges and there hold the estimates*
$$\|u_k - u\| \le M_0 p_k, \quad \|u_k' - u'\| \le M_1 p_k, \quad \|u_k'' - u''\| \le M_2 p_k, \tag{2.16}$$
where u is the exact solution of the problem (2.1)*, and M_0, M_1, M_2 are given by* (2.3)*.*

To illustrate the theoretical results, we consider the problem (2.1) with some particular cases of boundary conditions. Problems with such boundary conditions have been considered by Yao & Feng (2002), Feng & Liu (2005), Hopkins & Kosmatov (2007), Li & Li ((2017), Bai (2008). Applying our approach to the examples taken from these papers often yield superior qualitative results, such as the establishment of the existence and uniqueness of solution while these authors achieved the existence only, and better solution estimates.

2.2 Numerical methods for third order nonlinear BVPs

In this section, we propose iterative methods on discrete level of second- and third-order accuracy for the problem
$$\begin{aligned} u^{(3)}(t) &= f(t, u(t), u'(t), u''(t)), \quad 0 < t < 1, \\ u(0) &= 0, u'(0) = 0, u'(1) = 0. \end{aligned} \tag{2.17}$$

This is a special case of the problem (2.1). The iterative method on continuous level has been described in the previous section. In order to construct the corresponding discrete iterative methods, we cover the interval $[0, 1]$ by the uniform grid $\bar{\omega}_h = \{t_i = ih, \ h = 1/N, i = 0, 1, ..., N\}$ and denote by $\Phi_k(t), U_k(t), Y_k(t), Z_k(t)$ the grid functions defined on the grid $\bar{\omega}_h$ and approximating the functions $\varphi_k(t), u_k(t), y_k(t), z_k(t)$ on this grid, respectively.

First, consider the following discrete iterative method, named **Method 1**:

1. Given
$$\Phi_0(t_i) = f(t_i, 0, 0, 0),\ i = 0, ..., N. \tag{2.18}$$
2. Knowing $\Phi_k(t_i),\ k = 0, 1, ...;\ i = 0, ..., N$, compute approximately the integrals (2.13) by the trapezoidal rule
$$\begin{aligned} U_k(t_i) &= \sum_{j=0}^{N} h\rho_j G_0(t_i, t_j)\Phi_k(t_j), \quad Y_k(t_i) = \sum_{j=0}^{N} h\rho_j G_1(t_i, t_j)\Phi_k(t_j), \\ Z_k(t_i) &= \sum_{j=0}^{N} h\rho_j G_2^*(t_i, t_j)\Phi_k(t_j),\ i = 0, ..., N, \end{aligned} \tag{2.19}$$
where
$$\rho_j = \begin{cases} 1/2,\ j = 0, N \\ 1,\ j = 1, 2, ..., N-1 \end{cases},\ G_2^*(t, s) = \begin{cases} s, & 0 \le s < t \le 1, \\ s - 1/2, & s = t, \\ s - 1, & 0 \le t < s \le 1. \end{cases} \tag{2.20}$$
3. Update
$$\Phi_{k+1}(t_i) = f(t_i, U_k(t_i), Y_k(t_i), Z_k(t_i)). \tag{2.21}$$

Theorem 2.2.6 (Error estimates)**.** *For the approximate solution of the problem* (2.17) *obtained by the discrete iterative method* (2.18)-(2.21) *on* $\bar{\omega}_h$ *we have the estimates*
$$\begin{aligned} &\|U_k - u\| \le M_0 p_k + O(h^2),\ \|Y_k - u'\| \le M_1 p_k + O(h^2), \\ &\|Z_k - u''\| \le M_2 p_k + O(h^2), \end{aligned}$$
where $M_0 = \frac{1}{12}, M_1 = \frac{1}{8}, M_2 = \frac{1}{2}$, *and* p_k *is defined by* (2.15).

Method 2:
The steps of this method are the same as of **Method 1** with an essential difference in Step 2 and now the number of grid points is even $N = 2n$, namely:
2': Knowing $\Phi_k(t_i),\ k = 0, 1, ...;\ i = 0, ..., N$, compute approximately the integrals by the modified Simpson rule
$$U_k(t_i) = F(G_0(t_i, .)\Phi_k(.)),\ Y_k(t_i) = F(G_1(t_i, .)\Phi_k(.)),\ Z_k(t_i) = F(G_2^*(t_i, .)\Phi_k(.)),$$
where
$$F(G_l(t_i, .)\Phi_k(.)) = \begin{cases} \sum_{j=0}^{N} h\rho_j G_l(t_i, t_j)\Phi_k(t_j) + \dfrac{h}{6}\Big(G_l(t_i, t_{i-1})\Phi_k(t_{i-1}) \\ \quad -2G_l(t_i, t_i)\Phi_k(t_i) + G_l(t_i, t_{i+1})\Phi_k(t_{i+1})\Big) \ \text{if } i \text{ is odd}, \\ \sum_{j=0}^{N} h\rho_j G_l(t_i, t_j)\Phi_k(t_j) \ \text{if } i \text{ is even}, \ l = 0, 1;\ i = 0, 1, ..., N. \end{cases}$$
$$\rho_j = \begin{cases} 1/3,\ j = 0, N \\ 4/3,\ j = 1, 3, ..., N-1 \\ 2/3,\ j = 2, 4, ..., N-2, \end{cases}$$
$F(G_2^*(t_i, .)\Phi_k(.))$ is computed in the same way as $F(G_l(t_i, .)\Phi_k(.))$ above, where G_l is replaced by G_2^* defined by (2.20).

Theorem 2.2.9 (Error estimates)**.** *Assume that $f(t, x, y, z)$ has all continuous partial derivatives up to fourth order in $\mathcal{D}_M$. Then for the approximate solution of the problem* (2.17) *obtained by* **Method 2** *on $\bar{\omega}_h$ we have the estimates*

$$\|U_k - u\| \le M_0 p_k + O(h^3),\ \|Y_k - u'\| \le M_1 p_k + O(h^3),$$
$$\|Z_k - u''\| \le M_2 p_k + O(h^3).$$

For confirming the efficiency of the above discrete iterative methods, we conduct numerical experiments on some examples of the problems where exact solutions are either known or unknown. Below is a notable example:
Example 2.2.1. (Pandey 2016) Consider the problem

$$\begin{aligned} u'''(x) &= x^4 u(x) - u^2(x) + g(x),\ 0 < x < 1, \\ u(0) &= 0,\ u'(0) = -1,\ u'(1) = \sin(1), \end{aligned} \tag{2.22}$$

where $g(x) = -3\sin(x) - \cos(x)(x-1) - x^4(x-1)\sin(x) + (x-1)^2\sin^2(x)$. The exact solution is $u^*(x) = (x-1)\sin(x)$. The iterative process is continued until $\|\Phi_{k+1} - \Phi_k\| \le TOL$, TOL is a given tolerance. Results of the iterative methods are given in Table 2.1 below. Here $N+1$ is the number of grid points,

Table 2.1: Convergence in Example 2.2.1 with $TOL = 10^{-10}$

N	K	$Error_{trap}$	$Order$	$Error_{Simp}$	$Order$
8	7	9.9235e-04		9.7222e-04	
16	7	2.4732e-04	2.0045	1.3187e-04	2.8822
32	7	6.1782e-05	2.0011	1.6896e-05	2.9643
64	7	1.5443e-05	2.0003	2.1301e-06	2.9877
128	7	3.8605e-06	2.0001	2.6774e-07	2.9923
256	7	9.6511e-07	2.0000	3.3544e-08	2.9965
512	7	2.4128e-07	2.0000	4.1977e-09	2.9984

K is the number of iterations, $Error_{trap}$, $Error_{Simp}$ are errors $\|U_K - u^*\|$ of **Method 1** and **Method 2**, $Order$ is the order of convergence calculated by

$$Order = \log_2 \frac{\|U_K^{N/2} - u^*\|}{\|U_K^N - u^*\|},$$

the superscripts $N/2$ and N of U_K mean that U_K is computed on the grid with the corresponding number of grid points.

Pandey used iteration method to solve nonlinear system of equations arising after discretization of the problem by finite difference method. The iteration process is continued until $\|U_{k+1} - U_k\| \le 10^{-10}$. The number of iterations was not reported. The accuracy for some different N is given in Table 2.2.

Table 2.2: Pandey's results in Example 2.2.1

N	8	16	32	64
Error	0.11921225e-01	0.33391170e-02	0.87742222e-03	0.23732412e-03

It is clear that our discrete methods give better results than that of Pandey.

Chapter 3

Existence results and an iterative method for some nonlinear ODEs with integral boundary conditions

3.1 Existence results and an iterative method for fully third order nonlinear integral BVPs

Consider the boundary value problem

$$u'''(t) = f(t, u(t), u'(t), u''(t)), \quad 0 < t < 1, \tag{3.1}$$

$$u(0) = u'(0) = 0, \quad u(1) = \int_0^1 g(s)u(s)ds, \tag{3.2}$$

where $f : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}^+, g : [0, 1] \to \mathbb{R}^+$.

Similarly to the problems in the previous chapter, we reduce the problem (3.1)-(3.2) to an operator equation and then study the resulting equation. Denote by $\mathcal{B}$ the space of pairs $w = (\varphi, \alpha)^T$, where $\varphi \in C[0, 1], \alpha \in \mathbb{R}$, and equip it with the norm

$$\|w\|_{\mathcal{B}} = \max(\|\varphi\|, k|\alpha|), \tag{3.3}$$

where $\|\varphi\| = \max_{0 \le t \le 1} |\varphi(t)|$, k is a number, $k \ge 1$.
Define the operator $A : \mathcal{B} \to \mathcal{B}$ by

$$Aw = \begin{pmatrix} f(t, u(t), u'(t), u''(t) \\ \int_0^1 g(s)u(s)ds \end{pmatrix}, \tag{3.4}$$

where $u(t)$ is the solution of the problem

$$u'''(t) = \varphi(t), \quad 0 < t < 1, \tag{3.5}$$

$$u(0) = u'(0) = 0, \, u(1) = \alpha. \tag{3.6}$$

Thus, the problem (3.1)-(3.2) is reduced to the fixed point problem for A.
Denote by $G_0(t, s)$ the Green's function of the corresponding homogeneous problem of (3.5)-(3.6), by $G_1(t, s), G_2(t, s)$ its first and second derivative with respect to t, and

$$M_i = \max_{0 \le t \le 1} \int_0^1 |G_i(t, s)|ds, \quad i = 0, 1, 2.$$

We have $M_0 = \frac{2}{81}, M_1 = \frac{1}{18}, M_2 = \frac{2}{3}$. For any $M > 0$ define the domain

$$\begin{aligned}\mathcal{D}_M = \{(t,x,y,z) \mid 0 \le t \le 1,\ |x| \le (M_0 + \frac{1}{k})M,\\ |y| \le (M_1 + \frac{2}{k})M,\ |z| \le (M_2 + \frac{2}{k})M\}.\end{aligned} \tag{3.7}$$

Next, denote

$$C_0 = \int_0^1 g(s)ds,\ C_2 = \int_0^1 s^2 g(s)ds. \tag{3.8}$$

Theorem 3.1.1 (Existence of solution)**.** *Suppose that the function $f(t,x,y,z)$ is continuous and bounded by M in $\mathcal{D}_M$, that is,*

$$|f(t,x,y,z)| \le M \quad in\ \mathcal{D}_M \tag{3.9}$$

and

$$q_1 := kC_0M_0 + C_2 \le 1. \tag{3.10}$$

Then, the problem (3.1)-(3.2) *has a solution.*

Theorem 3.1.3 (Existence and uniqueness)**.** *Suppose that there exist numbers $M > 0, L_0, L_1, L_2 \ge 0$ such that*

(H1) $|f(t,x,y,z)| \le M,\ \forall (t,x,y,z) \in \mathcal{D}_M$.

(H2) $|f(t,x_2,y_2,z_2) - f(t,x_1,y_1,z_1)| \le L_0|x_2 - x_1| + L_1|y_2 - y_1| + L_2|z_2 - z_1|,\ \forall (t,x_i,y_i,z_i) \in \mathcal{D}_M,\ i = 1,2.$

(H3) $q := \max\{q_1, q_2\} < 1$, *where $q_1 = kC_0M_0 + C_2$ was defined as in* (3.10) *and*

$$q_2 = L_0(M_0 + \frac{1}{k}) + L_1(M_1 + \frac{2}{k}) + L_2(M_2 + \frac{2}{k}). \tag{3.11}$$

Then, the problem (3.1)-(3.2) *has a unique solution $u \in C^3[0,1]$.*

The conditions for the existence and uniqueness of positive solution are also established in this section.

Iterative method:

1. Given $w_0 = (\varphi_0, \alpha_0)^T \in B[0,M]$, say,

$$\varphi_0(t) = f(t,0,0,0),\ \alpha_0 = 0.$$

2. Knowing $\varphi_n(t)$ and $\alpha_n(t)$ $(n = 0,1,...)$, compute

$$u_n(t) = \int_0^1 G(t,s)\varphi_n(s)ds + \alpha_n t^2, \quad y_n(t) = \int_0^1 G_1(t,s)\varphi_n(s)ds + 2\alpha_n t,$$
$$z_n(t) = \int_0^1 G_2(t,s)\varphi_n(s)ds + 2\alpha_n.$$

3. Update

$$\varphi_{n+1}(t) = f(t,u_n(t),y_n(t),z_n(t)), \quad \alpha_{n+1} = \int_0^1 g(s)u_n(s)ds.$$

Theorem 3.1.5. *Under the assumptions of Theorem 3.1.3 the above iterative method converges, and for the approximate solution $u_n(t)$ and its derivatives there hold the estimates*

$$\|u_n - u\| \leq \left(M_0 + \frac{1}{k}\right) p_n d, \ \|u_n^{(i)} - u^{(i)}\| \leq \left(M_i + \frac{2}{k}\right) p_n d, \ i = 1, 2,$$

where $p_n = \frac{q^n}{1-q}$, $d = \|w_1 - w_0\|_{\mathcal{B}}$, $w_1 = (\varphi_1, \alpha_1)^T$.

Many examples of problems where exact solutions are either known or unknown are given in order to confirm the validity of the obtained theoretical results and the efficiency of the proposed iterative method. Below is an example where exact solution is unknown.
Example 3.1.4. Consider the problem

$$\begin{aligned} &u'''(t) = -(u^2 e^u + \frac{1}{5}\sin(u') + \frac{1}{8}\cos(u'') + 1), \quad 0 < t < 1, \\ &u(0) = u'(0) = 0, \ u(1) = \int_0^1 s^4 u(s) ds. \end{aligned}$$

With $M = 1.7, k = 4$, it can be verified that the conditions for the existence and uniqueness of solution are satisfied. This solution is found using the above iterative method after 6 iterations until the difference between two successive iterations is less than 10^{-4}.

3.2 Existence results and an iterative method for a fully fourth order nonlinear integral BVP

Consider the problem

$$u''''(t) = f(t, u(t), u'(t), u''(t), u'''(t)), \quad 0 < t < 1, \tag{3.12}$$

$$u'(0) = u''(0) = u'(1) = 0, \ u(0) = \int_0^1 g(s) u(s) ds, \tag{3.13}$$

where $f : [0, 1] \times \mathbb{R}^4 \to \mathbb{R}^+$, $g : [0, 1] \to \mathbb{R}^+$ are continuous functions.

As in the previous section, consider the space $\mathcal{B} = C[0, 1] \times \mathbb{R}$ of pairs $w = (\varphi, \mu)^T$, $\varphi \in C[0, 1], \mu \in \mathbb{R}$, and equip it with the norm

$$\|w\|_{\mathcal{B}} = \max(\|\varphi\|, r|\mu|), \ r \geq 1 \tag{3.14}$$

and define the operator A by

$$Aw = \begin{pmatrix} f(t, u(t), u'(t), u''(t), u'''(t)) \\ \int_0^1 g(s) u(s) ds \end{pmatrix}, \tag{3.15}$$

where $u(t)$ is the solution of the problem

$$u''''(t) = \varphi(t), \quad 0 < t < 1, \tag{3.16}$$

$$u'(0) = u''(0) = u'(1) = 0, \ \ u(0) = \mu. \tag{3.17}$$

Denote by $G_0(t,s)$ the Green's function of the corresponding homogeneous problem and by $G_i(t,s), i = 1,2,3$ its first, second and third derivatives with respect to t, and

$$M_i = \max_{0\le t\le 1} \int_0^1 |G_i(t,s)|ds, \quad i = 0,1,2,3.$$

It is easy to verify that $M_0 = 0.0139, M_1 = 0.0247, M_2 \le 0.1883, M_3 = 1.3333$. Also, we define

$$\begin{aligned}\mathcal{D}_M = \{(t,u,y,v,z) \mid 0 \le t \le 1,\ |u| \le (M_0 + \tfrac{1}{r})M,& \\ |y| \le M_1 M,\ |v| \le M_2 M,\ |z| \le M_3 M\}&\end{aligned} \tag{3.18}$$

and denote

$$C_0 = \int_0^1 g(s)ds > 0. \tag{3.19}$$

Theorem 3.2.3 (Existence and uniqueness)**.** *Suppose that there exist numbers $M > 0, L_0, L_1, L_2, L_3 \ge 0$ such that*

1. $|f(t,u,y,v,z)| \le M,\ \forall (t,u,y,v,z) \in \mathcal{D}_M$.
2. $|f(t,u_2,y_2,v_2,z_2) - f(t,u_1,y_1,v_1,z_1)| \le L_0|u_2-u_1| + L_1|y_2-y_1| + L_2|v_2 - v_1| + L_3|z_2-z_1|,\ \forall (t,u_i,y_i,v_i,z_i) \in \mathcal{D}_M,\ i = 1,2.$
3. $q := \max\{q_1, q_2\} < 1$, *where* $q_1 = rC_0M_0 + C_0$ *and*

$$q_2 = L_0(M_0 + \tfrac{1}{r}) + L_1M_1 + L_2M_2 + L_3M_3.$$

Then the problem has a unique solution $u \in C^4[0,1]$.

The existence and uniqueness of positive solution are also established.

Iterative method on continuous level:

1. Given

$$\varphi_0(t) = f(t,0,0,0,0),\ \mu_0 = 0 \tag{3.20}$$

2. Knowing $\varphi_k(t)$ and μ_k $(k = 0,1,...)$ compute

$$\begin{aligned} u_k(t) &= \int_0^1 G_0(t,s)\varphi_k(s)ds + \mu_k, \quad & y_k(t) &= \int_0^1 G_1(t,s)\varphi_k(s)ds,\\ v_k(t) &= \int_0^1 G_2(t,s)\varphi_k(s)ds, & z_k(t) &= \int_0^1 G_3(t,s)\varphi_k(s)ds,\end{aligned} \tag{3.21}$$

3. Update

$$\varphi_{k+1}(t) = f(t,u_k(t),y_k(t),v_k(t),z_k(t)), \quad \mu_{k+1} = \int_0^1 g(s)u_k(s)ds. \tag{3.22}$$

Theorem 3.2.5 (Convergence)**.** *The iterative method* (3.20)-(3.22) *converges and for the approximate solution* $u_k(t)$ *there hold estimates*

$$\|u_k - u\| \leq \left(M_0 + \frac{1}{r}\right) p_k d, \ \|u_k' - u'\| \leq M_1 p_k d,$$
$$\|u_k'' - u''\| \leq M_2 p_k d, \ \|u_k''' - u'''\| \leq M_3 p_k d.$$

where u *is the exact solution of the problem* (3.12)-(3.13), $p_k = \frac{q^k}{1-q}$, $d = \|w_1 - w_0\|_{\mathcal{B}}$ *and* r *is the number available in* (3.14).

Iterative method on discrete level:
Denote by $\Phi_k(t), U_k(t), Y_k(t), V_k(t), Z_k(t)$ the grid functions defined on the uniform grid $\bar{\omega}_h = \{t_i = ih, \ h = 1/N, i = 0, 1, ..., N\}$ approximating the functions $\varphi_k(t), u_k(t), y_k(t), v_k(t), z_k(t)$ and denote by $\hat{\mu_k}$ the approximation of μ_k. Consider the discrete iterative method:

1. Given
$$\Phi_0(t_i) = f(t_i, 0, 0, 0, 0), \ i = 0, ..., N; \quad \hat{\mu}_0 = 0$$
2. Knowing $\Phi_k(t_i), \ i = 0, ..., N$ and $\hat{\mu}_k$ $(k = 0, 1, ...)$ compute approximately the integrals (3.21) by trapezoidal rule
$$U_k(t_i) = \sum_{j=0}^{N} h\rho_j G_0(t_i, t_j)\Phi_k(t_j) + \hat{\mu}_k, \quad Y_k(t_i) = \sum_{j=0}^{N} h\rho_j G_1(t_i, t_j)\Phi_k(t_j),$$
$$V_k(t_i) = \sum_{j=0}^{N} h\rho_j G_2(t_i, t_j)\Phi_k(t_j), \quad Z_k(t_i) = \sum_{j=0}^{N} h\rho_j G_3^*(t_i, t_j)\Phi_k(t_j), \ i = 0, ..., N,$$
where $\rho_0 = \rho_N = 1/2; \rho_j = 1, j = 1, ..., N-1$ and
$$G_3^*(t,s) = \begin{cases} -(1-s)^2 + 1, & 0 \leq s < t \leq 1, \\ -(1-s)^2 + 1/2, & s = t, \\ -(1-s)^2, & 0 \leq t < s \leq 1. \end{cases}$$
3. Update
$$\Phi_{k+1}(t_i) = f(t_i, U_k(t_i), Y_k(t_i), V_k(t_i), Z_k(t_i)), \quad \hat{\mu}_{k+1} = \sum_{j=0}^{N} h\rho_j g(t_j) U_k(t_j).$$

Theorem 3.2.9 (Error estimates)**.** *Assume that the conditions in Theorem 3.2.3 are satisfied. Assume also that* $f(t, u, y, v, z)$ *has continuous derivatives up to second order and* $g(s) \in C^2[0, 1]$*. Then, for the approximate solution of the problem* (3.12), (3.13) *obtained by the discrete iterative method on uniform grid with grid size* h *there hold the estimates*

$$\begin{aligned} &\|U_k - u\| \leq \left(M_0 + \frac{1}{r}\right) p_k d + O(h^2), \ \|Y_k - u'\| \leq M_1 p_k d + O(h^2), \\ &\|V_k - u''\| \leq M_2 p_k d + O(h^2), \ \|Z_k - u'''\| \leq M_3 p_k d + O(h^2). \end{aligned} \tag{3.23}$$

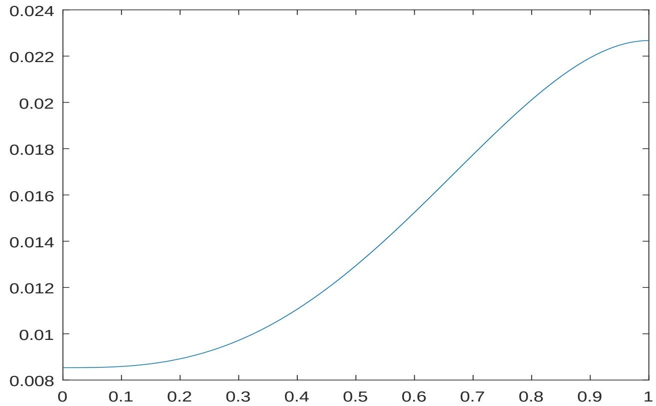

Figure 3.1: Graph of the approximate solution in Example 3.2.1

Many examples of problems where exact solutions are either known or unknown are given in order to confirm the validity of the obtained theoretical results and the efficiency of the proposed iterative method. Below is a notable example.

Example 3.2.3. (Benaicha &Haddouchi, 2016) Consider the problem

$$u''''(t) = -\sqrt{(1+u)} - \sin u, \quad 0 < t < 1,$$
$$u'(0) = u''(0) = u'(1) = 0,\ u(0) = \int_0^1 su(s)ds.$$

By using the above theoretical results, the problem can be proved to have unique positive solution, while Benaicha &Haddouchi could only show the existence of a positive solution. The approximate positive solution found by the above discrete method is depicted in Figure 3.1.

Chapter 4

Existence results and iterative methods for integro-differential and functional differential equations

4.1 Existence results and an iterative method for an integro-differential equation

In this section, we consider the problem

$$\begin{gathered} u^{(4)}(x) = f(x, u(x), u'(x), \int_0^1 k(x,t)u(t)dt), \\ u(0) = 0,\ u(1) = 0,\ u''(0) = 0,\ u''(1) = 0, \end{gathered} \tag{4.1}$$

where $f(x, u, v, z)$ and $k(x, t)$ are continuous functions.
Using the same methodology as in previous chapters, we introduce the operator A defined in the space $C[0, 1]$ by

$$(A\varphi)(x) = f(x, u(x), u'(x), \int_0^1 k(x,t)u(t)dt), \tag{4.2}$$

where $u(x)$ is the solution of the problem

$$\begin{gathered} u'''' = \varphi(x),\ 0 < x < 1, \\ u(0) = u''(0) = u(1) = u''(1) = 0. \end{gathered} \tag{4.3}$$

It can be verified that the study of the problem (4.3) can be reduced to the study of the fixed point of operator A. Denote by $G_0(t, s)$ the Green's function of the corresponding homogeneous problem and by $G_1(t, s)$ its first derivative with respect to t. Denote

$$M_i = \max_{0\le x\le 1} \int_0^1 |G_i(x,s)|ds, i = 0, 1, \quad M_2 = \max_{0\le x\le 1} \int_0^1 |k(x,s)|ds \tag{4.4}$$

and define the domain

$$\mathcal{D}_M = \{(x, u, v, z) \mid 0 \le x \le 1,\ |u| \le M_0M,\ |v| \le M_1M,\ |z| \le M_0M_2M\}.$$

Theorem 4.1.1 (Existence and uniqueness)**.** *Suppose that the function $k(x,t)$ is continuous in the square $[0, 1] \times [0, 1]$ and there exist numbers $M > 0$, $L_0, L_1, L_2 \ge 0$ such that:*

(i) *$f(x,u,v,z)$ is continuous in $\mathcal{D}_M$ and $|f(x,u,v,z)| \leq M$, $\forall (x,u,v,z) \in \mathcal{D}_M$.*

(ii) $|f(x_2,u_2,v_2,z_2) - f(x_1,u_1,v_1,z_1)| \leq L_0|u_2-u_1| + L_1|v_2-v_1| + L_2|z_2-z_1|$, $\forall (x_i,u_i,v_i,z_i) \in \mathcal{D}_M,\ i=1,2.$

(iii) $q = L_0M_0 + L_1M_1 + L_2M_0M_2 < 1.$

Then the problem (4.1) has a unique solution $u \in C^4[0,1]$ satisfying $|u(x)| \leq M_0M$, $|u'(x)| \leq M_1M$ for any $0 \leq x \leq 1$.

In order to study positive solutions of the problem, introduce the domain

$$\begin{aligned} \mathcal{D}_M^+ = \{(x,u,v,z) \mid 0 \leq x \leq 1,\ 0 \leq u \leq M_0M, \\ |v| \leq M_1M,\ |z| \leq M_0M_2M\}. \end{aligned} \tag{4.5}$$

and denote

$$S_M = \{\varphi \in C[0,1], 0 \leq \varphi(x) \leq M\}.$$

Theorem 4.1.2 (Positivity of solution)**.** *Suppose that the function $k(x,t)$ is continuous in the square $[0,1]\times[0,1]$ and there exist numbers $M>0$, $L_0, L_1, L_2 \geq 0$ such that:*

(i) *$f(x,u,v,z)$ is continuous in $\mathcal{D}_M^+$ and $0 \leq f(x,u,v,z) \leq M$, $\forall (x,u,v,z) \in \mathcal{D}_M^+$ and $f(x,0,0,0) \not\equiv 0$.*

(ii) $|f(x_2,u_2,v_2,z_2) - f(x_1,u_1,v_1,z_1)| \leq L_0|u_2-u_1| + L_1|v_2-v_1| + L_2|z_2-z_1|$, $\forall (x_i,u_i,v_i,z_i) \in \mathcal{D}_M^+,\ i=1,2.$

(iii) $q = L_0M_0 + L_1M_1 + L_2M_0M_2 < 1.$

Then the problem (4.1) has a unique positive solution $u \in C^4[0,1]$ satisfying $0 \leq u(x) \leq M_0M$, $|u'(x)| \leq M_1M$ for any $0 \leq x \leq 1$.

Iterative method

1. Given

$$\varphi_0(x) = f(x,0,0,0). \tag{4.6}$$

2. Knowing $\varphi_m(x)$ $(m=0,1,...)$, compute

$$\begin{aligned} u_m(x) &= \int_0^1 G_0(x,t)\varphi_m(t)dt, \quad v_m(x) = \int_0^1 G_1(x,t)\varphi_m(t)dt, \\ z_m(x) &= \int_0^1 k(x,t)u_m(t)dt. \end{aligned} \tag{4.7}$$

3. Update

$$\varphi_{m+1}(x) = f(x,u_m(x),v_m(x),z_m(x)). \tag{4.8}$$

Theorem 4.1.3 (Convergence)**.** *Under the assumptions of Theorem 4.1.1, the iterative method (4.6)-(4.8) converges and there hold the estimates*

$$\|u_m - u\| \leq M_0p_md,\ \|u'_m - u'\| \leq M_1p_md,$$

where u is the exact solution of the problem (4.1), $p_m = \frac{q^m}{1-q}$, $d = \|\varphi_1 - \varphi_0\|$..

Discrete iterative method

Denote by $\Phi_m(x), U_m(x), V_m(x), Z_m(x)$ the grid functions on the uniform grid $\bar{\omega}_h = \{x_i = ih,\ h = 1/N, i = 0, 1, ..., N\}$ approximating the functions $\varphi_m(x)$, $u_m(x), v_m(x), z_m(x)$. Consider the following discrete iterative method:

1. Given
$$\Phi_0(x_i) = f(x_i, 0, 0, 0),\ i = 0, ..., N. \tag{4.9}$$

2. Knowing $\Phi_m(x_i),\ m = 0, 1, ...;\ i = 0, ..., N,$, compute approximately the integrals (4.7) by trapezoidal rule
$$U_m(x_i) = \sum_{j=0}^{N} h\rho_j G_0(x_i, x_j)\Phi_m(x_j), \quad V_m(x_i) = \sum_{j=0}^{N} h\rho_j G_1(x_i, x_j)\Phi_m(x_j),$$
$$Z_m(x_i) = \sum_{j=0}^{N} h\rho_j k(x_i, x_j)U_m(x_j),\ i = 0, ..., N, \tag{4.10}$$
where ρ_j are the weights of trapezoidal rule.

3. Update
$$\Phi_{m+1}(x_i) = f(x_i, U_m(x_i), V_m(x_i), Z_m(x_i)). \tag{4.11}$$

Theorem 4.1.7 (Error estimates)**.** *Under the assumptions of Theorem 4.1.1 and $f(t, u, v, z)$ and $k(x, t)$ have all continuous partial derivatives up to second order. Then the approximate solution of the problem* (4.1) *is obtained using the above discrete iterative method on uniform grid with grid size h and there hold the estimates*
$$\|U_m - u\| \le M_0 p_m d + O(h^2),\ \|V_m - u'\| \le M_2 p_m d + O(h^2). \tag{4.12}$$

Many examples are given in order to confirm the validity of the obtained theoretical results and the efficiency of the proposed iterative method. Below is a notable example.
Example 4.1.2. Consider the problem (Wang, 2020)
$$\begin{aligned} &u^{(4)}(x) = \sin(\pi x)[(2 - u^2(x)) \int_0^1 tu(t)dt + 1], x \in (0, 1) \\ &u(0) = 0,\ u(1) = 0,\ u''(0) = 0,\ u''(1) = 0. \end{aligned} \tag{4.13}$$

By applying the above theoretical results, it can be proved that the problem has a unique solution $|u(x)| \le 0.0143,\ |u'(x)| \le 0.0458$ and on the grid with grid size $h = 0.01$ and stopping criterion $\|\Phi_m - \Phi_{m-1}\| \le 10^{-10}$ the solution is found after 7 iterations.

It is worth emphasizing that by the monotone method Wang could only prove the convergence of the iterative sequences to extremal solutions of the problem but not the existence and uniqueness of solution.

4.2 Existence results and an iterative method for functional differential equations

In this section, we consider the problem

$$\begin{aligned} u''' &= f(t, u(t), u(\varphi(t))), \quad t \in [0, a] \\ B_1[u] &= b_1, B_2[u] = b_2, B_3[u] = b_3, \end{aligned} \tag{4.14}$$

where $\varphi(t)$ is a continuous function mapping $[0, a]$ into itself, $B_1[u], B_2[u], B_3[u]$ are the boundary condition operators defined in (2.2).
In the space $C[a, b]$ define the operator A by

$$(A\psi)(t) = f(t, u(t), u(\varphi(t))), \tag{4.15}$$

where $u(t)$ is the solution of the problem

$$\begin{aligned} u'''(t) &= \psi(t), \quad 0 < t < a \\ B_1[u] &= b_1, B_2[u] = b_2, B_3[u] = b_3, \end{aligned} \tag{4.16}$$

Denote by $G(t, s)$ the Green's function of the corresponding homogeneous problem of the problem (4.16),

$$M_0 = \max_{0 \le t \le a} \int_0^a |G(t, s)| ds. \tag{4.17}$$

and $g(t)$ is the polynomial of second degree satisfying the boundary conditions

$$B_1[g] = b_1, B_2[g] = b_2, B_3[g] = b_3, \tag{4.18}$$

$$\mathcal{D}_M = \Big\{(t, u, v) \mid 0 \le t \le a; |u| \le \|g\| + M_0 M; |v| \le \|g\| + M_0 M\Big\}, \tag{4.19}$$

Theorem 4.2.2 (Existence and uniqueness)**.** *Suppose that:*

(i) $\varphi(t)$ is a continuous map from $[0, a]$ into $[0, a]$.

(ii) The function $f(t, u, v)$ is continuous and bounded by M in $\mathcal{D}_M$, that is

$$|f(t, u, v)| \le M \quad \forall (t, u, v) \in \mathcal{D}_M. \tag{4.20}$$

$f(t, u, v)$ satisfies the Lipschitz conditions in the variables u, v with the coefficients $L_1, L_2 \ge 0$ in $\mathcal{D}_M$, that is

$$\begin{aligned} |f(t, u_2, v_2) - f(t, u_1, v_1)| \le L_1|u_2 - u_1| + L_2|v_2 - v_1| \\ \forall (t, u_i, v_i) \in \mathcal{D}_M \ (i = 1, 2) \end{aligned} \tag{4.21}$$

(iv)

$$q := (L_1 + L_2)M_0 < 1. \tag{4.22}$$

Then the problem (4.14) has a unique solution $u(t) \in C^3[0, a]$ satisfying

$$|u(t)| \le \|g\| + M_0 M \quad \forall t \in [0, a]. \tag{4.23}$$

Iterative method

1. Given $\psi_0 \in B[0, M]$, say

$$\psi_0(t) = f(t, 0, 0). \tag{4.24}$$

2. Knowing $\psi_k(t)$ $(k = 0, 1, ...)$, compute

$$\begin{aligned} u_k(t) &= g(t) + \int_0^a G(t,s)\psi_k(s)ds, \\ v_k(t) &= g(\varphi(t)) + \int_0^a G(\varphi(t), s)\psi_k(s)ds. \end{aligned} \tag{4.25}$$

3. Update

$$\psi_{k+1}(t) = f(t, u_k(t), v_k(t)). \tag{4.26}$$

Theorem 4.2.3 (Convergence)**.** *Under the assumptions of Theorem 4.2.2 the above iterative method converges and there holds the estimate*

$$\|u_k - u\| \le M_0 p_k d,$$

where u is the exact solution of the problem (4.14) *and M_0 is given by* (4.17)*, $p_k = q^k/1-q,\ d = \|\psi_1 - \psi_0\|$.*

Denote by $\Phi_k(t), U_k(t), V_k(t)$ the grid functions on $\bar{\omega}_h$ approximating the functions $\psi_k(t), u_k(t), v_k(t)$ on this grid.

Discrete iterative method:

1. Given

$$\Psi_0(t_i) = f(t_i, 0, 0),\ i = 0, ..., N. \tag{4.27}$$

2. Knowing $\Psi_k(t_i),\ k = 0, 1, ...;\ i = 0, ..., N$, compute

$$\begin{aligned} U_k(t_i) &= g(t_i) + \sum_{j=0}^{N} h\rho_j G(t_i, t_j)\Psi_k(t_j), \\ V_k(t_i) &= g(\xi_i) + \sum_{j=0}^{N} h\rho_j G(\xi_i, t_j)\Psi_k(t_j), i = 0, ..., N, \end{aligned} \tag{4.28}$$

   where ρ_j are the weights of trapezoidal rule and $\xi_i = \varphi(t_i)$.

3. Update

$$\Psi_{k+1}(t_i) = f(t_i, U_k(t_i), V_k(t_i)). \tag{4.29}$$

Theorem 4.2.7 (Error estimates)**.** *Under the assumptions of Theorem 4.2.2, for the approximate solution of the problem* (4.14) *obtained by the iterative method* (4.27)-(4.29) *there holds the estimate*

$$\|U_k - u\|_{\bar{\omega}_h} \le M_0 p_k d + O(h^2).$$

Remark 4.2.4. For the discrete iterative method (4.24) -(4.26) we obtained $O(h^2)$ convergence. It is natural to think about the use of Gauss quadrature formulas to the integrals in (4.25) for higher accuracy but it is impossible because the nodes of Gauss quadrature formulas do not coincide with the grid nodes, where the solution of the problem is computed.

Many examples are given in order to confirm the validity of the obtained theoretical results and the efficiency of the proposed iterative method. Below is a notable example.

Example 4.2.1. Consider the problem

$$\begin{aligned} &u'''(t) = e^t - \frac{1}{4}u(t) + \frac{1}{4}u^2(\frac{t}{2}), \quad 0 < t < 1, \\ &u(0) = 1,\ u'(0) = 1,\ u'(1) = e \end{aligned} \tag{4.30}$$

with the exact solution $u(t) = e^t$.
It can be verified that the conditions in Theorem 4.2.3 are satisfied, therefore the problem has a unique solution. The results of convergence of the discrete iterative method are given in Table 4.1. Here, N is the number of grid points,

Table 4.1: The convergence in Example 4.2.1

N	h^2	K	$Error$
50	4.0000e-04	3	6.1899e-05
100	1.0000e-04	3	1.5475e-05
150	4.4444e-05	3	6.877 -06
200	2.5000e-05	3	3.8688e-06
300	1.1111e-05	3	1.7195e-06
400	6.2500e-06	3	9.6721e-07
500	4.0000e-06	3	6.1901e-07

K is the number of iterations performed until $\|\Psi_k - \Psi_{k-1}\|_{\bar{\omega_h}} \leq 10^{-10}$, $Error = \|U_K - u\|_{\bar{\omega_h}}$.

GENERAL CONCLUSIONS

In this thesis, we have successfully studied the existence, uniqueness of solutions and the iterative methods for solving some nonlinear boundary value problems for some high order differential equations including integro-differential and functional differential equations. The main achievements of the thesis include:

1. The establishment of the existence, uniqueness of solutions and positive solutions for third order nonlinear BVPs and the construction of numerical methods for finding the solutions; The proposal of discrete iterative methods of second and third order accuracy for solving third order nonlinear differential equations.

2. The establishment of the existence, uniqueness of solutions and construction of iterative methods for finding the solutions for nonlinear third and fourth order differential equations with integral boundary conditions.

3. The establishment of the existence, uniqueness of solutions and construction of numerical methods for finding the solutions of nonlinear integro-differential and functional differential equations.

The validity and applicability of the theoretical results and the effectiveness of the constructed iterative methods have been confirmed by many experimental examples.

The methodology throughout the thesis has been shown to be superior to those of many other authors due to its simplicity and coherence and can be applied to a wide range of boundary value problems for differential equations.

A weakness of this methodology is that it is only applicable to problems for differential equations with non-singular right-hand sides. Therefore, the future goals of the thesis are:

1. The further development of the above results for the case of singular right-hand sides and the case of unbounded domains.

2. The construction of iterative methods of higher order accuracy.

3. The study of the problems with nonlinear boundary conditions.

LIST OF THE WORKS OF THE AUTHOR RELATED TO THE DISSERTATION

[AL1] Q. A Dang , Q. L. Dang, A unified approach to fully third order nonlinear boundary value problems, J. Nonlinear Funct. Anal. 2020 (2020), Article ID 9, http://jnfa.mathres.org/archives/2136 (Scopus, Q3).

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