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**SOME ITERATIVE METHODS FOR
THE SPLIT FEASIBILITY AND RELATED
PROBLEMS**

**SUMMARY OF DISSERTATION ON APPLIED
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INTRODUCTION

The fixed point theory of nonexpansive mappings and their extensions play an important role not only in studying the theory of ordinary differential equations, partial differential equations, optimization problems, variational inequalities problem . . . but also in problems directly related to real-life problems such as: convex acceptance problem, multi-set separation and overlap acceptance problem. These problems arise from a number of practical problems such as: image recovery and processing problems, radiotherapy problems . . .

The basic methods for finding fixed points of a non-expansive map are Krasnosel'skii–Mann iterative method, Ishikawa iterative method, Halpern iterative method and the viscosity approximation method. The Krasnosel'skii–Mann iterative and Ishikawa iterative methods are weakly convergent while Halpern' iterative and the viscosity approximation method converge strongly in infinite dimensions space. The combination of these basic methods to obtain better modified methods has also been proposed.

The above methods are also used to approximate the solution for the multiple-sets split feasibility problem, the multiple-sets split equality problem and variational inequalities problem on fixed points for a family of nonexpensive mappings.

The goal of the thesis is to propose some new iterative methods to approximate the solution for the multiple-sets split feasibility problem, the multiple-sets split equality problem and variational inequalities problem over the common fixed points for a family of nonexpensive mappings, overcome some limitations of previous.

Problem 1. Multiple-sets split feasibility problem (MSSFP)

Let H_1 and H_2 be Hilbert spaces with inner products $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $A : H_1 \rightarrow H_2$ be a bounded linear mapping. Let C_i and Q_j be convex closed subsets, respectively, in H_1 và H_2 , with each $i \in J_1$ and $j \in J_2$ where, J_1 and J_2 are sets of indices, which can be finite or countably infinite. The MSSFP is for mulated a finding a point

$$x \in C := \bigcap_{i \in J_1} C_i \text{ such that } Ax \in Q := \bigcap_{j \in J_2} Q_j. \quad (\text{MSSFP})$$

when the sets J_1 and J_2 contain only an element the MSSFP becomes the split feasibility problem (SFP): find $x \in C$ such that $Ax \in Q$. The MSSFP

was first researched by Censor and Elfving

Problem MSSFP in finite-dimensional Hilbert spaces was first introduced by Censor and Elfving nãm 2005 for modeling inverse problems that arise from phase retrievals and in image reconstruction. Recently, it can also be used to model the intensity-modulated radiation therapy.

Denote by Γ the set of solutions for MSSFP. Throughout this paper, we assume that $\Gamma \neq \emptyset$.

For solving MSSFP in the cases that the cardinals of J_1 and J_2 , denoted, respectively, by $|J_1|$ and $|J_2|$, are countably infinite, i.e., $|J_1| = |J_2| = \mathbb{N}_+$, the set of all positive integers, or finite, i.e., $|J_1| = N$ and $|J_2| = M$ where N and M are some positive integers, several iterative methods were introduced by Buong, Takahashi, Xu, Wen, Yao, Wang . . . and references therein.

In the case that N and M are two any positive integers, to solve the MSSFP, Censor et al in { Y. Censor, T. Elfving, N. Knop, T. Bortfeld, The multiple-sets split feasibility problem and its applications for inverse problems. *Inverse Problems*. **21**, 2071-2084 (2005)} proposed an iterative method, based on the gradient projection one. This iterative method used a fixed step size restricted by a Lipschitz constant of a gradient mapping, which depends on $\|A\|$. To avoid the inconvenience of calculating the Lipschitz constant, in 2013, Zhao and Yang introduced a self-adaptive projection method by adopting Armijo-like searches. However, the iterative method needs an inner iteration number to have a suitable step size. Next, Zhao and Yang in { J. Zhao, Q. Yang, A simple projection method for solving the multiple-sets split feasibility problem. *Inverse Problems in Science and Engineering*. **21**(3), 537-546 (2013)} suggested a new self-adaptive way to compute directly the step size in each iteration, without estimating the Lipschitz constant or choosing the inner iteration number. The approach has been presented for the split feasibility problem (SFP), i.e. MSSFP with $N = M = 1$. On the other hand, in 2006, Xu showed that the MSSFP is equivalent to finding a common fixed point of a finite family of averaged mappings and proposed three iteration methods: (i) successive iteration method; (ii) simultaneous iteration method and (iii) cyclic iteration method. These iterative methods also used a fixed step size, which depends on the Lipschitz constant. The last two iterative methods with the self-adaptive step size have been recently studied by Zhao, Yang, Zhang et al in 2012, 2013 All the listed methods above converge weakly in infinite dimensional Hilbert spaces. In order to obtain a strongly convergent sequence from these methods, there exist several ways, one of which is to combine them with regularization methods. For solving the SFP

In 2010, Xu proposed Bruck and Bakushinsky type iterative correction method, defined as follows:

$$z^{k+1} = P_C(I - \gamma_k(A^*(I - P_Q)A + \alpha_k I))z^k, \quad z^1 \in H_1, \quad k \geq 1, \quad (0.1)$$

where, P_C and P_Q is a metric projection of H_1 and H_2 onto C and Q respectively, A^* is a dual mapping of A , positive parameters γ_k and α_k are small enough, such that $0 < \gamma_k \leq \alpha_k / (\|A\|^2 + \alpha_k)$ and $\alpha_k \rightarrow 0$ as $k \rightarrow \infty$. However, choosing parameters γ_k still depends on $\|A\|$. In 2017, Tian and Zhang [Ineq. Appl, 2017] proposed a self-adaptive iterative method for removing the dependence. In this study, γ_k is built as follows: $\gamma_k = \rho_k f(x^k) / \|A^*(I - P_Q)Ax^k\|^2$ vi $\varepsilon < \rho_k < 4 - \varepsilon$, $\varepsilon > 0$ small enough, where $f(x) = \frac{1}{2}\|(I - P_Q)Ax\|^2$, with condition (α) : $\alpha_k \in (0, 1)$ for all $k \geq 1$,

$\lim_{k \rightarrow \infty} \alpha_k = 0$ and $\sum_{k=1}^{\infty} \alpha_k = \infty$. However, proving this result is not completed because $\sum_{k=1}^{\infty} \gamma_k \alpha_k = +\infty$ when $\lim_{k \rightarrow \infty} f(x^k) = 0$ has not been proven.

There are two difficulties in implementing this method:

1. Must calculate infinite sum
2. Must calculate $\|A\|$

In [Acta App. Math, 2019], then difficulty 1 has been resolved by Nguyen Buong and et al.

Therefore, the first goal of the thesis is to provide a new iterative methods to approximate the solution for the MSSFP to overcome the second difficulty.

Problem 2. The multiple-sets split equality problem (MSSEP)

Let H_1, H_2 and H_3 be real Hilbert space with inner products $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let $A : H_1 \rightarrow H_3$ and $B : H_2 \rightarrow H_3$ be two bounded linear mappings. Let J_1, J_2 are sets of indices, $\{C_i\}_{i \in J_1}$ and $\{Q_j\}_{j \in J_2}$ are two families of convex, closed subsets in H_1 and H_2 respectively. The MSSEP is the problem of finding a poin

$$z = [x, y], x \in C := \bigcap_{i \in J_1} C_i \text{ and } y \in Q := \bigcap_{j \in J_2} Q_j \quad (0.2)$$

such that $Ax = By$.

Obviously, if $H_2 = H_3$ and $B = I$, then the MSSEP becomes the MSSFP problem. In particular, if the index sets J_1 and J_2 contains only one element then the MSSEP is the split equality problem, denoted as SEP: In 2013, This problem was first studied by Byrne and Moudafi [Working paper, 2013]. Then, Chen et al studied the problem in the case that $T = G^*G$, [Fixed Point Theory and Applications, 2014] and propose an iterative regularization method:

$$z^{k+1} = P_S(I - \gamma_k(T + \alpha_k I))z^k. \quad (0.3)$$

The second goal of the thesis is to solve this problem.

Problem 3. The variational inequalities problem in Banach space

Let E be a Banach space, $F : E \rightarrow E$ is a nonlinear mapping, C is a convex, closed subset of E . The variational inequalities problem (VIP), with mapping F and constraint set C in Banach space E is stated as follows:

$$\text{Find } p_* \in C \text{ such that } \langle Fp_*, j(p_* - p) \rangle \leq 0 \quad \forall p \in C, \quad (\text{VIP})$$

where, j is dual mapping of E . In this these, we consider the case when $C = \bigcap_{i \geq 1} \text{Fix}(T_i)$ the fixed point set of a nonexpensive mapping T_i defined on E .

When E is Hilbert space, then dual mapping j is unit mapping and then, variational inequalities problem (VIP) will become variational inequalities problem in Hilbert space.

We see that, The Ishikawa iterative method is formally an extension of the Krasnosel'skii–Mann iterative method. The convergene between these two is weak. However, there are examples showing the situation when we use the Ishikawa iteration method, this series of iterations converges to the solution of the problem, but when we use the Krasnosel'ski–Mann iteration method, it does not converge.

Combining the steepest-descent method with Ishikawa iterative one to approximate the solution for a class of variational inequalities in Banach space in order to obtain a strongly convergent sequence is a next research goal in this thesis.

The thesis includes 3 chapters.

Chapter 1: "Preliminaries". In this chapter presents some basic concepts and present some methods to approximate the solution for the fixed point problem, the multiple-sets split feasibility problem, the multiple-sets split equality problem.

Chapter 2: "Iterative regularization methods for approximate solutions of the multiple-sets split feasibility and the multiple-sets split equality problems". In this chapter, the thesis presents two methods to solve goals 1 and 2 above.

Chapter 3: "Steepest-descent Ishikawa iterative methods for a class of variational inequalities". In this chapter, the thesis presents two methods to solve goals 3 above.

Results of the thesis are reported at: XXIII National Conference on selected issues of Information and Communications Technology, Quang Ninh, 5–6/11/2020.

CHAPTER 1. PRELIMINARIES⁵

In this chapter, section 1.1 presents some basic concepts in Hilbert space and in Banach space.

Section 1.2, presents some methods to approximate the solution for the fixed point problem, the multiple-sets split feasibility problem, the multiple-sets split equality problem. These methods all have limitations that cause difficulties during implementation.

- (1) The problem of finding fixed points of a non-expansive mapping, the thesis proposes new method to overcome disadvantages such as weak convergence.
- (2) The multiple-sets split feasibility problem, The thesis presents the solution approximation method of Tian and Zhang has been proposed in 2017. However, the proof of the proposed results has not yet been completed.
- (3) The multiple-sets split equality problem, The thesis presents a method of Chen proposed in 2013. The difficulty of in this method is that it requires infinite summation during implementation. To date, there has been no research to address this issue.

The above issues are one of the reasons that led the author to the research that will be presented in chapters 2 and 3.

Section 1.3 of the thesis presents two practical applications of the above problems in medicine and in digital signal processing and image restoration.

CHAPTER 2. ITERATIVE REGULARIZATION METHODS FOR APPROXIMATE SOLUTIONS OF THE MULTIPLE-SETS SPLIT FEASIBILITY AND THE MULTIPLE-SETS SPLIT EQUALITY PROBLEMS

In this chapter, we propose two iterative regul methods for approximate solutions for the multiple-sets split feasibility and the multiple-sets split equality problems in real, infinite-dimensional Hilbert spaces. These methods strongly converge and has overcome the disadvantages of the methods presented in Chapter 1

The results of the chapter are written based on two scientific articles [CT2] and [CT3] in the List of published works of the thesis author.

2.1. The multiple-sets split feasibility problem

Let H_1 and H_2 be two real Hilbert spaces. The MSSFP is fomulated as follows.

$$\text{Find a point } x \in C := \bigcap_{i \in J_1} C_i \text{ sao cho } Ax \in Q := \bigcap_{j \in J_2} Q_j. \quad (\text{MSSFP})$$

where C_i and Q_j be two closed convex subsets in H_1 and H_2 , respectively and $A : H_1 \rightarrow H_2$ is a bounded linear mapping

2.1.1. The iterative regularization method of Lavrentiev' type

For solving the MSSFP, we introduce the first regularization method of Lavrentiev's type, described as,

$$F^k u^k + \alpha_k (u^k - x^+) = 0, \quad (2.1)$$

where,

$$F^k = I - U^k + A^*(I - V^k)A, \quad (2.2)$$

$$U^k = \frac{1}{\beta^k} \sum_{i=1}^k \beta_i P_{C_i}, \quad T_{\gamma_k, \alpha_k} = I - \gamma_k (A^*(I - V^k)A + \alpha_k I), \quad (2.3)$$

$$V^k = \frac{1}{\eta^k} \sum_{j=1}^k \eta_j P_{Q_j},$$

$x^+ \in H_1$ is a guess point in H_1 , for the parameters $\gamma_k, \alpha_k, \beta_i$ and η_j with $\beta^k = \beta_1 + \dots + \beta_k, \eta^k = \eta_1 + \dots + \eta_k$.

Satisfy the following assumptions:

- (a) $\gamma_k, \alpha_k \in (0, 1)$, $\lim_{k \rightarrow \infty} \gamma_k / \alpha_k = \lim_{k \rightarrow \infty} \alpha_k = 0$, $\alpha_{k+1} < \alpha_k$ and $\sum_{k=1}^{\infty} \gamma_k \alpha_k = \infty$.
- (b) $\lim_{k \rightarrow \infty} \tilde{\alpha}_k / (\gamma_k \alpha_k^2) = 0$ where $\tilde{\alpha}_k = (\alpha_{k-1} / \alpha_k) - 1$;
- (c) $\beta_i > 0$ for all $i \geq 1$ such that $\sum_{i=1}^{\infty} \beta_i = 1$ and $\lim_{k \rightarrow \infty} \beta_k / (\gamma_k \alpha_k^2) = 0$;
- (d) $\eta_j > 0$ for all $j \geq 1$ such that $\sum_{j=1}^{\infty} \eta_j = 1$ and $\lim_{k \rightarrow \infty} \eta_k / (\gamma_k \alpha_k^2) = 0$.

Remark 2.1.1. Examples of sequences, having properties (a)–(d) are: $\gamma_k = 1/(k+1)^a$, $\alpha_k = 1/(k+1)^b$, where $0 < b < a$ with $a + 2b < 1$, and $\eta_i = \beta_i = 1/(i(i+1))$

We have the following results when J_1 and J_2 are countably infinite.

Theorem 2.1.1. *Let H_1 and H_2 be two real Hilbert spaces and let A be a bounded linear mapping from H_1 into H_2 . Let $\{C_i\}_{i \in \mathbb{N}_+}$ and $\{Q_j\}_{j \in \mathbb{N}_+}$ be two infinite families of closed convex subsets in H_1 and H_2 , respectively. Assume that there hold conditions (c) and (d) with rejecting the limits. Then, we have:*

- (i) For each $\alpha_k > 0$, problem (2.1) has a unique solution u^k ;
- (ii) If $\Gamma \neq \emptyset$, where Γ denotes the solution set of the MSSFP, then $\lim_{k \rightarrow \infty} u^k = p_* \in \Gamma$, satisfying

$$\|p_* - x^+\| \leq \|p - x^+\| \quad \forall p \in \Gamma; \quad (2.4)$$

(iii)

$$\|u^k - u^{k-1}\| \leq d_k = \frac{2M_1}{\alpha_k} \left[\frac{\beta_k}{\beta^k} + \tilde{\alpha}_k + \frac{\eta_k}{\eta^k} \right] + \tilde{\alpha}_k (M_1 + \|x^+\|), \quad (2.5)$$

where M_1 is some positive constant.

Remark 1. Obviously, if $\{u^k\}$ converges strongly to some point \tilde{u} , where u^k is the solution of (2.1), and $\alpha_k \rightarrow 0$ as $k \rightarrow +\infty$, then $\Gamma \neq \emptyset$.

Theorem 2.1.2. *Let H_1, H_2, A, C_i and Q_j be as in theorem 2.1.1 with $\Gamma \neq \emptyset$. Assume that there hold conditions (a), (b), (c) and (d). Then, the sequence $\{z^k\}$, defined by*

$$z^{k+1} = (I - \gamma_k(F^k + \alpha_k(I - x^+)))z^k, \quad k \geq 1, \quad (2.6)$$

$z_1 \in H_1$ converges strongly to p_* , satisfying (2.4), where F^k defined by (2.2).

In the case that either one of $|J_1|$ and $|J_2|$ or both they are finite, We obtain the following theorems:

Theorem 2.1.3. *Let H_1, H_2 and A be as in Theorem 2.1.1. Let $\{C_i\}_{i=1}^N$ and $\{Q_j\}_{j \in \mathbb{N}_+}$ be two families of closed convex subsets in H_1 and H_2 , respectively,*

where N is any positive integer Assume that $\Gamma \neq \emptyset$ and there hold conditions (a), (b), (d) and

(c') $\beta_i > 0$ for $1 \leq i \leq N$ such that $\sum_{i=1}^N \beta_i = 1$.

Then, as $k \rightarrow \infty$, the sequence $\{z^k\}$, defined by

$$z^{k+1} = z^k - \gamma_k((I - U)z^k + A^*(I - V^k)Az^k + \alpha_k(z^k - x^+)), k \geq 1, z^1 \in H_1,$$

$$U = \sum_{i=1}^N \beta_i P_{C_i}, \quad V^k = \frac{1}{\eta^k} \sum_{j=1}^k \eta_j P_{Q_j},$$

converges strongly to p_* satisfying (2.4)

Theorem 2.1.4. Let H_1, H_2 and A be as in Theorem 2.1.1. Let $\{C_i\}_{i \in \mathbb{N}_+}$ and $\{Q_j\}_{j=1}^M$ be two families of closed convex subsets in H_1 and H_2 , respectively, where M is a position integer Assume that $\Gamma \neq \emptyset$ and there hold conditions (a), (b), (c) and (d'): $\eta_j > 0$ for $1 \leq j \leq M$ such that $\sum_{j=1}^M \eta_j = 1$.

Then, as $k \rightarrow \infty$, the sequence $\{z^k\}$, defined by

$$z^{k+1} = z^k - \gamma_k((I - U^k)z^k + A^*(I - V)Az^k + \alpha_k(z^k - x^+)), k \geq 1, z^1 \in H_1,$$

$$U^k = \frac{1}{\beta^k} \sum_{i=1}^k \beta_i P_{C_i}, \quad V = \sum_{j=1}^M \eta_j P_{Q_j},$$

converges strongly to p_* satisfying (2.4)

From Theorem 2.1.3 and 2.1.4, we have a result in the case that J_1, J_2 are finite.

Theorem 2.1.5. Let H_1, H_2 and A be as in Theorem 2.1.1. Let $\{C_i\}_{i=1}^N$ and $\{Q_j\}_{j=1}^M$ be two finite families of closed convex subsets in H_1 and H_2 , respectively. Assume that $\Gamma \neq \emptyset$ and there hold conditions (a), (b), (c') and (d'). Then, as $k \rightarrow \infty$, the sequence $\{z^k\}$, defined by

$$z^{k+1} = z^k - \gamma_k((I - U)z^k + A^*(I - V)Az^k + \alpha_k(z^k - x^+)), k \geq 1, z^1 \in H_1,$$

where U and V are defined in Theorems 3.3 and 3.4 respectively, converges strongly to p_* satisfying (2.4).

Remark 2.1.2. (a) In chapter 1 of this thesis two iterative regularization method to solve the (MSSFP) proposed by Xu and el al, has the form

$$z^{k+1} = P_C(I - \gamma_k(A^*(I - P_Q)A + \alpha_k I))z^k, \quad z^1 \in H_1, \quad k \geq 1, \quad (2.7)$$

where $0 < \gamma_k \leq \alpha_k / (\|A\|^2 + \alpha_k)$ at each iteration step depends on the norm off A . Calculating the norm of operator A is difficult, then, there will be difficulties in using the method (2.7).

- (b) Nguyen Buong and el al extended the method (2.7) to solve the MSSFP in the case of index sets J_1 and J_2 are finite:

$$z^{k+1} = U^k T_{\gamma_k, \alpha_k} z^k, \quad (2.8)$$

the parameter γ_k is chosen in dependen on $\|A\|$.

2.1.2. Numerical experiments

We consider MSSFP in real, finite-dimensional Hilbert spaces \mathbb{E}^m and \mathbb{E}^n with $C = \bigcap_{i=1}^{\infty} C_i$ end $Q = \bigcap_{j=1}^{\infty} Q_j$, where,

$$C_i = \left\{ x \in \mathbb{E}^n \mid a_1^i x_1 + a_2^i x_2 + \cdots + a_n^i x_n \leq b_i \right\}, \quad (2.9)$$

with $a_l^i, b_i \in (-\infty; +\infty)$, $1 \leq l \leq n$ end $i \in \mathbb{N}_+$,

$$Q_j = \left\{ y \in \mathbb{E}^m \mid \sum_{l=1}^m (y_l - a_l^j)^2 \leq R_j \right\}, \quad R_j > 0, \quad (2.10)$$

with $a_l^j \in (-\infty; +\infty)$, $1 \leq l \leq m$, $j \in \mathbb{N}_+$ and A is a 3×2 - matrix.

Example 2.1. In the first example, we consider the case $m = n = 2$, A is an identity matrix, with the numbers $a_1^i = 1/i$, $a_2^i = -1$ and $b_i = 0$ for all $i \geq 1$, $R_j = 1$, $a^j = (1/j; 0)$ for all $j \geq 1$ and $x^+ = (0, 0)$. Then, it is not difficult to verify that $x_* = (0; 0)$ is the unique minimum-norm solution of (2.9), (2.10). Since $A = I$, method (2.6) is written in the form

$$z^{k+1} = (1 - \gamma_k(2 + \alpha_k))z^k + \gamma_k(U^k z^k + V^k z^k). \quad (2.11)$$

Using method (2.11) with

$$\beta_i = \eta_i = 1/(i(i+1)), \quad \alpha_k = 1/(k+1)^{1/8}, \quad \gamma_k = 1/(k+1)^{1/2}$$

and a starting point $x^1 = (-3.0; 3.0)$, we obtain the following table of numerical results in Table 2.1.

Example 2.2. In the second example, we save $C_i, \beta_i, \eta_j, \gamma_k, \alpha_k$ and the starting point x^1 as in example 2.1. Where, thesis consider the case when $Q_j = \{y \in \mathbb{E}^3 : \|y - a^j\| \leq 1\}$ where $a^j = (1/(j+1); 1/(j+1); 1/(j+1))$ and A is a 3×2 -matrix with elements $a_{i1} = 1$, for $i = 1, 2, 3$, and zero for the others. Clearly, $x_* = (0; 0)$ is the unique minimum norm solution. The computational results, by using the method (2.6), are presented in the following numerical table, Table 2.2.

Table 2.1: numerical results of example 2.1 using method (2.11)

k	z_1^{k+1}	z_2^{k+1}	k	z_1^{k+1}	z_2^{k+1}
10	-0.0072951049	-0.0123790731	60	-0.0000003749	-0.0000006109
20	-0.0003743916	-0.0006192857	70	-0.0000001070	-0.0000001742
30	-0.0000421668	-0.0000692232	80	-0.0000000337	-0.0000000548
40	-0.0000070196	-0.0000114815	90	-0.0000000115	-0.0000000187
50	-0.0000014904	-0.0000014325	100	-0.0000000042	-0.0000000068

Table 2.2: numerical results of example 2.2 using method (2.6)

k	z_1^{k+1}	z_2^{k+1}	k	z_1^{k+1}	z_2^{k+1}
10	-0.0067281333	-0.0450293607	60	-0.0000189750	-0.0000189751
20	-0.0025241606	-0.0026043616	70	-0.0000078161	-0.0000078161
30	-0.0005405073	-0.0005415133	80	-0.0000034561	-0.0000034561
40	-0.0001513849	-0.0001514139	90	-0.0000016184	-0.0000016184
50	-0.0000504382	-0.0000504396	100	-0.0000007947	-0.0000007947

Remark 2.1.3. Assum, In case $m = n = 2$ and A is norm matrix, method (2.8) proposed by Buong et al, while projectors difined as (2.3) is difined by

$$x^{k+1} = U^k((1 - \gamma_k(1 + \alpha_k))x^k + \gamma_k V^k x^k). \quad (2.12)$$

Using the method (2.12), where $\gamma_k = 1/(1.05 + (1/k))$, $\alpha_k = 1/k$ condition (α) and the above datas, we have the results in table 2.3 and table 2.4

Table 2.3: numerical results of example 2.1 using the method (2.12)

k	x_1^{k+1}	x_2^{k+1}	k	x_1^{k+1}	x_2^{k+1}
1	0.0243902439	0.3658536585	100	0.0012390505	0.0083945251
10	0.0102553274	0.0694794968	500	0.0002695347	0.0018260888
20	0.0055344982	0.0374960376	1000	0.0001394192	0.0009445606
30	0.0038180428	0.0258671112	2000	0.0000720824	0.0004883558
40	0.0029249862	0.0198166827	3000	0.0000489994	0.0000331969

Put results illustrated in two Table 2.1, 2.2 and Table 2.3, 2.4, We see that both proposed theoretical methods are effective. Further, regularization methods in this thesis converge faster than results of Buong and et al in [Acta Appl. Math, 2019].

Table 2.4: numerical results of example 2.2 using the method (2.8)

k	x_1^{k+1}	x_2^{k+1}	k	x_1^{k+1}	x_2^{k+1}
1	0.6019388274	1.5365833659	100	0.0142047415	0.0363009852
10	0.1176994981	0.3004546610	500	0.0030934268	0.0078966734
20	0.0635189516	0.1621465290	1000	0.0016001024	0.0040846244
30	0.0438193443	0.1118588139	2000	0.0008272834	0.0021118284
40	0.0356981140	0.0856945566	3000	0.0005623615	0.0014355553

Example 2.3. Now, in the case that

$$\begin{aligned} a_{11} &= 0.1, a_{12} = 0.2, a_{21} = 0.2, \\ a_{22} &= 0.4, a_{31} = a_{32} = 0, \end{aligned}$$

example 2.2, considered above, has many solutions (MS), containing the zero point, as the minimal norm solution because $x^+ = 0$. The numerical results, calculated by (2.11) and (2.2) with the same data, are described in Table 2.5.

k	z_1^{k+1}	z_2^{k+1}	k	z_1^{k+1}	z_2^{k+1}
10	-0.0420263650	-0.0420003267	60	-0.0000380359	-0.0000380359
20	-0.0051177476	-0.0051176931	70	-0.0000156676	-0.0000156667
30	-0.0010841787	-0.0010841781	80	-0.0000069279	-0.0000069279
40	-0.0003034753	-0.0003034753	90	-0.0000032440	-0.0000032440
50	-0.0001011055	-0.0001011055	100	-0.0000015929	-0.0000012929

Table 2.5. Computational results by method (2.11) and (2.2), MS

Tables 2.2 and 2.5 show that, for the considered example with a unique solution or many solutions, method (2.11) and (2.2) converges well and, in the case that the problem has a unique solution, the method works a little better than in the other case.

2.2. The multiple-sets split equality problem

Let H_1, H_2 are Hilbert spaces; C_i and Q_j are two closed convex subsets in H_1 and H_2 , respectively.

We consider the MSSEP: find a point

$$x \in C := \bigcap_{i \in J_1} C_i \text{ and } y \in Q := \bigcap_{j \in J_2} Q_j \text{ such that } Ax = By. \quad (\text{MSSEP})$$

where $A : H_1 \rightarrow H_3$; $B : H_2 \rightarrow H_3$ are bounded, linear mappings.

Denote by Ω the set of solutions for Ω . Throughout this thesis, assume that $\Omega \neq \emptyset$.

2.2.1. The regularization method of Bakushinsky–Bruck' type

Extend iterative regularization method of Bakushinsky [Comput. Math. and Math. Physics., 2011] and Bruck [J. Math. Anal. Appl., 1974], thesis proposes a new iterative regularization method for the MSSEP in infinite dimensional Hilbert spaces. Start from an arbitrary initial approximation $z^1 \in H$, the next approximations is determined by:

$$z^{k+1} = U_k T_{\gamma_k, t_k} z^k, \quad (2.13)$$

where

$$U_k = \frac{1}{\tilde{\beta}_k} \sum_{i=1}^k \beta_i P_{S_i}, \quad T_{\gamma_k, t_k} = I - \gamma_k [G^* G + t_k I], \quad (2.14)$$

γ_k, t_k, β_i are positive number and $\tilde{\beta}_k = \beta_1 + \dots + \beta_k$.

Remark 2.2.1. In this new proposal, at each iteration step of which only a finite number of sets from the families is used. As the consequence, this result is better than some previously proposed methods

Let's assume that parameters γ_k, t_k, β_i satisfy the conditions

(t) $t_k \in (0, 1)$ for all k , $\lim_{k \rightarrow \infty} t_k = 0$ and $\sum_{k=1}^{\infty} t_k = \infty$.

(β) $\beta_i > 0$ for all i and $\sum_{i=1}^{\infty} \beta_i = 1$.

(γ) $\gamma_k \in (0, 2/(\|A\|^2 + t_k))$, $\liminf_{k \rightarrow \infty} \gamma_k > 0$ and $\lim_{k \rightarrow \infty} (\gamma_{k+1} - \gamma_k) = 0$.

Lemma 2.2.1. Let H_1, H_2 and H_3 be three real Hilbert and let $A : H_1 \rightarrow H_3$ and $B : H_2 \rightarrow H_3$ be bounded linear mappings. Then, for a fixed number $\gamma \in (0, 2/(\|G\|^2 + 2\alpha))$, where, $G = [A - B] : H = H_1 \times H_2 \rightarrow H$, and α is a number in $(0, 1)$, the mapping $T_{\gamma, t} := I - \gamma[G^*G + tI]$ is a contraction with coefficient $1 - \gamma t$, $t \in (0, 1)$. When $t = 0$, then $T_\gamma := I - \gamma G^*G$ is nonexpansive.

Lemma 2.2.2. Let H is Hilbert space and let G be a bounded linear mapping on H . Then, $\text{Zer}G := \{z \in H \mid Gz = 0\} = \text{Fix}(T_\gamma)$ where T_γ is defined in Lemma 2.2.1 for any positive real number γ .

Lemma 2.2.3. The solution set Ω of MSSEP equals to the solution set of variational inequalities

$$\text{Find } z_* \in S \text{ such that } \langle Tz_*, z - z_* \rangle \geq 0 \quad \forall z \in S, \quad (\text{VIP})$$

with $T = G^*G$.

Theorem 2.2.1. *Let H_1, H_2, H_3, A and B be as in Lemma 2.2.1. Let C_i and Q_j , for each $i \in J_1$ and each $j \in J_2$ with $J_1 = J_2 = \mathbb{N}_+$, be closed convex subsets in H_1 and H_2 , respectively. Assume that there hold conditions (γ) , (β) and (t) . Then, Then, the sequence $\{z^k\}$, defined by (2.13) and (2.14), as $k \rightarrow \infty$, converges strongly to a MSSEP*

In the case that either one or both the sets J_1 and J_2 are finite, we obtain the following results.

Theorem 2.2.2. *Let H_1, H_2, H_3, A, B, C_i and Q_j with $J_1 = \{1, \dots, N\}$, $J_2 = \{1, \dots, M\}$ and $N < M$ be as in Theorem 2.2.1. Assume that there hold conditions (γ) and (t) . Then, as $k \rightarrow \infty$, the sequence $\{z^k\}$, defined by*

$$z^{k+1} = UT_{\gamma_k, t_k} z^k, k \geq 1, z^1 \in H, U = \sum_{i=1}^M \beta_i P_{S_i},$$

converges to a solution of MSSEP when $k \rightarrow \infty$, where, $C_i = C_N$, $i = N + 1, \dots, M$, $\beta_i > 0$ and $\sum_{i=1}^M \beta_i = 1$.

In the case that only J_1 is finite, $J_2 = \mathbb{N}_+$, by setting $C_i = C_N$, $i = N + 1, \dots, \infty$, we return to the case in Theorem 2.2.1. In the case that only J_2 is finite, is similar.

Remark 2.2.2. (a) We can express method (2.13) in terms of x and y as follows: for any starting point $x^1 \in H_1$ and $y^1 \in H_2$,

$$\begin{cases} v^k = Ax^k - By^k, \\ x^{k+1} = \tilde{U}_k((1 - \gamma_k t_k)x^k - \gamma_k A^* v^k), \\ y^{k+1} = \tilde{V}_k((1 - \gamma_k t_k)y^k + \gamma_k B^* v^k), \end{cases} \quad (2.15)$$

where \tilde{U}_k is defined in (2.3) and $\tilde{V}_k = (1/\tilde{\beta}_k) \sum_{i=1}^k \beta_i P_{Q_i}$.

(b) We will use method (2.15) with $H_3 = H_2$ and $B = I$ for MSSFP with $J_1 = J_2 = \mathbb{N}_+$, we get a new iterative regularization method: for any starting point $x^1 \in H_1$ and $y^1 \in H_2$,

$$\begin{cases} v^k = Ax^k - y^k, \\ x^{k+1} = \tilde{U}_k((1 - \gamma_k t_k)x^k - \gamma_k A^* v^k), \\ y^{k+1} = \tilde{V}_k((1 - \gamma_k t_k)y^k + \gamma_k v^k). \end{cases} \quad (2.16)$$

Under conditions (γ) , (β) and (t) , the sequences $\{x^k\}$ defined by (2.16) converge strongly to x_* , solving the MSSFP when $k \rightarrow \infty$.

Clearly, method (2.16) is different from (2.8) with projector defined by (2.3).

- (c) Use iterative regularization method (2.16) for SFP, we also see that this method is completely different from Yao's method et al the problems published in 2012.

2.2.2. Numerical experiments

We consider MSSEP with $C = \cap_{i=1}^{\infty} C_i$ and $Q = \cap_{j=1}^{\infty} Q_j$, where

$$C_i = \{x \in \mathbb{E}^n : \tilde{a}_1^i x_1 + \tilde{a}_2^i x_2 + \cdots + \tilde{a}_n^i x_n \leq b_i\},$$

$\tilde{a}_j^i, b_i \in (-\infty; +\infty)$, for $1 \leq j \leq n$ and $i \in \mathbb{N}_+$,

$$Q_j = \{y \in \mathbb{E}^m : \sum_{l=1}^m (y_l - \bar{a}_l^j)^2 \leq r_j\}, \quad r_j > 0,$$

$\bar{a}_l^j \in (-\infty; +\infty)$, for $1 \leq l \leq m$ and $j \in \mathbb{N}_+$, A and B are $p \times n$ - and $p \times m$ -matrices, respectively.

For computation, we consider the case $H_1 = \mathbb{E}^2$, $H_2 = \mathbb{E}^3$ and $H_3 = \mathbb{E}^4$ with $\tilde{a}_1^i = 1/i$, $\tilde{a}_2^i = -1$ and $b_i = 0$ for all $i \geq 1$; $r_j = 1$ and $\bar{a}^j = (1/(j+1); 1/(j+1); 1/(j+1))$ for all $j \geq 1$; and

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Then, it is not difficult to verify that $z_* = [x_*, y_*]$, where $x_* = (0; 0)$ and $y_* = (0; 0; 0)$, is the unique minimum-norm solution of MSSEP problem with the above data. Using method (2.57) with $\gamma_k = 0.05 + 0.05/k$, $\beta_i = 1/(i(i+1))$, $\alpha_k = 1/k$ and a starting point $z^1 = [x^1, y^1]$ where $x^1 = (-2.0; -2.0)$ and $y^1 = (-2.0; -2.0; -2.0)$, we obtain the (2.6) numerical table of the values $\|z^k - z_*\| = \sqrt{\|x^k - x_*\|^2 + \|y^k - y_*\|^2}$.

k	$\ z^{k+1} - z_*\ $	k	$\ z^{k+1} - z_*\ $
10	0.0126467309	100	0.0000712257
20	0.0011458555	200	0.0000058032
30	0.0006146665	300	0.0000005387
40	0.0004108535	400	0.0000000527
50	0.0002940086	500	0.0000000053

Conclusions

In this chapter, we suggest new iterative regularization methods for solving the MSSFP and MSSEP. These methods delete some difficulties, that exist in the literature such as the iterative parameter is chosen in dependence

of $\|A\|$ or the methods do not contain infinite sums. We also give some numerical examples for illustration.

CHAPTER 3. STEEPEST-DESCENT ISHIKAWA ITERATIVE METHODS FOR A CLASS OF VARIATIONAL INEQUALITIES

In this chapter, we propose an iterative method to approximate a solution of a class of variational inequalities in the case that the feasibility set is the set of common fixed points of an infinite family of nonexpansive mappings on a Banach space E and the involving mapping is η strong accretive and l -Lipschitz continuous on E .

The results of the chapter are written based on two scientific articles [CT1] in the List of published works of the thesis author.

3.1. Steepest-descent Ishikawa iterative methods for a class of variational inequalities in Banach space

The class of variational inequalities in Banach space E , considered in this chapter, is to find $p_* \in C$ such that

$$\langle Fp_*, j(p_* - p) \rangle \leq 0 \quad \forall p \in C, \quad (3.1)$$

where, $\langle x, x^* \rangle$ is used instead $x^*(x)$ with $x \in E$, $x^* \in E^*$ and j is the normalized, dual mapping of E , F is an η strong accretive and l -Lipschitz continuous mapping on E

3.1.1. Variational inequalities problem on the fixed point set of a nonexpansive mapping

When $C := \text{Fix}(T)$ where T is a nonexpansive mapping on E , problem (3.1) becomes

$$\text{find } p_* \in \text{Fix}(T) \text{ such that } \langle Fp_*, j(p_* - p) \rangle \leq 0 \quad \forall p \in \text{Fix}(T), \quad (3.2)$$

where E is an either uniformly smooth or strictly convex reflexive Banach space with a uniformly Gâteaux differentiable norm, the mapping $F : E \rightarrow E$ is η - j -strongly accretive and γ -strictly pseudocontractive mapping on E . For finding a solution of class variational inequalities (3.2), in this thesis we give a combination of steepest-descent method with Ishikawa one. The iterative algorithm is built as follows: with arbitrary initial point $x^1 \in E$, The next iterate is determined by

$$x^{k+1} = (I - t_k F)T^k x^k, \quad k \geq 1, \quad (3.3)$$

where,

$$T^k = (1 - \beta_k)I + \beta_k T [(1 - \alpha_k)I + \alpha_k T], \quad k \geq 1, \quad (3.4)$$

and parameters t_k , β_k and α_k satisfy the conditions:

(t) $t_k \in (0, 1)$, $\lim_{k \rightarrow \infty} t_k = 0$ và $\sum_{k=1}^{\infty} t_k = \infty$.

(β) $\beta_k \in [a, b] \subset (0, 1)$ for all $k \geq 1$.

(α) $\alpha_k \in [0, \bar{a}]$, with $\bar{a} \in (0, 1)$, $\forall k \geq 1$ và $\alpha_k \rightarrow 0$ when $k \rightarrow \infty$.

The following theorem confirms the strong convergence of iterative sequence (3.3) and is fully proven in the thesis.

Theorem 3.1.1. *Cho $F : E \rightarrow E$ be an η -strongly accretive and γ -strictly pseudocontractive mapping on an either uniformly smooth or real reflexive and strictly convex Banach space E , having a uniformly Gâteaux differentiable norm, such that $\eta + \gamma > 1$ and $T : E \rightarrow E$ be a nonexpansive mapping on E with $\text{Fix}(T) \neq \emptyset$. Assume that t_k, β_k and α_k satisfy conditions (t), (β) and (α), respectively. Then, the sequence $\{x^k\}$, defined by (3.3) with T^k in (3.4) converges strongly to p_* , solving (3.2).*

Remark 3.1.1. (a) Theorem 3.1.1 has still value for the following method:
 $y^1 \in E$ is any element and

$$y^{k+1} = T^k(I - t_k F)y^k, \quad k \geq 1, \quad (3.5)$$

with the same conditions on E, F, T, t_k, β_k and α_k .

(b) We take $F = I - f$ with $f = a'I$ for a fixed number $a' \in (0, 1)$. Then, F is an η -strongly accretive and γ -strictly pseudocontractive mapping on E with some positive numbers η and γ such that $\eta + \gamma > 1$.

Replace F by $I - f = (1 - a')I$ in (3.3), we get the following algorithm:

$$x^{k+1} = (1 - t'_k)T^k x^k, \quad k \geq 1, \quad (3.6)$$

where $t'_k = t_k(1 - a')$.

Theorem 3.1.2. *Let T be a nonexpansive mapping on an either uniformly smooth or strictly convex reflexive Banach space E with a uniformly Gâteaux differentiable norm. Assume that t_k, β_k and α_k satisfy conditions (t), (β) and (α), respectively. Fix a real number $a' \in (0, 1)$. Then, the sequence $\{x^k\}$, generated by (3.6), converges strongly to a point in $\text{Fix}(T)$.*

Remark 3.1.2. (a) Next, we consider the case, when T is a nonexpansive mapping on a closed and convex subset Q of E . Clearly, with the starting point $x^1 \in Q$, for any point $x^k \in Q$, $T^k x^k \in Q$. Thus, if the set Q contains the original point of E then $x^{k+1} \in Q$, because $x^{k+1} = \tau_k T^k x^k$ with $\tau_k = 1 - t'_k \in (0, 1)$. It means that method (3.6) is well defined for any $x^1 \in Q$, and hence, Theorem 3.1.2 has value in this case. In the case that the set Q does not contain the original point of E , we

take $f = a'I + (1 - a')u$ with a fixed $u \in Q$. It is easy to see that $F = I - f$ is also η -strongly accretive and γ -strictly pseudocontractive such that $\eta + \gamma > 1$. Then, instead of (3.6), we obtain the Halpern Ishikawa method,

$$\begin{cases} x^1 \in Q, \text{ any element,} \\ x^{k+1} = t'_k u + (1 - t'_k)T^k x^k, \quad k \geq 1, \end{cases} \quad (3.7)$$

that is method' Qin et al [J. Math. Anal. Appl., 2008] with re-denoting $t_k := t'_k$. Clearly, t_k satisfies condition (t) if and only if t'_k is so. Method (3.7), by Theorem 3.1.2, converges strongly in a uniformly smooth or strictly convex reflexive Banach space E , meantime, method of Quin above needs stronger conditions

$$\sum_{k=1}^{\infty} |t_{k+1} - t_k| < \infty, \sum_{k=1}^{\infty} |\alpha_{k+1} - \alpha_k| < \infty, \sum_{k=1}^{\infty} |\beta_{k+1} - \beta_k| < \infty. \quad (3.8)$$

compared to the method proposed in the thesis

- (b) Let $\tilde{a} > 1$ and let f be an \tilde{a} -co-coercive accretive mapping on E , i.e.,

$$\langle fx - fy, j(x - y) \rangle \geq \tilde{a} \|fx - fy\|^2, \quad \forall x, y \in E.$$

It is easily seen that f is a contraction with constant $1/\tilde{a} \in (0, 1)$, and hence, $F := I - f$ is an η -strongly accretive mapping with $\eta = 1 - (1/\tilde{a})$. Moreover,

$$\begin{aligned} \langle Fx - Fy, j(x - y) \rangle &= \|x - y\|^2 - \langle fx - fy, j(x - y) \rangle \\ &\leq \|x - y\|^2 - \tilde{a} \|fx - fy\|^2 \\ &\leq \|x - y\|^2 - \gamma \|(I - F)x - (I - F)y\|^2, \end{aligned}$$

for any $\gamma \in (0, \tilde{a}]$. Taking any fixed $\gamma \in ((1/\tilde{a}), \tilde{a}]$ we get that F is a γ -strictly pseudocontractive mapping with $\eta + \gamma > 1$. Next, by replacing F by $I - f$ in (3.5), the thesis obtain a new viscosity approximation Ishikawa method,:

$$y^{k+1} = T^k(t_k f y^k + (1 - t_k)y^k), \quad y^1 \in E, \quad k \geq 1, \quad (3.9)$$

that is an improved modification of Quin' method and different from (3.8). Obviously, if f is an \tilde{a} -co-coercive accretive mapping on Q , a closed convex subset of E , then method (3.9) is also well defined for any $y^1 \in Q$.

For a given α -co-coercive accretive mapping f , we can obtain an \tilde{a} -co-coercive accretive mapping \tilde{f} with $\tilde{a} > 1$ by considering $\tilde{f} := \beta f$ with a positive number $\beta < \alpha$.

3.1.2. Variational inequalities problem on the set of common fixed points of an infinite family of nonexpansive mappings

In this section, consider (3.1) in the case that $C = \bigcap_{i \geq 1} \text{Fix}(T_i) \neq \emptyset$, where $\{T_i\}$ is an infinite family of nonexpansive mappings on E , it means that:

$$\text{find } p_* \in \bigcap_{i \geq 1} \text{Fix}(T_i) \text{ such that } \langle Fp_*, j(p_* - p) \rangle \leq 0 \quad \forall p \in \bigcap_{i \geq 1} \text{Fix}(T_i). \quad (3.10)$$

Let T^k be defined as follows:

$$T^k = (1 - \beta_k)I + \beta_k W^k ((1 - \alpha_k)I + \alpha_k W^k), \quad (3.11)$$

where $\{W^k\}$ is a sequence of nonexpansive mappings that satisfies the conditions:

- (i) Exist $Wx := \lim_{k \rightarrow \infty} W^k x$ for all $x \in E$ and if $\bigcap_{i \geq 1} \text{Fix}(T_i) \neq \emptyset$ then $\text{Fix}(W) = \bigcap_{i \geq 1} \text{Fix}(T_i)$.
- (ii) $\lim_{k \rightarrow \infty} \sup_{x \in B} \|W^k x - Wx\| = 0$ with B is a bounded subset.

Remark 3.1.3. we see that $S^k = \sum_{i=1}^k \gamma_i T_i / \tilde{\gamma}_k$ with $\tilde{\gamma}_k = \gamma_1 + \dots + \gamma_k$ and $V^k = T'_1 \dots T'_k$ where, $T'_i = \gamma_i I + (1 - \gamma_i)T_i$ vi $\gamma_i \in (0, \infty)$ such that $\sum_{i=1}^{\infty} \gamma_i = \tilde{\gamma} < \infty$ also satisfies the conditions (i) and (ii) as in W^k .

In this thesis we propose a new method to solve the problem (3.10). That is the combination of steepest-descent method with Ishikawa method. We show that One of the special cases of the newly proposed method is the Halpern iterative method.

Theorem 3.1.3. *Let F is η - j -co-coercive accretive and γ -strictly pseudo-contractive mapping on an either uniformly smooth or strictly convex reflexive Banach space E with a uniformly Gâteaux differentiable norm, such that $\eta + \gamma > 1$ and $\{T_i\}$ be an infinite family of nonexpansive mappings on E such that $\bigcap_{i \geq 1} \text{Fix}(T_i) \neq \emptyset$. Assuming that t_k, β_k and α_k satisfy the respective conditions (t), (β) and (α). Then, sequence $\{x^k\}$, determined by (3.3) with T^k in (3.11), is strongly convergence to solution p_* of problem (3.10).*

Remark 3.1.4. (a) Remarks 3.1.1 and 3.1.2 have still value when T^k is defined by (3.11).

- (b) Taking $\alpha_k = 0$ in (3.3) and (3.11), we obtain the steepest-descent Krasnoselskii-Mann method and its extension to an infinite family of nonexpansive mappings T_i on E , that is the method

$$x^{k+1} = (I - t_k F)((1 - \beta_k)I + \beta_k W^k)x^k, \quad k \geq 1,$$

and its equivalent formula is

$$x^{k+1} = ((1 - \beta_k)I + \beta_k W^k)(I - t_k F)x^k, \quad k \geq 1, \quad (3.12)$$

(See remark 3.1.1). Replacing F in (3.12) by $(1 - a')I$, we get the method

$$y^{k+1} = ((1 - \beta_k)I + \beta_k W^k)(1 - t'_k)y^k, \quad k \geq 1.$$

strong convergence of which was proved by Shehu in [Taiwanese J. Math., 2015] in uniformly convex and uniformly smooth Banach spaces under conditions (t), (β), $\sum_{k=1}^{\infty} \limsup_{x \in B} \|W^{k+1}x - W^kx\| = 0$ and (i) in the definition of W^k . Marino and Muglia [Optim. Lett., 2015] replacing (ii) in the definition of W^k by $\lim_{k \rightarrow \infty} \|W^{k+1}x - W^kx\| = 0$ uniformly in $x \in B$ and combining the steepest-descent method with the Krasnosel'skii-Mann one, studied the methods

$$\begin{aligned} x^{k+1} &= \beta_k x^k + (1 - \beta_k)(I - t_k D)W^k x^k \quad \text{v\`a} \\ x^{k+1} &= \beta_k (I - t_k D)x^k + (1 - \beta_k)W^k x^k, \quad k \geq 1, \end{aligned} \quad (3.13)$$

in a setting Hilbert space H , where D is η -strongly monotone and L -Lipschitz continuous. Strong convergence of (3.13) is proved under conditions (t) with $\lim_{k \rightarrow \infty} |t_k - t_{k+1}|/t_{k+1} = 0$, $\beta_k \in (0, \bar{\alpha}]$ with $\lim_{k \rightarrow \infty} |\beta_k - \beta_{k+1}|/\beta_{k+1} = 0$ and additional condition on constructing W^k from the given family $\{T_i\}$. We note that the mappings $V^k = T'_1 \cdots T'_k$ where $T'_i = \gamma_i I + (1 - \gamma_i)T_i$ with $\gamma_i \in (0, \infty)$ such that $\sum_{i=1}^{\infty} \gamma_i = \tilde{\gamma} < \infty$ and $S^k = \sum_{i=1}^k \gamma_i T_i / \tilde{\gamma}_k$ with $\tilde{\gamma}_k = \gamma_1 + \cdots + \gamma_k$ also satisfy conditions (i) and (ii) in the definition of W^k give in the proposals of Buong and et al. the first author et al. introduced the methods,

$$\begin{aligned} x^{k+1} &= (1 - \beta_k)x^k + \beta_k S^k (I - t_k F)x^k \quad \text{v\`a} \\ x^{k+1} &= (1 - \beta_k)S^k x^k + \beta_k (I - t_k F)x^k, \end{aligned}$$

strong convergence of which have been investigated in strictly convex reflexive Banach spaces with a G\^ateaux differentiable norm under conditions (t) and (β).

- (c) In 2012, Li studied also method [fixed point Theory 2012], where T^k defined in (3.11) with W^k -mapping of Shimoji and Takahashi Katchang and Kumam proposed the method:

$$x^{k+1} = t_k \gamma f(x^k) + (I - t_k A)T^k x^k, \quad k \geq 1,$$

Table 3.1: Computational results by (3.24) and (3.27) with $W^k = T_k$.

k	x_1^{k+1}	x_2^{k+1}	k	x_1^{k+1}	x_2^{k+1}
10	1.1363636364	0.6411155490	100	1.0148514851	0.9431215161
20	1.0714285714	0.7700827178	200	1.0074626866	0.9707901594
30	1.0483870968	0.8326554114	300	1.0049833887	0.9803526365
40	1.0365853659	0.8687796127	400	1.0037406484	0.9851987678
50	1.0294117647	0.8921748170	500	1.0029940120	0.9881273689

a modification of the method of Li above and proved that it converges in the Banach space with a weak continuous duality mapping j under conditions (t), $\lim_{k \rightarrow \infty} \beta_k = 0$ and $\lim_{k \rightarrow \infty} \alpha_k = 0$, where A is a strongly positive bounded linear mapping on E and γ is a some positive constant.

3.2. Numerical experiments

Obviously, for the family of nonexpansive mappings $T_i = (1 - 1/(i+1))I$ with $E = \mathbb{R}^1$, we have that $\cap_{i \geq 1} Fix(T_i) = \{0\}$ and $\lim_{k \rightarrow \infty} T_k x = Ix$ for each $x \in \mathbb{R}^1$. Thus, condition (i) in the definition of W^k is not satisfied, because $Fix(I) = \mathbb{R}^1$.

It is easy to see that the family $\{T_i = P_{C_i}\}$, where P_{C_i} is the metric projection of $H = \mathbb{E}^2$, an Euclidian space, onto the set $C_i = \{x = (x_1, x_2) \in H : a_i \leq x_2 \leq b_i\}$ with $a_i = 1 - 1/(i+1)$ and $b_i = 2 + 1/(i+1)$ for all $i \geq 1$, satisfies conditions (i) and (ii) in the definition of W^k . In this case, we have that $C = \cap_{i=1}^{\infty} C_i = \{x \in \mathbb{E}^2 : 1 \leq x_2 \leq 2\}$ and we can take $W^k = T_k$ for all $k \geq 1$. Taking $u = (1.0; 0.0)$, we have that the solution of (3.26) $p_* = (1.0; 1.0)$. The computational results by method (3.24) and T^k in (3.27) with starting point $x^1 = (2.5; 2.5)$, $t_k = 1/(k+1)$, $\beta_k = 0.2 + 1/(k+1)$ and $\alpha_k = 1/(k+1)$ are given in Table 3.1.

In the case that $a_i = 1 + 1/(i+1)$, we have $C = \{x \in \mathbb{E}^2 : 1.5 \leq x_2 \leq 2\}$ and $p_* = (1.0; 1.5)$. Moreover, condition (i) in the definition of W^k for T_k , i.e. $W^k = T_k$, does not hold. For computation by (3.24), we use $W^k = S^k$ in (3.27) where $S^k = \sum_{i=1}^k \gamma_i T_i / \tilde{\gamma}_k$ vi $\tilde{\gamma}_k = \gamma_1 + \dots + \gamma_k$ vi $\gamma_i = 1/i(i+1)$. with $\gamma_i = 1/i(i+1)$. The results of computation are given in Table 3.2.

The numerical results show the effectiveness of the method.

Conclude

Chapter 3 of the Thesis proposes a method to solve the problem of variational inequalities on a fixed set of points of one or a family of non-

Table 3.2: Computational results by (3.24) and (3.27) with $W^k = T_k$.

k	x_1^{k+1}	x_2^{k+1}	k	x_1^{k+1}	x_2^{k+1}
10	0.8226906920	0.9967100188	100	0.8216765320	1.3503455533
20	0.8116106625	1.1196844726	200	0.8261485102	1.4207098495
30	0.8123975068	1.1852032060	300	0.8280615950	1.4464230799
40	0.8142620005	1.2298614455	400	0.8291386059	1.4595495405
50	0.8160321266	1.2628985966	500	0.8298349294	1.4675113528

expanded mappings, proves the strong convergence of the proposed method. The chapter proposed new results are extensions of known results with simpler conditions and give strong convergence results, The convergence speed of the iterative methods has also been illustrated with a clear numerical example.

CONCLUSION AND ²³FUTURE RESEARCH DIRECTIONS

CONCLUSION

The thesis has achieved the following results:

1. Propose iterative regularization methods to approximate a solution for the multiple-sets split feasibility problem in real Hilbert spaces, that converges strong under some conditions an parameter and give some numerical examples (see [CT2] in List of published projects). The effectiveness of the proposed method is that the iterative parameter γ_k is selected independent on $\|A\|$.
2. Suggest iterative regularization methods to approximate a solution for the multiple-sets split equality problem in real Hilbert spaces, strong convergence of the method is proved with numerical experemnts for illustration (see [CT3] in List of published projects). In this method, we use only finite summ elements, that is different from the existence in the literatere.
3. We introduce a new iterative method, a combination of the steepest-descent method with the Ishikawa one for solving a variational inequalities over the set of common fixed point of an infinite family of nonexpansive mappings and give numerical example for illustration (see [CT1] in List of published projects).

FURTHER RESEARCH DIRECTIONS

In the next stage:

1. We extend the results in Chapters 2 and 3 to the case T_i, U_i is a pseudo contraction mapping in Hilbert space.
2. Research on iterative correction methods Extragradient' type for a class of variational inequalities with F is η -strong monotony and L -Lipschitz continuous.
3. Research on the combination of inertial components and iterative regularization methods to increase the convergence speed of this method.

LIST OF THE PUBLICATIONS RELATED TO THE DISSERTATION

1. Buong Ng., Anh Ng.T.Q., Binh K.T., (2020), Steepest-Descent Ishikawa Iterative Methods for a Class of Variational Inequalities in Banach Spaces, *Filomat*, (2020), 34 (5), 1557–1569. (SCI-E, Q2).
2. Buong Ng., Hoai P.T.T, Binh K.T, (2020), New Iterative regularization methods for the multiple-sets split feasibility problem, *Journal of Computational and Applied Mathematics* 388(3), 113291. DOI 10-1016/j cam 2020. (SCI, Q2).
- 3 . Buong Ng., Anh Ng.T.Q., Binh, K.T., 2020, Iterative methods for the multiple-sets split equality problem in Hilbert spaces, *Proceedings of the 23th National Conference:Some selected issues of Information and Communications Technology -- Quang Ninh*, 5--6/11/2020, 151-157