MINISTRY OF EDUCATION AND TRAINING

VIETNAM ACADEMY OF SIENCE AND TECHNOLOGY

**GRADUATE UNIVERSITY OF SIENCE AND TECHNOLOGY**

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# SOME ITERATIVE METHODS FOR THE SPLIT FEASIBILITY AND RELATED PROBLEMS

**SUMMARY OF DISSERTATION ON APPLIED MATHEMATICS**

**Code: 9 46 01 12**

***Hanoi – 2024***

The dissertation is completed at: Graduate University of Science and Technology, Vietnam Academy of Science and Technology

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The dissertation will be examined by Examination Board of Graduate University of Science and Technology, Vietnam Academy of Science and Technology at 9.00 am, July 19, 2024.

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2) National Library of Vietnam

# LIST OF THE PUBLICATIONS RELATED TO THE DISSERTATION

1. Buong Ng., Anh Ng.T.Q., Binh K.T., (2020), Steepest-Descent Ishikawa Iterative Methods for a Class of Variational Inequalities in Banach Spaces, Filomat, (2020), 34 (5), 1557–1569. (SCI-E, Q2).

2. Buong Ng., Hoai P.T.T, Binh K.T, (2020), New Iterative regularization methods for the multiple-sets split feasibility problem, Journal of Computational and Applied Mathematics 388(3), 113291. DOI 10-1016/j cam 2020. (SCI, Q2).

3 . Buong Ng., Anh Ng.T.Q., Binh, K.T., 2020, Iterative methods for the multiple-sets split equality problem in Hilbert spaces, Proceedings of the 23th National Conference:Some selected issues of Information and Communications Technology -- Quang Ninh, 5--6/11/2020, 151--157

## INTRODUCTION

The fixed point theory of nonexpansive mappings and their extensions play an important role not only in studying the theory of ordinary differential equations, partial differential equations, optimization problems, variational inequalities problem ... but also in problems directly related to real-life problems such as: convex feasibility problem, multi-set split and split equality problem. These problems arise from a number of practical problems such as: image recovery and processing problems, radiotherapy problems ...

The basic methods for finding fixed points of a non-expansive map are Krasnosel'skii–Mann iterative method, Ishikawa iterative method, Halpern iterative method and the viscosity approximation method. The Krasnosel'skii–Mann iterative and Ishikawa iterative methods are weakly convergent while Halpern' iterative and the viscosity approximation method converge strongly in infinite dimensions space. The combination of these basic methods to obtain better modified methods has also been proposed.

The above methods are also used to approximate the solution for the multiple-sets split feasibility problem, the multiple-sets split equality problem and variational inequalities problem on fixed points for a family of nonexpensive mappings.

The goal of the thesis is to propose some new iterative methods to approximate a solution for the multiple-sets split feasibility problem, the multiple-sets split equality problem and variational inequality problem over the common fixed points of a family of nonexpensive mappings, overcome some limitations of previous.

### Problem 1. Multiple-sets split feasibility problem (MSSFP)

Let $H_1$ and $H_2$ be Hilbert spaces with inner products $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $A : H_1 \to H_2$ be a bounded linear mapping. Let $C_i$ and $Q_j$ be convex closed subsets, respectively, in $H_1$ và $H_2$, with each $i \in J_1$ and $j \in J_2$ where, $J_1$ and $J_2$ are sets of indices, which can be finite or countably infinite. The MSSFP is formulated a finding a point

$$x \in C := \bigcap_{i \in J_1} C_i \text{ such that } Ax \in Q := \bigcap_{j \in J_2} Q_j. \qquad \text{(MSSFP)}$$

When the sets $J_1$ and $J_2$ contain only an element the MSSFP becomes the split feasibility problem (SFP): find $x \in C$ such that $Ax \in Q$. The MSSFP was first researched by Censor and Elfving

Problem MSSFP in finite-dimensional Hilbert spaces was first introduced by Censor and Elfving năm 2005 for modeling inverse problems that arise from phase retrievals and in image reconstruction. Recently, it can also be used to model the intensity-modulated radiation therapy.

For solving MSSFP in the cases that the cardinals of $J_1$ and $J_2$, denoted, respectively, by $|J_1|$ and $|J_2|$, are countably infinite, i.e., $|J_1| = |J_2| = \mathbb{N}_+$, the set of all positive integers, or finite, i.e., $|J_1| = N$ and $|J_2| = M$ where $N$ and $M$ are some positive integers, several iterative methods were introduced by Buong, Takahashi, Xu, Wen, Yao, Wang ... and references therein.

In the case that $N$ and $M$ are two any positive integers, to solve the MSSFP, Censor et al in { Y. Censor, T. Elfving, N. Knop, T. Bortfeld, The multiple-sets split feasibility problem and its applications for inverse problems. Inverse Problems. **21**, 2071-2084 (2005)} proposed an iterative method, based on the gradient projection one. This iterative method used a fixed step size restricted by a Lipschitz constant of a gradient mapping, which depends on $\|A\|$. To avoid the inconvenience of calculating the Lipschitz constant, in 2013, Zhao and Yang introduced a self-adaptive projection method by adopting Armijo-like searches. However, the iterative method needs an inner iteration number to have a suitable step size. Next, Zhao and Yang in { J. Zhao, Q. Yang, A simple projection method for solving the multiple-sets split feasibility problem. Inverse Problems in Science and Engineering. **21**(3), 537-546 (2013)} suggested a new self-adaptive way to compute directly the step size in each iteration, without estimating the Lipschitz constant or choosing the inner iteration number. The approach has been presented for the SFP, i.e. MSSFP with $N = M = 1$. On the other hand, in 2006, Xu showed that the MSSFP is equivalent to finding a common fixed point of a finite family of averaged mappings and proposed three iteration methods: (i) successive iteration method; (ii) simultaneous iteration method and (iii) cyclic iteration method. These iterative methods also used a fixed step size, which depends on the Lipschitz constant. The last two iterative methods with the self-adaptive step size have been recently studied by Zhao, Yang, Zhang et al in 2012, 2013 .... All the listed methods above converge weakly in infinite dimensional Hilbert spaces. In order to obtain a strongly convergent sequence from these methods, there exist several ways, one of which is to combine them with regularization methods. For solving the SFP

In 2010, Xu proposed Bruck and Bakushinsky type iterative regularization method, difined as follows:

$$z^{k+1} = P_C(I - \gamma_k(A^*(I - P_Q)A + \alpha_k I))z^k, \; z^1 \in H_1, \; k \geq 1, \tag{0.1}$$

where, we denotes the identify map $P_C$ and $P_Q$ are metric projections of $H_1$ and $H_2$ onto $C$ and $Q$ respectively, $A^*$ is the dual mapping of $A$, positive parameters $\gamma_k$ and $\alpha_k$ are small enough, such that and $0 < \gamma_k \leq \alpha_k/(\|A\|^2 + \alpha_k)$ and $\alpha_k \to 0$ as $k \to \infty$. However, choosing parameters $\gamma_k$ still depends on $\|A\|$. In 2017, Tian and Zhang [Ineq. Appl, 2017] proposed a self-adaptive iterative method for removing the dependence. In this study, $\gamma_k$ is built as follows: $\gamma_k = \rho_k f(x^k)/\|A^*(I - P_Q)Ax^k\|^2$ vi $\varepsilon < \rho_k < 4 - \varepsilon$, $\varepsilon > 0$ small enough, where $f(x) = \frac{1}{2}\|(I - P_Q)Ax\|^2$, with condition $(\alpha)$: $\alpha_k \in (0, 1)$ for all $k \geq 1$, $\lim\limits_{k\to\infty} \alpha_k = 0$ and $\sum\limits_{k=1}^{\infty} \alpha_k = \infty$. However, proving this result is not completed because $\sum\limits_{k=1}^{\infty} \gamma_k \alpha_k = +\infty$ when $\lim\limits_{k\to\infty} f(x^k) = 0$ has not been proven.

There are two difficulties in implementing this method:
1. Must calculate infinite sum
2. Must calculate $\|A\|$

In [Acta App. Math, 2019], then difficulty 1 has been resolved by Nguyen Buong and et al.

**Therefore, the first goal of the thesis is to provide a new iterative methods to approximate a solution of the MSSFP, that overcomes the second difficulty.**

**Problem 2. The multiple-sets split equality problem (MSSEP)**

Let $H_1$, $H_2$ and $H_3$ be real Hilbert spaces with inner products $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear mappings. Let $J_1$, $J_2$ are sets of indices, $\{C_i\}_{i\in J_1}$ and $\{Q_j\}_{j\in J_2}$ are two families of convex, closed subsets in $H_1$ and $H_2$ respectively. The MSSEP is the problem of finding a point

$$\begin{aligned} z = [x, y],\, x \in C := \cap_{i\in J_1} C_i \text{ and } y \in Q := \cap_{j\in J_2} Q_j \\ \text{such that } Ax = By. \end{aligned} \tag{0.2}$$

Obviously, if $H_2 = H_3$ and $B = I$, then the MSSEP becomes the MSSFP problem. In particular, if the index sets $J_1$ and $J_2$ contains only one element then the MSSEP is the split equality problem, denoted as SEP: In

2013, This problem was first studied by Byrne and Moudafi [Working paper, 2013]. Then, Chen et al studied the problem in the case that $T = G^*G$, [Fixed Point Theory and Applications, 2014] and propose an iterative regularization method:

$$z^{k+1} = P_S(I - \gamma_k(T + \alpha_k I))z^k. \tag{0.3}$$

**The second goal of the thesis is to solve this problem.**

**Problem 3. The variational inequalities problem in Banach space**

Let $E$ be a Banach space, $F : E \to E$ is a nonlinear mapping, $C$ is a convex, closed subset of $E$. The variational inequality problem (VIP), with mapping $F$ and constraint set $C$ in Banach space $E$ is stated as follows:

$$\text{Find } p_* \in C \text{ such that} \langle Fp_*, j(p_* - p)\rangle \leq 0 \quad \forall p \in C, \tag{VIP}$$

where, $j$ is the norm alized duality mapping of $E$. In this these, we consider the case when $C = \cap_{i\geq 1} Fix(T_i)$ the fixed point set of a nonexpensive mapping $T_i$ defined on $E$.

When $E$ is Hilbert space, $j$ is identify mapping and then, variational inequality problem (VIP) will become variational inequalities problem in Hilbert space.

We see that, The Ishikawa iterative method is formally an extension of the Krasnosel'skii–Mann iterative method. The convergene between these two is weak. However, there are examples showing the situation when we use the Ishikawa iteration method, this iterative sequence converges to the solution of the problem, but when we use the Krasnosel'ski–Mann iteration method, it does not converge.

**Combining the steepest-descent method with Ishikawa iterative one to approximate the solution for a class of variational inequalities in Banach space in order to obtain a strongly convergent sequence is a as research goal in this thesis.**

The thesis includes 3 chapters.

Chapter 1: "Preliminaries". In this chapter we present some basic concepts and some methods to approximate the solution for the fixed point problem, the multiple-sets split feasibility problem, the multiple-sets split equality problem.

Chapter 2: "Iterative regularization methods for approximate solutions of the multiple-sets split feasibility and the multiple-sets split equality problems". In this chapter, the thesis presents two methods to solve goals 1 and 2 above.

Chapter 3: "Steepest-descent Ishikawa iterative methods for a class of variational inequalities". In this chapter, the thesis presents two methods to solve the third goal above .

Results of the thesis are reported at: XXIII National Conference on selected issues of Information and Communications Technology, Quang Ninh, 5–6/11/2020.

In this chapter, section 1.1 gives some basic concepts in Hilbert and Banach spaces.

Section 1.2, presents some methods to approximate the solution for a fixed point problem, the multiple-sets split feasibility problem, the multiple-sets split equality problem. These methods all have limitations that cause difficulties during implementation.

(1) The problem of finding fixed points of a non-expansive mapping, the thesis proposes new menthod to overcome disadvantages such as weak convergence.
(2) The multiple-sets split feasibility problem, The thesis presents the solution approximation method of Tian and Zhang that has been proposed in 2017. However, the proof of the proposed results has not yet been completed.
(3) The multiple-sets split equality problem, The thesis presents a method of Chen proposed in 2013. The difficulty of this method is that it requires infinite summation during implementation. To date, there has been no research to address this issue.

The above issues are one of the reasons that led the author to the research that will be presented in chapters 2 and 3.

Section 1.3 of the thesis presents two practical applications of the above problems in medicine and in digital signal processing and in image restoration.

# CHAPTER 2. ITERATIVE REGULARIZATION METHODS FOR APPROXIMATE SOLUTIONS OF THE MULTIPLE-SETS SPLIT FEASIBILITY AND THE MULTIPLE-SETS SPLIT EQUALITY PROBLEMS

In this chapter, we propose two iterative regularization methods for approximating solutions for the multiple-sets split feasibility and the multiple-sets split equality problems in real, infinite-dimensional Hilbert spaces. These methods strongly converge and have overcome the disadvantages of the methods presented in Chapter 1

The results of the chapter are written based on two scientific articles [2] and [3] in the List of published works of the thesis author.

## 2.1. The multiple-sets split feasibility problem

Let $H_1$ and $H_2$ be two real Hilbert spaces. The MSSFP is fomulated as follows.

$$\text{Find a point } x \in C := \bigcap_{i \in J_1} C_i \text{ sao cho } Ax \in Q := \bigcap_{j \in J_2} Q_j. \qquad \text{(MSSFP)}$$

where $C_i$ and $Q_j$ be two closed convex subsets in $H_1$ and $H_2$, respectivery and $A : H_1 \to H_2$ is a bounded linear mapping

### 2.1.1. The iterative regularization method of Lavrentivev' type

For solving the MSSFP, we first introduce the regularization method of Lavrentivev's type, described as follows,

$$F^k u^k + \alpha_k (u^k - x^+) = 0, \tag{2.1}$$

where,

$$F^k = I - U^k + A^*(I - V^k)A, \tag{2.2}$$

$$U^k = \frac{1}{\beta^k} \sum_{i=1}^{k} \beta_i P_{C_i}, \; V^k = \frac{1}{\eta^k} \sum_{j=1}^{k} \eta_j P_{Q_j}, \tag{2.3}$$

$x^+ \in H_1$ is a guess point in $H_1$, the parameters $\gamma_k, \alpha_k, \beta_i$ and $\eta_j$ with $\beta^k = \beta_1 + \cdots + \beta_k$, $\eta^k = \eta_1 + \cdots + \eta_k$,
satisfy the following assumptions:

(a) $\gamma_k, \alpha_k \in (0,1)$, $\lim\limits_{k\to\infty} \gamma_k/\alpha_k = \lim\limits_{k\to\infty} \alpha_k = 0$, $\alpha_{k+1} < \alpha_k$ and $\sum\limits_{k=1}^{\infty} \gamma_k \alpha_k = \infty$.

(b) $\lim_{k\to\infty} \tilde{\alpha}_k/(\gamma_k \alpha_k^2) = 0$ where $\tilde{\alpha}_k = (\alpha_{k-1}/\alpha_k) - 1$;

(c) $\beta_i > 0$ for all $i \geq 1$ such that $\sum_{i=1}^{\infty} \beta_i = 1$ and $\lim_{k\to\infty} \beta_k/(\gamma_k \alpha_k^2) = 0$;
(d) $\eta_j > 0$ for all $j \geq 1$ such that $\sum_{j=1}^{\infty} \eta_j = 1$ and $\lim_{k\to\infty} \eta_k/(\gamma_k \alpha_k^2) = 0$.

**Remark 2.1.1.** Examples of sequences, having properties (a)–(d) are: $\gamma_k = 1/(k+1)^a$, $\alpha_k = 1/(k+1)^b$, where $0 < b < a$ with $a + 2b < 1$, and $\eta_i = \beta_i = 1/(i(i+1))$

We have the following results when $J_1$ and $J_2$ are countably infinite.

**Theorem 2.1.1.** *Let $H_1$ and $H_2$ be two real Hilbert spaces and let $A$ be a bounded linear mapping from $H_1$ into $H_2$. Let $\{C_i\}_{i\in\mathbb{N}_+}$ and $\{Q_j\}_{j\in\mathbb{N}_+}$ be two infinite families of closed convex subsets in $H_1$ and $H_2$, respectively. Assume that there hold conditions (c) and (d) with rejecting the limits. Then, we have:*

*(i) For each $\alpha_k > 0$, problem (2.1) has a unique solution $u^k$;*
*(ii) If $\Gamma \neq \emptyset$, where $\Gamma$ denotes the salution set of the MSSFP, then $\lim_{k\to\infty} u^k = p_* \in \Gamma$, satisfying*

$$\|p_* - x^+\| \leq \|p - x^+\| \ \forall p \in \Gamma; \tag{2.4}$$

*(iii)*

$$\|u^k - u^{k-1}\| \leq d_k = \frac{2M_1}{\alpha_k}\left[\frac{\beta_k}{\beta^k} + \tilde{\alpha}_k + \frac{\eta_k}{\eta^k}\right] + \tilde{\alpha}_k(M_1 + \|x^+\|), \tag{2.5}$$

*where $M_1$ is some positive constant.*

**Remark** 1. Obviously, if $\{u^k\}$ converges strongly to some point $\tilde{u}$, where $u^k$ is the solution of (2.1), and $\alpha_k \to 0$ as $k \to +\infty$, then $\Gamma \neq \emptyset$.

In algorithm (2.1), the non-linear equation (2.1) has only theoretical meaning, the calculation of its solution is very difficult. Algorithm (2.11) is constructed according to the following theorem, which is to convert algorithm (2.1) into iterative sequence (2.11), then the calculation will be much more feasible. Now we consider the following theorem and will prove the strong convergence of algorithm (2.11).

**Theorem 2.1.2.** *Let $H_1, H_2, A, C_i$ and $Q_j$ be as in theorem 2.1.1 with $\Gamma \neq \emptyset$. Assume that there hold conditions (a), (b), (c) and (d). Then, the sequence $\{z^k\}$, defined by*

$$z^{k+1} = (I - \gamma_k(F^k + \alpha_k I))z^k + \gamma_k \alpha_k x^+, \ k \geq 1, \tag{2.6}$$

*$z_1 \in H_1$ converges strongly to $p_*$, satisfying (2.4), where $F^k$ defined by (2.2).*

In the case that either one of $|J_1|$ and $|J_2|$ or both they are finite, We obtain the following theorems:

**Theorem 2.1.3.** *Let $H_1, H_2$ and $A$ be as in Theorem 2.1.1. Let $\{C_i\}_{i=1}^N$ and $\{Q_j\}_{j\in\mathbb{N}_+}$ be two families of closed convex subsets in $H_1$ and $H_2$, respectively, where $N$ is any positive integer Assume that $\Gamma \neq \emptyset$ and there hold conditions (a), (b), (d) and*
*(c′) $\beta_i > 0$ for $1 \leq i \leq N$ such that $\sum_{i=1}^N \beta_i = 1$.*
*Then, as $k \to \infty$, the sequence $\{z^k\}$, defined by*

$$z^{k+1} = z^k - \gamma_k((I-U)z^k + A^*(I-V^k)Az^k + \alpha_k(z^k - x^+)), k \geq 1, z^1 \in H_1, \tag{2.7}$$

*in this*

$$U = \sum_{i=1}^N \beta_i P_{C_i}, \qquad V^k = \frac{1}{\eta^k}\sum_{j=1}^k \eta_j P_{Q_j},$$

*converges strongly to $p_*$ satisfying* (2.4)

**Theorem 2.1.4.** *Let $H_1, H_2$ and $A$ be as in Theorem 2.1.1. Let $\{C_i\}_{i\in\mathbb{N}_+}$ and $\{Q_j\}_{j=1}^M$ be two families of closed convex subsets in $H_1$ and $H_2$, respectively, where $M$ is a position integer Assume that $\Gamma \neq \emptyset$ and there hold conditions (a), (b), (c) and (d′): $\eta_j > 0$ for $1 \leq j \leq M$ such that $\sum_{j=1}^M \eta_j = 1$.*
*Then, as $k \to \infty$, the sequence $\{z^k\}$, defined by*

$$z^{k+1} = z^k - \gamma_k((I-U^k)z^k + A^*(I-V)Az^k + \alpha_k(z^k - x^+)), k \geq 1, z^1 \in H_1, \tag{2.8}$$

*in this*

$$U^k = \frac{1}{\beta^k}\sum_{i=1}^k \beta_i P_{C_i}, \qquad V = \sum_{j=1}^M \eta_j P_{Q_j},$$

*converges strongly to $p_*$ satisfying* (2.4)

From Theorem 2.1.3 and 2.1.4, we have a result in the case that $J_1, J_2$ are finite.

**Theorem 2.1.5.** *Let $H_1, H_2$ and $A$ be as in Theorem 2.1.1. Let $\{C_i\}_{i=1}^N$ and $\{Q_j\}_{j=1}^M$ be two finite families of closed convex subsets in $H_1$ and $H_2$, respectively. Assume that $\Gamma \neq \emptyset$ and there hold conditions (a), (b), (c′) and (d′). Then, as $k \to \infty$, the sequence $\{z^k\}$, defined by*

$$z^{k+1} = z^k - \gamma_k((I-U)z^k + A^*(I-V)Az^k + \alpha_k(z^k - x^+)), k \geq 1, z^1 \in H_1, \tag{2.9}$$

*where $U$ and $V$ are defined in Theorems 3.3 and 3.4 respectively, converges strongly to $p_*$ satisfying* (2.4).

**Remark 2.1.2.** (a) In chapter 1 of this thesis two iterative regularization method to solve the (MSSFP) proposed by Xu and el al, has the form

$$z^{k+1} = P_C(I - \gamma_k(A^*(I - P_Q)A + \alpha_k I))z^k, \ z^1 \in H_1, \ k \geq 1, \quad (2.10)$$

where $0 < \gamma_k \leq \alpha_k/(\|A\|^2 + \alpha_k)$ at each iteration step depends on the norm off $A$. Calculating the norm of operator $A$ is difficult, then, there will be difficulties in using the method (2.10).

(b) Nguyen Buong and el al extended the method (2.10) to solve the MSSFP in the case of index sets $J_1$ and $J_2$ are finite:

$$z^{k+1} = U^k T_{\gamma_k,\alpha_k} z^k, \quad (2.11)$$

in this

$$T_{\gamma_k,\alpha_k} = I - \gamma_k(A^*(I - V^k)A + \alpha_k I),$$

the parameter $\gamma_k$ is chosen in dependen on $||A||$.

### 2.1.2. Numerical experiments

We consiter MSSFP in real, finite-dimensional Hilbert spaces $\mathbb{E}^m$ and $\mathbb{E}^n$ with $C = \cap_{i=1}^{\infty} C_i$ end $Q = \cap_{j=1}^{\infty} Q_j$, where,

$$C_i = \Big\{ x \in \mathbb{E}^n \ \Big| \ a_1^i x_1 + a_2^i x_2 + \cdots + a_n^i \leq b_i \Big\}, \quad (2.12)$$

with $a_l^i, b_i \in (-\infty; +\infty)$, $1 \leq l \leq n$ end $i \in \mathbb{N}_+$,

$$Q_j = \Big\{ y \in \mathbb{E}^m \ \Big| \ \sum_{l=1}^{m} (y_l - a_l^j)^2 \leq R_j \Big\}, \ R_j > 0, \quad (2.13)$$

with $a_l^j \in (-\infty; +\infty)$, $1 \leq l \leq m$, $j \in \mathbb{N}_+$ and $A$ is a $3 \times 2$ - matrix.

**Example 2.1.** In the first example, we consider the case $m = n = 2$, $A$ is an identity matrix, with the numbers $a_1^i = 1/i, a_2^i = -1$ and $b_i = 0$ for all $i \geq 1$, $R_j = 1$, $a^j = (1/j; 0)$ for all $j \geq 1$ and $x^+ = (0, 0)$. Then, it is not difficult to verify that $x_* = (0; 0)$ is the unique minimum-norm solution of (2.12), (2.13). Since $A = I$, method (2.6) is written in the form

$$z^{k+1} = (1 - \gamma_k(2 + \alpha_k))z^k + \gamma_k(U^k z^k + V^k z^k). \tag{2.14}$$

Using method (2.14) with

$$\beta_i = \eta_i = 1/(i(i+1)),\ \alpha_k = 1/(k+1)^{1/8},\ \gamma_k = 1/(k+1)^{1/2}$$

and a starting point $x^1 = (-3.0; 3.0)$, we obtain the following table of numerical results in Table 2.1.

Table 2.1: numerical results of example 2.1 using method (2.14)

| $k$ | $z_1^{k+1}$ | $z_2^{k+1}$ | $k$ | $z_1^{k+1}$ | $z_2^{k+1}$ |
|---|---|---|---|---|---|
| 10 | -0.0072951049 | -0.0123790731 | 60 | -0.0000003749 | -0.0000006109 |
| 20 | -0.0003743916 | -0.0006192857 | 70 | -0.0000001070 | -0.0000001742 |
| 30 | -0.0000421668 | -0.0000692232 | 80 | -0.0000000337 | -0.0000000548 |
| 40 | -0.0000070196 | -0.0000114815 | 90 | -0.0000000115 | -0.0000000187 |
| 50 | -0.0000014904 | -0.0000014325 | 100 | -0.0000000042 | -0.0000000068 |

**Example 2.2.** In the second example, we save $C_i, \beta_i, \eta_j, \gamma_k, \alpha_k$ and the starting point $x^1$ as in exemple 2.1. Where, thesis consider the case when $Q_j = \{y \in \mathbb{E}^3 : \|y - a^j\| \leq 1\}$ where $a^j = (1/(j+1); 1/(j+1); 1/(j+1))$ and $A$ is a $3 \times 2$-matrix with elements $a_{i1} = 1$, for $i = 1, 2, 3$, and zero for the others. Clearly, $x_* = (0; 0)$ is the unique minimum norm solution. The computational results, by using the method (2.6), are presented in the following numerical table, Table 2.2.

**Remark 2.1.3.** Assum, In case $m = n = 2$ and $A$ is norm matrix, method (2.11) propoced by Ng. Buong et al, while projectors difened as (2.3) is difened by

$$x^{k+1} = U^k((1 - \gamma_k(1 + \alpha_k))x^k + \gamma_k V^k x^k). \tag{2.15}$$

Using the method (2.15), where $\gamma_k = 1/(1.05 + (1/k))$, $\alpha_k = 1/k$ condition $(\alpha)$ and the above datas, we have the results in table 2.3 and table 2.4
Put results illustrated in two Table 2.1, 2.2 and Table 2.3, 2.4, We see that both proposed theoretical methods are effective. Further, regularization methods in this thesis converge faster than results of Buong and et al in [Acta Appl. Math, 2019].

Table 2.2: numerical results of example 2.2 using method (2.6)

| $k$ | $z_1^{k+1}$ | $z_2^{k+1}$ | $k$ | $z_1^{k+1}$ | $z_2^{k+1}$ |
|---|---|---|---|---|---|
| 10 | -0.0067281333 | -0.0450293607 | 60 | -0.0000189750 | -0.0000189751 |
| 20 | -0.0025241606 | -0.0026043616 | 70 | -0.0000078161 | -0.0000078161 |
| 30 | -0.0005405073 | -0.0005415133 | 80 | -0.0000034561 | -0.0000034561 |
| 40 | -0.0001513849 | -0.0001514139 | 90 | -0.0000016184 | -0.0000016184 |
| 50 | -0.0000504382 | -0.0005043396 | 100 | -0.0000007947 | -0.0000007947 |

Table 2.3: numerical results of example 2.1 using the method (2.15)

| $k$ | $x_1^{k+1}$ | $x_2^{k+1}$ | $k$ | $x_1^{k+1}$ | $x_2^{k+1}$ |
|---|---|---|---|---|---|
| 1 | 0.0243902439 | 0.3658536585 | 100 | 0.0012390505 | 0.0083945251 |
| 10 | 0.0102553274 | 0.0694794968 | 500 | 0.0002695347 | 0.0018260888 |
| 20 | 0.0055344982 | 0.0374960376 | 1000 | 0.0001394192 | 0.0009445606 |
| 30 | 0.0038180428 | 0.0258671112 | 2000 | 0.0000720824 | 0.0004883558 |
| 40 | 0.0029249862 | 0.0198166827 | 3000 | 0.0000489994 | 0.0000331969 |

**Example 2.3.** Now, in the case that

$$a_{11} = 0.1, a_{12} = 0.2, a_{21} = 0.2,$$
$$a_{22} = 0.4, a_{31} = a_{32} = 0,$$

example 2.2, considered above, has many solutions (MS), containing the zero point, as the minimal norm solution because $x^+ = 0$. The numerical results, calculated by (2.11) and (2.2) with the same data, are described in Table 2.5.

Table 2.4: numerical results of example 2.2 using the method (2.11)

| $k$ | $x_1^{k+1}$ | $x_2^{k+1}$ | $k$ | $x_1^{k+1}$ | $x_2^{k+1}$ |
|---|---|---|---|---|---|
| 1 | 0.6019388274 | 1.5365833659 | 100 | 0.0142047415 | 0.0363009852 |
| 10 | 0.1176994981 | 0.3004546610 | 500 | 0.0030934268 | 0.0078966734 |
| 20 | 0.0635189516 | 0.1621465290 | 1000 | 0.0016001024 | 0.0040846244 |
| 30 | 0.0438193443 | 0.1118588139 | 2000 | 0.0008272834 | 0.0021118284 |
| 40 | 0.0356981140 | 0.0856945566 | 3000 | 0.0005623615 | 0.0014355553 |

| $k$ | $z_1^{k+1}$ | $z_2^{k+1}$ | $k$ | $z_1^{k+1}$ | $z_2^{k+1}$ |
|---|---|---|---|---|---|
| 10 | -0.0420263650 | -0.0420003267 | 60 | -0.0000380359 | -0.0000380359 |
| 20 | -0.0051177476 | -0.0051176931 | 70 | -0.0000156676 | -0.0000156667 |
| 30 | -0.0010841787 | -0.0010841781 | 80 | -0.0000069279 | -0.0000069279 |
| 40 | -0.0003034753 | -0.0003034753 | 90 | -0.0000032440 | -0.0000032440 |
| 50 | -0.0001011055 | -0.0001011055 | 100 | -0.0000015929 | -0.0000012929 |

Table 2.5. Computational results by method (2.11) and (2.2), MS

Tables 2.2 and 2.5 show that, for the considered example with a unique solution or many solutions, method (2.11) and (2.2) converges well and, in the case that the problem has a unique solution, the method works a little better than in the other case.

### 2.2. The multiple-sets split equality problem

Let $H_1, H_2$ and $H_3$ be real Hilbert spaces; $C_i$ and $Q_j$ are two closed convex subsets in $H_1$ and $H_2$, respectively.

We consider the MSSEP: find a point

$$x \in C := \bigcap_{i \in J_1} C_i \text{ and } y \in Q := \bigcap_{j \in J_2} Q_j \text{ such that } Ax = By. \qquad \text{(MSSEP)}$$

where $A : H_1 \to H_3$; $B : H_2 \to H_3$ are bounded, linear mappings.

Denote by $\Omega$ the set of solutions for $\Omega$. Throughout this thesis, assume that $\Omega \neq \emptyset$.

#### 2.2.1. The iterative regularization method of Bakushinsky–Bruck' type

By extending the iterative regularization method of Bakushinsky [Comput. Math. and Math. Physics., 2011] and Bruck [J. Math. Anal. Appl., 1974], thesis proposes a new iterative regularization method for the MSSEP in infinite dimensional Hilbert spaces. Start from an arbitrary initial point $z^1 \in H$, the next approximations is determined by:

$$z^{k+1} = U_k T_{\gamma_k, t_k} z^k, \qquad (2.16)$$

where

$$U_k = \frac{1}{\tilde{\beta}_k} \sum_{i=1}^{k} \beta_i P_{S_i}, \ T_{\gamma_k, t_k} = I - \gamma_k [G^* G + t_k I], \qquad (2.17)$$

$\gamma_k$, $t_k$, $\beta_i$ are positive number and $\tilde{\beta}_k = \beta_1 + \cdots + \beta_k$.

**Remark 2.2.1.** In this method, at each iteration step only a finite number of sets from the families is used. So,this result is better than some previously proposed methods

Let's assume that parameters $\gamma_k$, $t_k$, $\beta_i$ satisfy the conditions

**(t)** $t_k \in (0,1)$ for all $k$, $\lim\limits_{k\to\infty} t_k = 0$ and $\sum\limits_{k=1}^{\infty} t_k = \infty$.

**($\beta$)** $\beta_i > 0$ for all $i$ and $\sum\limits_{i=1}^{\infty} \beta_i = 1$.

**($\gamma$)** $\gamma_k \in (0, 2/(\|A\|^2 + t_k))$, $\lim\inf\limits_{k\to\infty} \gamma_k > 0$ and $\lim\limits_{k\to\infty}(\gamma_{k+1} - \gamma_k) = 0$.

**Lemma 2.2.1.** *Let $H_1$, $H_2$ and $H_3$ be three real Hilbert spaces and let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be bounded linear mappings. Then, for a fixed number $\gamma \in (0, 2/(\|G\|^2+2\alpha))$, where, $G = [A - B] : H = H_1 \times H_2 \to H$, and $\alpha$ is a number in $(0,1)$, the mapping $T_{\gamma,t} := I - \gamma[G^*G + tI]$ is a contraction with coefficient $1 - \gamma t$, $t \in (0,1)$. When $t = 0$, then $T_\gamma := I - \gamma G^*G$ is nonexpansive.*

**Lemma 2.2.2.** *Let $H$ is Hilbert space and let $G$ be a bounded linear mapping on $H$. Then, $\mathrm{Zer}G := \{z \in H \mid Gz = 0\} = \mathrm{Fix}(T_\gamma)$ where $T_\gamma$ is defined in Lemma 2.2.1 for any positive real number $\gamma$.*

**Lemma 2.2.3.** *The solution set $\Omega$ of* MSSEP *coincides with to the solution set of the variational inequality*

$$\textit{Find } z_* \in S \textit{ such that } \langle Tz_*, z - z_*\rangle \geq 0 \ \forall z \in S, \qquad \text{(VIP)}$$

*with $T = G^*G$.*

**Theorem 2.2.1.** *Let $H_1$, $H_2$, $H_3$, $A$ and $B$ be as in Lemma 2.2.1. Let $C_i$ and $Q_j$, for each $i \in J_1$ and each $j \in J_2$ with $J_1 = J_2 = \mathbb{N}_+$, be closed convex subsets in $H_1$ and $H_2$, respectively. Assume that there hold conditions* **(t)**, *($\beta$) and ($\gamma$). Then, the sequence $\{z^k\}$, defined by (2.16) and (2.17), as $k \to \infty$, converges strongly to a solution of the MSSEP*

In the case that either one or both the sets $J_1$ and $J_2$ are finite, we obtain the following results.

**Theorem 2.2.2.** *Let $H_1, H_2, H_3, A, B,$ be as in lemma 2.2.1. Let $C_i$ and $Q_j$, for each $i \in J_1$ and each $j \in J_2$ be closed convex subsets in $H_1, H_2$ respectively, in this $J_1 = \{1, \ldots, N\}$, $J_2 = \{1, \ldots, M\}$ and $N < M$ . Assume* *that there hold conditions* $(\gamma)$ *and* $(t)$*. Then, as* $k \to \infty$*, the sequence* $\{z^k\}$*, defined by*

$$z^{k+1} = UT_{\gamma_k,t_k} z^k, k \geq 1,\ z^1 \in H,\ U = \sum_{i=1}^{M} \beta_i P_{S_i},$$

*converges to a solution of MSSEP when* $k \to \infty$*, where,* $C_i = C_N$*,* $i = N+1, \ldots, M$*,* $\beta_i > 0$ *and* $\sum_{i=1}^{M} \beta_i = 1$.

In the case that only $J_1$ is finite, $J_2 = \mathbb{N}_+$, by setting $C_i = C_N$, $i = N+1, \ldots, \infty$, we return to the case in Theorem 2.2.1. In the case that only $J_2$ is finite, is similar.

**Remark 2.2.2.** (a) We can express method (2.16) in terms of $x$ and $y$ as follows: for any starting point $x^1 \in H_1$ and $y^1 \in H_2$,

$$\begin{cases} v^k = Ax^k - By^k, \\ x^{k+1} = \tilde{U}_k\big((1-\gamma_k t_k)x^k - \gamma_k A^* v^k\big), \\ y^{k+1} = \tilde{V}_k\big((1-\gamma_k t_k)y^k + \gamma_k B^* v^k\big), \end{cases} \tag{2.18}$$

where $\tilde{U}_k$ is defined in (2.3) and $\tilde{V}_k = (1/\tilde{\beta}_k) \sum_{i=1}^{k} \beta_i P_{Q_i}$.

(b) We will use method (2.18) with $H_3 = H_2$ and $B = I$ for MSSFP with $J_1 = J_2 = \mathbb{N}_+$, we get a new iterative regularization method: for any starting point $x^1 \in H_1$ and $y^1 \in H_2$,

$$\begin{cases} v^k = Ax^k - y^k, \\ x^{k+1} = \tilde{U}_k\big((1-\gamma_k t_k)x^k - \gamma_k A^* v^k\big), \\ y^{k+1} = \tilde{V}_k\big((1-\gamma_k t_k)y^k + \gamma_k v^k\big). \end{cases} \tag{2.19}$$

Under conditions $(\gamma)$, $(\beta)$ and $(t)$, the sequences $\{x^k\}$ defined by (2.19) converge strongly to $x_*$, solving the MSSFP when $k \to \infty$. Clearly, method (2.19) is different from (2.11) with projecter defined by (2.3).

(c) Use iterative regularization method (2.19) for SFP, we also see that this method is completely different from Yao's method et al the problems published in 2012.

### 2.2.2. Numerical experiments

We consider MSSEP with $C = \cap_{i=1}^{\infty} C_i$ and $Q = \cap_{j=1}^{\infty} Q_j$, where

$$C_i = \{x \in \mathbb{E}^n : \tilde{a}_1^i x_1 + \tilde{a}_2^i x_2 + \cdots + \tilde{a}_n^i \leq b_i\},$$

$\tilde{a}_j^i, b_i \in (-\infty; +\infty)$, for $1 \leq j \leq n$ and $i \in \mathbb{N}_+$,

$$Q_j = \{y \in \mathbb{E}^m : \sum_{l=1}^{m} (y_l - \overline{a}_l^j)^2 \leq r_j\},\ r_j > 0,$$

$\overline{a}_l^j \in (-\infty; +\infty)$, for $1 \leq l \leq m$ and $j \in \mathbb{N}_+$, $A$ and $B$ are $p \times n$- and $p \times m$-matrices, respectively.

For computation, we consider the case $H_1 = \mathbb{E}^2$, $H_2 = \mathbb{E}^3$ and $H_3 = \mathbb{E}^4$ with $\tilde{a}_1^i = 1/i, \tilde{a}_2^i = -1$ and $b_i = 0$ for all $i \geq 1$; $r_j = 1$ and $\overline{a}^j = (1/(j+1); 1/(j+1); 1/(j+1))$ for all $j \geq 1$; and

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Then, it is not difficult to verify that $z_* = [x_*, y_*]$, where $x_* = (0; 0)$ and $y_* = (0; 0; 0)$, is the unique minimum-norm solution of MSSEP problem with the above data. Using method (2.57) with $\gamma_k = 0.05 + 0.05/k$, $\beta_i = 1/(i(i+1))$, $\alpha_k = 1/k$ and a starting point $z^1 = [x^1, y^1]$ where $x^1 = (-2.0; -2.0)$ and $y^1 = (-2.0; -2.0; -2.0)$, we obtain the (2.6) numerical table of the values $\|z^k - z_*\| = \sqrt{\|x^k - x_*\|^2 + \|y^k - y_*\|^2}$.

| $k$ | $\|z^{k+1} - z_*\|$ | $k$ | $\|z^{k+1} - z_*\|$ |
|---|---|---|---|
| 10 | 0.0126467309 | 100 | 0.0000712257 |
| 20 | 0.0011458555 | 200 | 0.0000058032 |
| 30 | 0.0006146665 | 300 | 0.0000005387 |
| 40 | 0.0004108535 | 400 | 0.0000000527 |
| 50 | 0.0002940086 | 500 | 0.0000000053 |

## Conclusions

In this chapter, we suggest new iterative regularization methods for solving the MSSFP and MSSEP. These methods delete some difficulties, that exist in the literature such as the iterative parameter is chosen in dependence of $\|A\|$ or the methods do not contain infinite summs. We also give some numerical examples for illustration.

# CHAPTER 3. STEEPEST-DESCENT ISHIKAWA ITERATIVE METHODS FOR A CLASS OF VARIATIONAL INEQUALITIES

In this chapter, we propose an iterative method to approximate a solution of a class of variational inequalities in the case that the feasibility set is the set of comemon fixed points of an infinite family of nonexpansive mappings on a Banach space $E$ and the involving mapping is $\eta$ - strongly accretive and $l$-Lipschitz continuons on $E$.

The results of the chapter are written based on two scientific articles [1] in the List of published works of the thesis author.

## 3.1. Steepest-descent Ishikawa iterative methods for a class of variational inequalities in Banach space

The class of variational inequalities in Banach space $E$, considired in this chapter, is to find $p_* \in C$ such that

$$\langle Fp_*, j(p_* - p)\rangle \leq 0 \quad \forall p \in C, \tag{3.1}$$

where, $\langle x, x^*\rangle$ is used instead $x^*(x)$ with $x \in E$, $x^* \in E^*$ and $j$ is the normalized duality mapping of $E$, $F$ is an $\eta$ strong accretive and $l$-Lipschitz continuons mapping on $E$

### 3.1.1. Variational inequality problem on the fixed point set of a nonexpansive mapping

When $C := \mathrm{Fix}(T)$ where $T$ is a nonexpansive mapping on $E$, problem (3.1) becomes

$$\text{find } p_* \in \mathrm{Fix}(T) \text{ such that } \langle Fp_*, j(p_* - p)\rangle \leq 0 \quad \forall p \in \mathrm{Fix}(T), \tag{3.2}$$

where $E$ is an either uniformly smooth or strictly convex reflexive Banach space with a uniformly Gâteaux differentiable norm, the mapping $F : E \to E$ is $\eta$-$j$-strongly accretive and $\gamma$-strictly pseudocontractive mapping on $E$. For finding a solution of class variational inequalities (3.2), in this thesis we give a combination of steepest-descent method with Ishikawa one. The iterative algorithm is built as follows: with arbitrary initial point $x^1 \in E$, the next iterate is determined by

$$x^{k+1} = (I - t_k F)T^k x^k, \ k \geq 1, \tag{3.3}$$

where,

$$T^k = (1 - \beta_k)I + \beta_k T\big[(1 - \alpha_k)I + \alpha_k T\big], \ k \geq 1, \tag{3.4}$$

and parameters $t_k$, $\beta_k$ and $\alpha_k$ satisfy the following conditions:

**(t)** $t_k \in (0, 1)$, $\lim_{k\to\infty} t_k = 0$ và $\sum_{k=1}^{\infty} t_k = \infty$.
**(β)** $\beta_k \in [a, b] \subset (0, 1)$ for all $k \geq 1$.
**(α)** $\alpha_k \in [0, \overline{a}]$, with $\overline{a} \in (0, 1)$, $\forall k \geq 1$ và $\alpha_k \to 0$ when $k \to \infty$.

The following theorem confirms the strong convergence of iterative sequence (3.3) and is fully proven in the thesis.

**Theorem 3.1.1.** *Cho $F : E \to E$ be an $\eta$-strongly accretive and $\gamma$-strictly pseudocontractive mapping on an either uniformly smooth or real reflexive and strictly convex Banach space $E$, having a uniformly Gâteaux differentiable norm, such that $\eta+\gamma > 1$ and $T : E \to E$ be a nonexpansive mapping on $E$ with $Fix(T) \neq \emptyset$. Assume that $t_k, \beta_k$ and $\alpha_k$ satisfy conditions (t), (β) and (α), respectively. Then, the sequence $\{x^k\}$, defined by (3.3) with $T^k$ in (3.4) converges strongly to $p_*$, solving* (3.2).

**Remark 3.1.1.** (a) Theorem 3.1.1 has still value for the following method: $y^1 \in E$ is any element and

$$y^{k+1} = T^k(I - t_k F)y^k, \; k \geq 1, \tag{3.5}$$

with the same conditions on $E, F, T, t_k, \beta_k$ and $\alpha_k$.

(b) We take $F = I - f$ with $f = a'I$ for a fixed number $a' \in (0, 1)$. Then, $F$ is an $\eta$-strongly accretive and $\gamma$-strictly pseudocontractive mapping on $E$ with some positive numbers $\eta$ and $\gamma$ such that $\eta + \gamma > 1$. Replace $F$ by $I - f = (1 - a')I$ in (3.3), we get the following algorithm:

$$x^{k+1} = (1 - t'_k)T^k x^k, \; k \geq 1, \tag{3.6}$$

where $t'_k = t_k(1 - a')$.

**Theorem 3.1.2.** *Let $T$ be a nonexpansive mapping on an either uniformly smooth or strictly convex reflexive Banach space $E$ with a uniformly Gâteaux differentiable norm. Assume that $t_k, \beta_k$ and $\alpha_k$ satisfy conditions (t), (β) and (α), respectively. Fix a real number $a' \in (0, 1)$. Then, the sequence $\{x^k\}$, generated by (3.6), converges strongly to a point in $Fix(T)$.*

**Remark 3.1.2.** (a) Next, we consider the case, when $T$ is a nonexpansive mapping on a closed and convex subset $Q$ of $E$. Clearly, with the starting point $x^1 \in Q$, for any point $x^k \in Q$, $T^k x^k \in Q$. Thus, if the set $Q$ contains the original point of $E$ then $x^{k+1} \in Q$, because $x^{k+1} = \tau_k T^k x^k$ with $\tau_k = 1 - t'_k \in (0, 1)$. It means that method (3.6) is well defined for any $x^1 \in Q$, and hence, Theorem 3.1.2 has value in this case. In the case that the set $Q$ does not contain the original point of $E$, we take $f = a'I + (1 - a')u$ with a fixed $u \in Q$. It is easy to see that $F = I - f$ is also $\eta$-strongly accretive and $\gamma$-strictly pseudocontractive such that $\eta + \gamma > 1$. Then, instead of (3.6), we obtain the Halpern Ishikawa method,

$$\begin{cases} x^1 \in Q, \quad \text{any element,} \\ x^{k+1} = t'_k u + (1 - t'_k) T^k x^k, \ k \geq 1, \end{cases} \tag{3.7}$$

that is method' Qin et al [J. Math. Anal. Appl., 2008] with redenoting $t_k := t'_k$. Clearly, $t_k$ satisfies condition **(**$t$**)** if and only if $t'_k$ is so. Method (3.7), by Theorem 3.1.2, converges strongly in a uniformly smooth or strictly convex reflexive Banach space $E$, meantime, method of Quin above needs stronger conditions

$$\sum_{k=1}^{\infty} |t_{k+1} - t_k| < \infty, \sum_{k=1}^{\infty} |\alpha_{k+1} - \alpha_k| < \infty, \sum_{k=1}^{\infty} |\beta_{k+1} - \beta_k| < \infty. \tag{3.8}$$

compared to the method proposed in the thesis

(b) Let $\tilde{a} > 1$ and let $f$ be an $\tilde{a}$-co-coercive accretive mapping on $E$, i.e.,

$$\langle fx - fy, j(x - y) \rangle \geq \tilde{a} \|fx - fy\|^2, \ \forall x, y \in E.$$

It is easily seen that $f$ is a contraction with constant $1/\tilde{a} \in (0, 1)$, and hence, $F := I - f$ is an $\eta$-strongly accretive mapping with $\eta = 1 - (1/\tilde{a})$. Moreover,

$$\begin{aligned} \langle Fx - Fy, j(x - y) \rangle &= \|x - y\|^2 - \langle fx - fy, j(x - y) \rangle \\ &\leq \|x - y\|^2 - \tilde{a} \|fx - fy\|^2 \\ &\leq \|x - y\|^2 - \gamma \|(I - F)x - (I - F)y\|^2, \end{aligned}$$

for any $\gamma \in (0, \tilde{a}]$. Taking any fixed $\gamma \in ((1/\tilde{a}), \tilde{a}]$ we get that $F$ is a $\gamma$-strictly pseudocontractive mapping with $\eta + \gamma > 1$. Next, by replacing $F$ by $I - f$ in (3.5), the thesis obtain a new viscosity approximation Ishikawa method,:

$$y^{k+1} = T^k (t_k f y^k + (1 - t_k) y^k), \ y^1 \in E, \ k \geq 1, \tag{3.9}$$

that is an improved modification of Quin' method and different from (3.8). Obviously, if $f$ is an $\tilde{a}$-co-coercive accretive mapping on $Q$, a closed convex subset of $E$, then method (3.9) is also well defined for any $y^1 \in Q$.

For a given $\alpha$-co-coercive accretive mapping $f$, we can obtain an $\tilde{\alpha}$-co-coercive accretive mapping $\tilde{f}$ with $\tilde{\alpha} > 1$ by considering $\tilde{f} := \beta f$ with a positive number $\beta < \alpha$.

### 3.1.2. Variational inequality problem on the set of common fixed points of an infiniti family of nonexpansive mappings

In this section, consider (3.1) in the case that $C = \cap_{i\geq 1}\mathrm{Fix}(T_i) \neq \emptyset$, where $\{T_i\}$ is an infinite family of nonexpansive mappings on $E$, it means that:

$$\text{find } p_* \in \cap_{i\geq 1}\mathrm{Fix}(T_i) \text{ such that } \langle Fp_*, j(p_* - p)\rangle \leq 0 \quad \forall p \in \cap_{i\geq 1}\mathrm{Fix}(T_i). \tag{3.10}$$

Let $T^k$ be defined as follows:

$$T^k = (1-\beta_k)I + \beta_k W^k\big((1-\alpha_k)I + \alpha_k W^k\big), \tag{3.11}$$

where $\{W^k\}$ is a sequence of nonexpensive mappings that satisfies the conditions:

($i$) Exist $Wx := \lim_{k\to\infty} W^k x$ for all $x \in E$ and if $\cap_{i\geq 1}\mathrm{Fix}(T_i) \neq \emptyset$ then $\mathrm{Fix}(W) = \cap_{i\geq 1}\mathrm{Fix}(T_i)$.

($ii$) $\lim_{k\to\infty} \sup_{x\in B} \|W^k x - Wx\| = 0$ with $B$ is a bounded subset.

**Remark 3.1.3.** we see that $S^k = \sum_{i=1}^{k} \gamma_i T_i/\tilde{\gamma}_k$ with $\tilde{\gamma}_k = \gamma_1 + \cdots + \gamma_k$ and $V^k = T_1' \cdots T_k'$ where, $T_i' = \gamma_i I + (1-\gamma_i)T_i$ vi $\gamma_i \in (0,\infty)$ such that $\sum_{i=1}^{\infty} \gamma_i = \tilde{\gamma} < \infty$ also satisfies the conditions ($i$) and ($ii$) as in $W^k$.

In this thesis we propose a new method to solve the problem (3.10). That is the combination of steepest-descent method with Ishikawa method. We show that One of the special cases of the newly proposed method is the Halpern iterative method.

**Theorem 3.1.3.** *Let $F$ is $\eta$-$j$-co-coercive accretive and $\gamma$-strictly pseudo-contractive mapping on an either uniformly smooth or strictly convex reflexive Banach space $E$ with a uniformly Gâteaux differentiable norm, suth that $\eta + \gamma > 1$ and $\{T_i\}$ be an infiniti family of nonexpansive mappings on $E$ suth that $\cap_{i\geq 1}\mathrm{Fix}(T_i) \neq \emptyset$. Assuming that $t_k$, $\beta_k$ and $\alpha_k$ satisfy the respective conditions* ($t$), ($\beta$) *and* ($\alpha$). *Then, sequence $\{x^k\}$, determined by (3.3) with $T^k$ in (3.11), is strongly convergence to solution $p_*$ of problem* (3.10).

**Remark 3.1.4.** (a) Remarks 3.1.1 and 3.1.2 have still value when $T^k$ is defined by (3.11).

(b) Taking $\alpha_k = 0$ in (3.3) and (3.11), we obtain the steepest-descent Krasnoselskii-Mann method and its extension to an infinite family of nonexpansive mappings $T_i$ on $E$, that is the method

$$x^{k+1} = (I - t_k F)((1-\beta_k)I + \beta_k W^k)x^k, \ k \geq 1,$$

and its equivalent formula is

$$x^{k+1} = \big((1-\beta_k)I + \beta_k W^k\big)(I - t_k F)x^k, \ k \geq 1, \tag{3.12}$$

(See remark 3.1.1). Replacing $F$ in (3.12) by $(1-a')I$, we get the method

$$y^{k+1} = \big((1-\beta_k)I + \beta_k W^k\big)(1 - t'_k)y^k, \ k \geq 1.$$

strong convergence of which was proved by Shehu in [Taiwanese J. Math., 2015] in uniformly convex and uniformly smooth Banach spaces under conditions $(t)$, $(\beta)$, $\sum_{k=1}^{\infty} \lim_{k\to\infty} \sup_{x\in B} \|W^{k+1}x - W^k x\| = 0$ and $(i)$ in the definition of $W^k$. Marino and Muglia [Optim. Lett., 2015] replacing (ii) in the definition of $W^k$ by $\lim_{k\to\infty} \|W^{k+1}x - W^k x\| = 0$ uniformly in $x \in B$ and combining the steepest-descent method with the Krasnosel'skii-Mann one, studied the methods

$$\begin{aligned} x^{k+1} &= \beta_k x^k + (1-\beta_k)(I - t_k D)W^k x^k \text{ và} \\ x^{k+1} &= \beta_k (I - t_k D)x^k + (1-\beta_k)W^k x^k, \ k \geq 1, \end{aligned} \tag{3.13}$$

in a setting Hilbert space $H$, where $D$ is $\eta$-strongly monotone and $L$-Lipschitz continuous. Strong convergence of (3.13) is proved under conditions $(t)$ with $\lim_{k\to\infty} |t_k - t_{k+1}|/t_{k+1} = 0$, $\beta_k \in (0, \overline{a}]$ with $\lim_{k\to\infty} |\beta_k - \beta_{k+1}|/\beta_{k+1} = 0$ and additional condition on constructing $W^k$ from the given family $\{T_i\}$. We note that the mappings $V^k = T'_1 \cdots T'_k$ where $T'_i = \gamma_i I + (1-\gamma_i)T_i$ with $\gamma_i \in (0,\infty)$ such that $\sum_{i=1}^{\infty} \gamma_i = \tilde{\gamma} < \infty$ and $S^k = \sum_{i=1}^{k} \gamma_i T_i / \tilde{\gamma}_k$ with $\tilde{\gamma}_k = \gamma_1 + \cdots + \gamma_k$ also satisfy conditions (i) and (ii) in the definition of $W^k$ give in the proposals of Buong and et al. the first author et al. introduced the methods,

$$\begin{aligned} x^{k+1} &= (1-\beta_k)x^k + \beta_k S^k (I - t_k F)x^k \text{ và} \\ x^{k+1} &= (1-\beta_k)S^k x^k + \beta_k (I - t_k F)x^k, \end{aligned}$$

Table 3.1: Computational results by (3.24) and (3.27) with $W^k = T_k$.

| $k$ | $x_1^{k+1}$ | $x_2^{k+1}$ | $k$ | $x_1^{k+1}$ | $x_2^{k+1}$ |
|---|---|---|---|---|---|
| 10 | 1.1363636364 | 0.6411155490 | 100 | 1.0148514851 | 0.9431215161 |
| 20 | 1.0714285714 | 0.7700827178 | 200 | 1.0074626866 | 0.9707901594 |
| 30 | 1.0483870968 | 0.8326554114 | 300 | 1.0049833887 | 0.9803526365 |
| 40 | 1.0365853659 | 0.8687796127 | 400 | 1.0037406484 | 0.9851987678 |
| 50 | 1.0294117647 | 0.8921748170 | 500 | 1.0029940120 | 0.9881273689 |

strong convergence of which have been investigated in strictly convex reflexive Banach spaces with a Gâteaux differentiable norm under conditions $(t)$ and $(\beta)$.

(c) In 2012, Li studied also method [fixed point Theory 2012], where $T^k$ defined in (3.11) with $W^k$-mapping of Shimoji and Takahashi Katchang and Kumam proposed the method:

$$x^{k+1} = t_k\gamma f(x^k) + (I - t_k A)T^k x^k,\ \ k \geq 1,$$

a modification of the method of Li above and proved that it converges in the Banach space with a weak continuous duality mapping $j$ under conditions $(t)$, $\lim_{k\to\infty}\beta_k = 0$ and $\lim_{k\to\infty}\alpha_k = 0$, where $A$ is a strongly positive bounded linear mapping on $E$ and $\gamma$ is a some positive constant.

## 3.2. Numerical experiments

Obviously, for the family of nonexpansive mappings $T_i = (1-1/(i+1))I$ with $E = \mathbb{R}^1$, we have that $\cap_{i\geq 1} Fix(T_i) = \{0\}$ and $\lim_{k\to\infty} T_k x = Ix$ for each $x \in \mathbb{R}^1$. Thus, condition (i) in the definition of $W^k$ is not satisfied, because $Fix(I) = \mathbb{R}^1$.

It is easy to see that the family $\{T_i = P_{C_i}\}$, where $P_{C_i}$ is the metric projection of $H = \mathbb{E}^2$, an Euclidian space, onto the set $C_i = \{x = (x_1, x_2) \in H : a_i \leq x_2 \leq b_i\}$ with $a_i = 1-1/(i+1)$ and $b_i = 2+1/(i+1)$ for all $i \geq 1$, satisfies conditions (i) and (ii) in the definition of $W^k$. In this case, we have that $C = \cap_{i=1}^{\infty} C_i = \{x \in \mathbb{E}^2 : 1 \leq x_2 \leq 2\}$ and we can take $W^k = T_k$ for all $k \geq 1$. Taking $u = (1.0; 0.0)$, we have that the solution of (3.26) $p_* = (1.0; 1.0)$. The computational results by method (3.24) and $T^k$ in (3.27) with starting point $x^1 = (2.5; 2.5)$, $t_k = 1/(k+1)$, $\beta_k = 0.2+1/(k+1)$ and $\alpha_k = 1/(k+1)$ are given in Table 3.1.

Table 3.2: Computational results by (3.24) and (3.27) with $W^k = T_k$.

| $k$ | $x_1^{k+1}$ | $x_2^{k+1}$ | $k$ | $x_1^{k+1}$ | $x_2^{k+1}$ |
|---|---|---|---|---|---|
| 10 | 0.8226906920 | 0.9967100188 | 100 | 0.8216765320 | 1.3503455533 |
| 20 | 0.8116106625 | 1.1196844726 | 200 | 0.8261485102 | 1.4207098495 |
| 30 | 0.8123975068 | 1.1852032060 | 300 | 0.8280615950 | 1.4464230799 |
| 40 | 0.8142620005 | 1.2298614455 | 400 | 0.8291386059 | 1.4595495405 |
| 50 | 0.8160321266 | 1.2628985966 | 500 | 0.8298349294 | 1.4675113528 |

In the case that $a_i = 1+1/(i+1)$, we have $C = \{x \in \mathbb{E}^2 : 1.5 \le x_2 \le 2\}$ and $p_* = (1.0; 1.5)$. Moreover, condition (i) in the definition of $W^k$ for $T_k$, i.e. $W^k = T_k$, does not hold. For computation by (3.24), we use $W^k = S^k$ in (3.27) where $S^k = \sum_{i=1}^{k} \gamma_i T_i/\tilde{\gamma}_k$ vi $\tilde{\gamma}_k = \gamma_1 + \cdots + \gamma_k$ vi $\gamma_i = 1/i(i+1)$. with $\gamma_i = 1/i(i+1)$. The results of computation are given in Table 3.2.

The numerical results show the effectiveness of the method.

**Conclude**

Chapter 3 of the Thesis proposes a method to solve the problem of variational inequalities on a fixed set of points of one or a family of non-expanded mappings, proves the strong convergence of the proposed method. The chapter proposed new results that are extensions of known results with simpler conditions and give strong convergence results, The convergence rate of the iterative methods has also been illustrated with a clear numerical example.

## CONCLUSION

The thesis has achieved the following results:

1. Propose iterative regularization methods to approximate a solution for the multiple-sets split feasibility problem in real Hilbert spaces, that converges strong under some conditions an parameter and give some numerical examples (see [2] in List of published projects). The effectiveness of the proposed method is that the iterative parameter $\gamma_k$ is selected independent on $\|A\|$.
2. Suggest iterative regularization methods to approximate a solution for the multiple-sets split equality problem in real Hilbert spaces, strong convergence of the method is proved with numerical experements for illustration (see [3] in List of published projects). In this method, we use only finite summ elements, that is different from the existence in the literatere.
3. We introduce a new iterative method, a combination of the steepest-descent method with the Ishikawa one for solving a variational inequalities over the set of common fixed point of an infinite family of nonexpansive mappings and give numerical example for illustration (see [1] in List of published projects).

## FURTHER RESEARCH DIRECTIONSurther

In the next stage:

1. We extend the results in Chapters 2 and 3 to the case $T_i$, $U_i$ is a pseudo contraction mapping in Hilbert space.
2. Research on iterative correction methods Extragradient' type for a class of variational inequalities with $F$ is $\eta$-strong monotony and $L$-Lipschitz continuous.
3. Research on the combination of inertial components and iterative regularization methods to increase the convergence speed of this method.