MINISTRY OF EDUCATION AND TRAINING

VIETNAM ACADEMY OF SCIENCE AND TECHNOLOGY

GRADUATE UNIVERSITY OF SCIENCE AND TECHNOLOGY

THAM NHU PHONG

ON THE FAST REACTION LIMIT OF NONLINEAR BULK-SURFACE REACTION-DIFFUSION SYSTEMS

MASTER'S THESIS IN MATHEMATICS

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SUPERVISORS :
1. Assoc. Prof. Dr. Hoang The Tuan
2. Dr. Tang Quoc Bao

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Commitment

I declare that this thesis is the result of a research process under the supervision of Assoc. Prof. Hoang The Tuan and Dr. Tang Quoc Bao. All information and ideas cited from other authors have their sources clearly stated. I completely agree being responsible for these commitments.

Hanoi, September 2024

Thẩm Như Phong

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Author

Thâm Như Phong

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LIST OF ABBREVIATIONS AND SYMBOLS

$\mathbb{R}^n$	The n-dimensional Euclidean space
$\|x\|$	The Euclidean norm of x in Euclidean space
$x \cdot y$	Inner product of x, y in Euclidean space
$\\|x\\|_X$	Norm in Banach space X
$(x, y)_H$	Inner product in Hilbert space H
$L^p(\Omega)$	The space of functions $f : \Omega \to \mathbb{R}$ that $p-$integrable
$L^p(\Gamma)$	The space of functions $f : \partial\Omega := \Gamma \to \mathbb{R}$ that $p-$integrable
$H^1(\Omega)$	Sobolev space of functions with weak derivative in $L^2(\Omega)$
$H^1(\Gamma)$	Sobolev space of functions with weak derivative in $L^2(\Gamma)$
X^*	dual space of X
$H^{-1}(\Omega)$	Dual space of $H^1(\Omega)$
$H^{-1}(\Gamma)$	Dual space of $H^1(\Gamma)$
$\langle x, y\rangle_{X^* \times X}$	Pairing between $x \in X^*$ and $y \in X$
$\langle x, y\rangle_\Omega$	Pairing between $x \in H^{-1}(\Omega)$ and $y \in H^1(\Omega)$
$\langle x, y\rangle_\Gamma$	Pairing between $x \in H^{-1}(\Gamma)$ and $y \in H^1(\Gamma)$
$L^2(0, T; X)$	Space of functions $f : [0, T] \to X$, with $\int_0^T \\|f(t)\\|_X^2 dt$ is finite
$\nu(x)$	unit outward normal vector at $x \in \Gamma := \partial\Omega$ of the domain Ω
∇f	Gradient operator of f
Δf	Laplace operator of f
$\nabla_\Gamma f$	Tangential gradient operator of f on Γ
$\Delta_\Gamma f$	Laplace–Beltrami operator of f on Γ

INTRODUCTION

Motivation

Fast reaction limits (or instantaneous limits) is a topic that has been developed in recent years. In particular, the main idea is that in a series of reactions that there are some reactions, which occur very quickly, then we will approximate those reactions by their equilibrium states, which reduce the numbers of reaction and variable, and make the system become simpler. In fact, the first research on this topic dates back to the beginning of the last century through discoveries in the field of biochemistry with the article by Michaelis and Menten [1], from which it quickly became a technique that popular in technical chemistry and related fields. However, although it is widely applied in practical problems, a rigorous mathematical proof of this method has not been paid attention for a long time. Moreover, abusing this method can lead to incorrect results, with a counterexample given in [2]. Therefore, a detailed proof of this reduction idea, even in just for a system contains one reaction, is being interested.

Besides, early researches on the fast reaction limits were mainly for the systems of ordinary differential equations, which were intended to simplify the problem by considering the concentration at all locations to be the same (or also called homogeneity). For realistic models, the inhomogeneous property, for example, spatial diffusion, often have to be taken into account. It led to new studies on the fast reaction limits for systems of partial differential equations, starting from 1980 with Evans's paper studying a system of two diffusion equations in the 3-dimensional case (see [3]). Since then, this research direction has become popular not only because of its applications but also theoretically when it comes to many interesting mathematical structures, some typical works include, [4, 5, 6, 7]. In particular, the study of fast reaction limits for bulk-surface reaction-diffusion systems (or volume-surface reaction-diffusion systems) has received a lot of attention in recent years when the system is used for modeling in a number of fields such as population biology, materials science or cell biology (see, e.g., [8, 9, 10]). Here, an example is article [6], where the authors consider the reaction equation:

$$U \underset{k}{\overset{k}{\leftrightarrows}} V$$

with U is a substance on volume Ω, V is a substance on boundary $\Gamma := \partial\Omega$ and k is the reaction rate. Denote (u_k, v_k) the concentration of U and V, corresponding to reaction rate k. It has been shown in article [6] that

$$(u_k, v_k) \to (w, w|_\Gamma)$$

strongly in $L^2(0, T; H^1(\Omega) \times H^1(\Gamma))$ as $k \to \infty$, where w is the weak solution of a heat equation with dynamical boundary and $w|_\Gamma$ is the trace of w on the boundary Γ. At the end of the article, the authors have given an open question in nonlinear case, which means considering the stoichiometrics in the chemical reaction are arbitrary positive numbers, what can we conclude about the fast reaction limits. In this thesis, we will have a result about this open question.

Main content

Consider an open, bounded, and connected set $\Omega \subset \mathbb{R}^n$, we will research the asymptotic behavior of the reversible chemical reactions with the form

$$\alpha U \underset{k}{\overset{k}{\leftrightarrows}} \beta V \tag{1}$$

when the reaction rate $k > 0$. Similar to the case $\alpha = \beta = 1$ as above, U is a (volume-)substance on Ω, V is a (surface-)substance on $\Gamma := \partial\Omega$ (we assume that the boundary Γ is smooth enough) and the stoichiometric coefficients $\alpha, \beta \geq 1$ (in the view of chemistry, we can assume that $\alpha, \beta \in \mathbb{Z}^+$).

The reaction (1) is motivated from a process called asymmetric stem cell division, where there is a reaction between the proteins in cell cortex and cell cytoplasm. An example is in SOP stem cells of Drosophila, the division operates around a key protein called Lgl (Lethal giant larvae) and the chemical reaction is the reaction between the Lpl protein on cytoplasm and the one on membrane (see [8] and its references).

To rewrite reaction (1) in a mathematical problem, we set $u(x,t)$ and $v(x,t)$ stand for the concentration of U and V at position x and time t. Due to the mass action law and Fick's second law, we can model (1) by the following reaction-diffusion system problem

$$\begin{cases} u_t - d_u \Delta u = 0, & x \in \Omega, t > 0, \\ d_u \nabla u \cdot \nu = -\frac{\alpha}{\epsilon}(u^\alpha - v^\beta), & x \in \Gamma, t > 0, \\ \partial_t v - d_v \Delta_\Gamma v = \frac{\beta}{\epsilon}(u^\alpha - v^\beta), & x \in \Gamma, t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \Omega, \\ v(x,0) = v_0(x) \geq 0, & x \in \Gamma, \end{cases} \tag{2}$$

where we change the parameter $\epsilon = 1/k$. In the above system, positive constants d_u and d_v are the diffusion coefficients of U in Ω and V in Γ, respectively. We use the notation Δ for Laplace operator on Ω, Δ_Γ for the Laplace–Beltrami operator on Γ and ν for the outward pointing unit normal vector field on the boundary Γ. The initial condition (u_0, v_0) are some non-negative function defined on Ω and Γ, which should be bounded on corresponding area. The second and third equations of (2) are coupled by the rate function $u^\alpha - v^\beta$, which is a consequence of mass action law. The system (2) is a bulk-surface reaction-diffusion system (or volume-surface reaction-diffusion system) with nonlinear boundary coupling, and we call nonlinear bulk-surface-reaction-diffusion system for short. From the system (2), we can show that it satisfies the following mass conservation law

$$\alpha \int_\Omega u(x,t)dx + \beta \int_\Gamma v(x,t)dS = \alpha \int_\Omega u_0(x)dx + \beta \int_\Gamma v_0(x)dS = M_0,$$

for each $t > 0$, and M_0 is some non-negative constant.

In this thesis, we will concentrate on two questions. The first one is about the fast reaction limit, that is, the reaction rate $k \to \infty$, or, ϵ tend to 0, how the solution of (2) converges. Let (u_ϵ, v_ϵ) be the unique weak solution of the system (see [10]), we will show that there exists a subsequene of $\{(u_\epsilon, v_\epsilon)\}$, converges to (w, z) as the parameter $\epsilon \to 0$, where $z = (w|_\Gamma)^{\alpha/\beta}$ and w is the solution to a heat equation with nonlinear dynamic boundary condition, see (3).

On the second question, we will consider on the convergence rate. In detail, we will show that in the case $\alpha = \beta$, and adding some technical assumption, we will have a estimate in term ϵ for $|u_\epsilon - w|$ and $|v_\epsilon - z|$ (in the L^2- norm).

Methodology

In this thesis, we will prove the fast reaction limits by using functional analytic approach. In this method, most of recent articles have done with three main steps (see [4],[6],[11]). The first one is formally guessing the limit, base on the structure of system. In the system (2), formally, one can expect that $u_\epsilon \to w$ and $v_\epsilon \to z$ as $\epsilon \to 0$, in a certain sense, and due to the reactions in (2), it is expected that $u_\epsilon^\alpha - v_\epsilon^\beta \to 0$ on Γ, as the reaction rate tends to infinity, which means the limit $w^\alpha = z^\beta$ on Γ. Therefore, from the original system (2), by substituting (u, v) by $(w, (w|_\Gamma)^{\alpha/\beta})$, and combining the second and third equation to remove the parameter ϵ, the formal limit system of the original system (2) as $\epsilon \to 0$ would be

$$\begin{cases} \partial_t w - d_u \Delta w = 0, & x \in \Omega, t > 0, \\ d_u \nabla w \cdot \nu = -\frac{\alpha}{\beta}[\partial_t(w^{\alpha/\beta}) - d_v \Delta_\Gamma (w^{\alpha/\beta})], & x \in \Gamma, t > 0, \\ w(x,0) = u_0(x), & x \in \Omega, \\ w|_\Gamma(x,0) = v_0^{\beta/\alpha}(x), & x \in \Gamma. \end{cases} \tag{3}$$

This a heat equation with nonlinear dynamic boundary conditions.

The next step is to prove uniform estimations using the L^p approach, entropy function, and so on, which is based on the structure of the system (see [12]). In the last step, by these estimations, we will take the limit, such as applying Aubin–Lions Lemma (see Lemma 1.1) and then define the solution of the limit system. In these step, the second step is the most difficult since we do not have any global method for this one, and also we do not know which estimation is required.

For the second problem, the convergence rate, we use a similar calculation in [7] by setting $U = u - w$ and $V = v - z$, and then estimates the L^2-norm of them with their time derivative. During the calculation, we will need some technical assumptions.

Thesis structure

This thesis will be structured as following: In the first chapter, we will present some important knowledge for the result, including Sobolev spaces, Aubin–Lions Lemma and the existence of solution for system (2). Then, on Chapter 2, we show that the convergence. After that, the limit system and convergence rate will be shown on Chapter 3. Finally, we will discuss some questions related to this problem.

Chapter 1

PRELIMINARIES

The target of this chapter is to present some basic notations, definitions and theorems, which are necessary for the following chapters.

1.1 L^p spaces

A vector $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$, the Euclidean norm is given by

$$|x| = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

For two vectors $x, y \in \mathbb{R}^n$, $x = (x_1, \ldots, x_n)^T$ and $y = (y_1, \ldots, y_n)^T$, the inner product between x and y is given by

$$x \cdot y = \sum_{i=1}^{n} x_i y_i.$$

Otherwise, if it is a norm defined for a Banach space X, we will use the notation $\|\cdot\|_X$, and, for an inner product for a Hilbert space H, we denote by $(\cdot, \cdot)_H$.

Definition 1.1 ([13]). *Consider an open subset Ω of $\mathbb{R}^n$, we define $L^p(\Omega)$*

is the space of all functions $f : \Omega \to \mathbb{R}$ *with the finite* L^p-*norm*

$$\|f\|_{L^p(\Omega)} := \left(\int_\Omega |f(x)|^p dx \right)^{1/p}$$

if $1 \leq p < \infty$. *For the case* $p = \infty$, *we use the norm*

$$\|f\|_{L^\infty(\Omega)} := \operatorname*{ess\,sup}_{x \in \Omega} |f(x)|.$$

We call a property that holds almost everywhere on Ω, if it is true on Ω except a (or union of) subset(s) that has measure equal to 0, and written short as "a.e." (see [13, Section 1.4]).

Definition 1.2 ([14]). *Let* U, V *be open subset of* $\mathbb{R}^n$. *We write* $U \subset\subset V$, *if* $U \subset \bar{U} \subset V$, *and we say the subset* U *us compactly contained in* V.

Definition 1.3 ([14]). *The functional sapce* $L^p_{loc}(\Omega)$ *contains all functions* $f : \Omega \to \mathbb{R}$, *such that for every* $K \subset\subset \Omega$, $f \in L^p(K)$.

Theorem 1.1 (Hölder's inequality, [14]). *Assume* $1 \leq p, q \leq \infty$, $1/p + 1/q = 1$. *Then, if* $u \in L^p(\Omega), v \in L^q(\Omega)$, *we have*

$$\int_\Omega |uv| dx \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}.$$

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two Banach spaces, the product space $X \times Y$ is the space that contains all the elements (x, y), with $x \in X$ and $y \in Y$. In this case, we use the norm in $X \times Y$

$$\|(x, y)\|_{X \times Y} := \|x\|_X + \|y\|_Y.$$

Next, we state three important convergence theorems.

Theorem 1.2 (Fatou's Lemma, [14]). *Assume the functions* $\{f_k\}_{k=1}^\infty$ *are* *non-negative and measurable, we have*

$$\int_\Omega \liminf_{k\to\infty} f_k dx \le \liminf_{k\to\infty} \int_\Omega f_k dx.$$

Remark 1.1. *The Fatou's lemma still works if we change the non-negative condition to a lower bound or an upper bound condition. Indeed, if there exists an integrable function g, where for all k, $f_k \ge -g$ a.e.; we can apply the original Fatou's lemma to the sequence of functions $f_k + g \ge 0$. We obtain*

$$\int_\Omega \liminf_{k\to\infty}(f_k + g)dx \le \liminf_{k\to\infty} \int_\Omega (f_k + g)dx.$$

So,

$$\int_\Omega \liminf_{k\to\infty} f_k dx \le \liminf_{k\to\infty} \int_\Omega f_k dx.$$

For the upper bound case $f_k \le g$, we apply to sequence $\{g - f_k\}$, which we need to change $\liminf$ *to* $\limsup$ *and change the sign of inequality to get*

$$\int_\Omega \limsup_{k\to\infty} f_k dx \ge \limsup_{k\to\infty} \int_\Omega f_k dx.$$

Theorem 1.3 (Monotone Convergence Theorem, [14]). *Consider increasing sequence of non-negative functions $\{f_k\}_{k=1}^\infty$, that is, $0 \le f_i \le f_j$ (where $i \le j$), we have*

$$\lim_{k\to\infty} \int_\Omega f_k dx = \int_\Omega \lim_{k\to\infty} f_k dx.$$

Theorem 1.4 (Dominated Convergence Theorem, [14]). *Assume the functions $\{f_k\}_{k=1}^\infty$ are integrable and satisfy $\lim_{k\to\infty} f_k = f$. Suppose also $|f_k| \le g$ a.e. for all k, we have*

$$\lim_{k\to\infty} \int_\Omega f_k dx = \int_\Omega \lim_{k\to\infty} f_k dx = \int_\Omega f dx.$$

Definition 1.4 ([15]). *We denote X^* the dual space of X, which is the space that contains all the linear functional on X. The norm on X is the* *operator norm*

$$\|f\|_{X^*} = \sup_{\|x\|_X \leq 1} \langle f, x \rangle_{X^* \times X},$$

which $\langle f, x \rangle_{X^* \times X} = f(x)$ *is the pairing of* $f \in X^*$ *and* $x \in X$.

Remark 1.2. *Let* $1 < p < \infty$*, the dual sapce of* $L^p(\Omega)$ *is* $L^q(\Omega)$*, where* $1/p + 1/q = 1$

1.2 Sobolev spaces

In this section, we will briefly present the theory of Sobolev spaces, which is a powerful tool of functional analysis to work on problems in partial differential equations.

1.2.1 Sobolev spaces on bounded domains

We first start with the Sobolev space on the domain Ω, where Ω is an open, bounded subset of $\mathbb{R}^n$. Sobolev space comes from the idea of defining a functional space that is Banach but still has some smoothness properties, or its function has a weak derivative - a weakened form of the classical derivative.

Definition 1.5 (Weak derivative, [14])**.** *Denote* $C_c^\infty(\Omega)$ *the space of infinitely differentiable functions with compact support in* Ω*. Suppose that* $u, v \in L^1_{loc}(\Omega)$ *and a multi-index* $\alpha = (\alpha_1, \ldots, \alpha_n)$*. We say that* v *is the* α^{th}*- weak partial derivative of* u*, written as*

$$D^\alpha u = v,$$

if the following equation holds for all $\varphi \in C_c^\infty(\Omega)$

$$\int_\Omega u D^\alpha \varphi dx = (-1)^{|\alpha|} \int_\Omega \varphi v dx.$$

With the weak derivative mentioned above, we now give the definition for Sobolev spaces

Definition 1.6 (Sobolev spaces, [14]). *With $k \in \mathbb{N}, 1 \leq p < \infty$, the Sobolev space*

$$W^{k,p}(\Omega)$$

contains all function $u : \Omega \to \mathbb{R}$ such that for each multi-index α, with $|\alpha| \leq k$, u has the α^{th}-weak derivative $D^\alpha u$ and it belongs to $L^p(\Omega)$.

Notice that with $k = 0$, the Sobolev space $W^{0,p}(\Omega) = L^p(\Omega)$. Next, we go to the norm of Sobolev space.

Proposition 1.1 (Norm of Sobolev spaces, [14]). *Sobolev space $W^{k,p}(\Omega)$ is a Banach space, equipped by the following norm*

$$\|u\|_{W^{k,p}(\Omega)} := \left(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^p dx \right)^{1/p}.$$

Notation: In the thesis, we mostly work in the Sobolev spaces with $p = 2$, and we use the notation:

$$H^k(\Omega) := W^{k,2}(\Omega) \quad (k \in \mathbb{Z}^+),$$

where the letter H is used since it is a Hilbert space.

Proposition 1.2 ([14]). *The Sobolev space $H^k(\Omega)$ is a Hilbert space, with the inner product*

$$(u, v)_{H^k(\Omega)} := \sum_{|\alpha| \leq k} \int_\Omega (D^\alpha u)(D^\alpha v) dx.$$

One of the most significant difference between $L^p(\Omega)$ and $W^{k,p}(\Omega)$ spaces is that Sobolev spaces allow us to work with the boundary of Ω via the trace theorems. Before stating a theorem, we need to define the smooth domain (or domain with smooth boundary).

Definition 1.7 (Smooth domain, [16]). *An open set $\Omega \subset \mathbb{R}^n$ is C^k if for each point $x_0 \in \Gamma := \partial\Omega$, there exists a system of coordinates $(y_1, \ldots, y_{n-1}, y_n) \equiv (\mathbf{y}', y_n)$, with origin x_0 and a ball $B(x_0, r)$ and a function ϕ (on y-coordinate), defined in a neighborhood $N_{x_0} \subset \mathbb{R}^{n-1}$ of $\mathbf{y}' = 0'$, satisfies the conditions*

- $\phi \in C^k(N_{x_0})$, $\phi(0') = 0$;
- $\Gamma \cap B(x_0, r) = \{(\mathbf{y}', y_n) : y_n = \phi(\mathbf{y}'), \mathbf{y}' \in N_{x_0}\}$;
- $\Omega \cap B(x_0, r) = \{(\mathbf{y}', y_n) : y_n > \phi(\mathbf{y}'), \mathbf{y}' \in N_{x_0}\}$.

A domain is Lipschitz if the mapping ϕ is Lipschitz.

Remark 1.3. *From here, a boundary Γ of the domain Ω is C^k (Lipschitz) if the domain is C^k (Lipschitz).*

Theorem 1.5 (Trace theorem, [14]). *Assume Ω is bounded and C^1, there exists a bounded linear operator $T : W^{1,p}(\Omega) \to L^p(\Gamma)$, with $1 \le p < \infty$, such that*

- $Tu = u|_\Gamma$ *if* $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$.
- *For all* $u \in W^{1,p}(\Omega)$: $\|Tu\|_{L^p(\Gamma)} \le C\|u\|_{W^{1,p}(\Omega)}$, *where C is not depend on u.*

From here, we use the notation $u|_\Gamma$, which stands for the trace of $u \in H^1(\Omega)$.

Remark 1.4. *The space $L^p(\Gamma)$ in the above theorem is the space of all function $u : \Gamma \to \mathbb{R}$ with the finite $L^p(\Gamma)-$norm*

$$\|u\|_{L^p(\Gamma)} := \left(\int_\Gamma u^p dS\right)^{1/p}$$

where $1 \leq p < \infty$, and the integration is defined by localization (see, e.g., [16]). For $p = \infty$, we use the essential supremum norm

$$\|u\|_{L^\infty(\Gamma)} := \operatorname*{ess\,sup}_{x\in\Gamma} |u(x)|.$$

1.2.2 Sobolev spaces on surfaces

As discussed in the introduction, we not only work with functions on domain Ω but also the functions defined on its boundary. Assume that the boundary Γ of the domain Ω is at least C^2. A way to define the Sobolev spaces on Γ is to consider the boundary as a compact manifold, where various books have mentioned it (see, e.g., [17] and [18]). In this thesis, we will present another way to establish Sobolev spaces $W^{k,p}(\Gamma)$ on the boundary Γ, which is summarized from [19] and [20]. The authors define the Sobolev space via the weak derivative in these articles. Denote $\nu(x) = (\nu_1(x), \ldots, \nu_n(x))$ is the unit outward normal vector at $x \in \Gamma$ (see [14]), we first introduce the tangential gradient and Laplace–Beltrami operator.

Definition 1.8 (Tangential gradient and Laplace–Beltrami operator, [20])**.** *Let $f : \Gamma \to \mathbb{R}$ be differentiable, the tangential gradient of f at point $x \in \Gamma$ is defined by the projection*

$$\nabla_\Gamma f(x) = \nabla \bar{f}(x) - [\nabla \bar{f}(x) \cdot \nu(x)]\nu(x),$$

where $\bar{f}$ is a smooth extension of f to an n dimensional neighborhood U of Γ (about constructing $\bar{f}$, see, e.g., [20, Section 2.3]) and ∇ is the normal gradient in $\mathbb{R}^n$.

The Laplace–Beltrami operator is the Laplace operator on Γ, given by

$$\Delta_\Gamma f = \nabla_\Gamma \cdot \nabla_\Gamma f = \sum_{i=1}^{n} \bar{D}_i \bar{D}_i f,$$

and $\bar{D}_i$ is the coordinate of tangential gradient

$$\nabla_\Gamma f(x) = (\bar{D}_1 f(x), \ldots, \bar{D}_n f(x)).$$

Let $H : \Gamma \to \mathbb{R}$, given by

$$H(x) = \sum_{i=1}^{n} \bar{D}_i \nu_i(x) \quad \text{for } x \in \Gamma$$

be the mean curvature, the weak derivative and Sobolev spaces on surfaces can be defined as below.

Definition 1.9 (Weak derivative and Sobolev spaces on boundary, [20])**.** *A function $f \in L^1(\Gamma)$ has the weak derivative $v_i = \bar{D}_i f \in L^1(\Gamma)$; if for all test function $\varphi \in C^1(\Gamma)$ with compact support $\overline{\{x \in \Gamma \,|\, \varphi(x) \neq 0\}} \subset \Gamma$, the following equation holds*

$$\int_\Gamma f \bar{D}_i \varphi dS = -\int_\Gamma \varphi v_i dS + \int_\Gamma f \varphi H \nu_i dS.$$

Then, the Sobolev space $W^{1,p}(\Gamma)$ is defined by

$$W^{1,p}(\Gamma) = \{f \in L^p(\Gamma) | \bar{D}_i f \in L^p(\Gamma), i = 1, 2, .., n\},$$

with the norm

$$\|f\|_{W^{1,p}(\Gamma)} = \left(\|f\|^p_{L^p(\Gamma)} + \|\nabla_\Gamma f\|^p_{L^p(\Gamma)}\right)^{1/p}.$$

We will use the notation $H^1(\Gamma) := W^{1,2}(\Gamma)$ for this Hilbert space.

1.2.3 Spaces involving time

In this part, we will present the theory in spaces of function that map a time interval into a Banach space X. These functions are essential since we are studying the parabolic partial differential equations. The idea is to think of a function $u(x,t)$ as a family of functions $u(t)$, that each one is constructed as a function belongs to X.

Definition 1.10 (Measurable function, [14]). *A function $f:[0,T]\to X$ is called measurable if there exists a sequence of functions $\{s_k\}$ such that*

$$s_k(t)\to f(t) \text{ for a.e. } 0\le t\le T,$$

and s_k has form

$$s_k(t)=\sum_{i=1}^{m}\lambda_{E_i}(t)u_i,\ t\in[0,T],$$

where E_i is a Lebesgue measurable subset of $[0,T]$, λ_{E_i} is the indicator function on E_i and $u_i\in X$.

We mow recall the definition of the L^p-space of the Banach-space valued functions.

Definition 1.11 (Bochner spaces, [14]). *The space $L^p(0,T;X)$ contains all measurable functions $u:[0,T]\to X$ that has finite norm*

$$\|u\|_{L^p(0,T;X)}:=\left(\int_0^T\|u\|_X^p\right)^{1/p}$$

for $1\le p<\infty$. For $p=\infty$, the norm is defined by

$$\|u\|_{L^\infty(0,T;X)}:=\operatorname*{ess\,sup}_{t\in[0,T]}\|u\|_X.$$

Definition 1.12 ([14]). *The space $C([0,T];X)$ contains all continuous functions $u:[0,T]\to X$ with the norm*

$$\|u\|_{C([0,T];X)}:=\sup_{t\in[0,T]}\|u\|_X.$$

We can expand to the definition of (strongly) differentiable continuous functions $C^k([0,T],X)$ by the Fréchet derivative (see [16, Section 7.11]).

The weak derivative of Banach-space valued functions is given by

Definition 1.13 (Weak derivative, [14]). *For $u \in L^1(0,T;X)$, a function $v \in L^1(0,T;X)$ is called weak time derivative of u, and written: $u' = v$, if*

$$\int_0^T \phi'(t)u(t)dt = -\int_0^T \phi(t)v(t)dt$$

for all test functions $\phi \in C_c^\infty(0,T)$, the space of all infinitely differentiable functions with compact support on $(0,T)$.

In the next chapters, we will work with the spaces that the functions locate in Hilbert space $H^1(\Omega)$ (or $H^1(\Gamma)$) and its time derivative belongs to its dual space $H^{-1}(\Omega)$ (or $H^{-1}(\Gamma)$) (we denote $H^{-1}(\Omega) := H^1(\Omega)^*$). The following is an essential theorem about what happens if we have a space with properties as above.

Theorem 1.6 ([13]). *Suppose that $u \in L^2(0,T;H^1(\Omega))$, and $u' \in L^2(0,T; H^{-1}(\Omega))$. Then, $u \in C([0,T];L^2(\Omega))$, and we have the identity*

$$\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} = 2\langle u'(t), u(t)\rangle_{H^{-1}(\Omega)\times H^1(\Omega)}$$

for a.e. $t \in [0,T]$. The notation $\langle u'(t), u(t)\rangle_{H^{-1}(\Omega)\times H^1(\Omega)}$ denotes the paring between $u'(t) \in H^{-1}(\Omega)$ and $u(t) \in H^1(\Omega)$.

Remark 1.5. *Beside u', the notations u_t, $\partial_t u$ or $\frac{\partial}{\partial t}u$ also present the (weak) time derivative of u.*

Remark 1.6. *The theorem also works when we replace $H^{-1}(\Omega), L^2(\Omega)$ and $H^1(\Omega)$ by V^*, H and V, respectively, where the relation $V \subset H$ is dense and continuous (see, e.g., [13]). In the thesis, we will work with the triple $H^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$ (and the corresponding in Γ). For the rest of this thesis, we will use the notation: $\langle \cdot,\cdot\rangle_\Omega$ for the pairing in $H^{-1}(\Omega)$ and $H^1(\Omega)$; and $\langle \cdot,\cdot\rangle_\Gamma$ for the pairing in $H^{-1}(\Gamma)$ and $H^1(\Gamma)$.*

Remark 1.7. *The identity in the Theorem 1.6 can be proved by using a smooth functions sequence* $\{u_n\}$ *that approximates* u *(e.g., mollifier function). For a.e.* $t \in [0, T]$*, we have*

$$\|u_n(t_2)\|^2_{L^2(\Omega)} - \|u_n(t_1)\|^2_{L^2(\Omega)} = 2\int_{t_1}^{t_2} \langle u_n'(\tau), u_n(\tau)\rangle_\Omega d\tau.$$

Then, the conclusion is obtained by taking the limit $n \to \infty$ *(see full proof in [13]). Due to the proof, we can expand to calculate the time derivative of the following function*

$$F[u](t) := \int_\Omega u^\alpha(x, t)dx,$$

where $\alpha \in \mathbb{Z}^+$*,* u *is non-negative,* $u \in L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^\infty(\Omega))$ *and* $u' \in L^2(0, T; H^{-1}(\Omega))$*. We have for a.e.* $0 \le t \le T$

$$\frac{d}{dt}\int_\Omega u^\alpha dx = \alpha\langle u'(t), u(t)\rangle_\Omega.$$

At the end of this section, we present a theorem about the dual space of $L^p(0, T; X)$.

Theorem 1.7 (Dual space, [13])**.** *Let* X *is reflexive,* $p, q > 1$ *and* $1/p + 1/q = 1$*, the dual space of* $L^p(0, T; X)$ *can be identified with the space* $L^q(0, T; X^*)$*.*

1.3 Weak convergence and embedding theorems

1.3.1 Weak convergence

Definition 1.14 (Weak convergence, [14])**.** *Let denote* X *is a real Banach space. A sequence* $\{u_k\} \subset X$ *is called weakly converges to* $u \in X$*, notation with the "half-arrow"*

$$u_k \rightharpoonup u,$$

if it provides

$$\lim_{k\to\infty} \langle v, u_k \rangle_{X^*\times X} = \langle v, u \rangle_{X^*\times X}$$

for all v in X^*.

Remark 1.8. *The strong convergence implies the weak convergence and the limit of weak convergence is unique (see, e.g., [16]).*

Theorem 1.8 (Uniform boundedness implies weak convergence, [14])**.** *Consider a Hilbert space H and a sequence of functions $\{u_k\}_{k=1}^{\infty}$ bounded uniformly in H. Then, there exists a subsequence of $\{u_k\}$ that converges weakly in H.*

1.3.2 Embedding theorems

Definition 1.15 (Compact embedding, [14])**.** *Consider Banach spaces $X \subset Y$, X is called compactly embedded in Y if the following conditions are satisfied*

- *The embedding X into Y is continuous, i.e $\|x\|_X \leq C\|x\|_Y$ for all $x \in X$ and constant C do not depend on x.*
- *Every bounded sequence in X has a subsequence that converges in Y.*

We need an important result about the compact embedding of Sobolev space.

Theorem 1.9 ([16])**.** *The embedding $H^1(\Omega)$ into $L^2(\Omega)$ is compact, with Ω is an open, bounded subset of $\mathbb{R}^n$, and the boundary is C^1.*

Similar to the above theorem, we have an analogous theorem for the boundary Γ, which is a case of Proposition 3.4, chapter 4 in [18]

Theorem 1.10 ([18])**.** *The embedding $H^1(\Gamma)$ into $L^2(\Gamma)$ is compact.*

Next, we state the Aubin–Lions Lemma (or Aubin– Lions–Simon lemma), which is a essential tool for nonlinear fast reaction limit problems.

Lemma 1.1 (Aubin–Lions, [21]). *For $1 < p, q < \infty$, we denote*

$$W := \{u \in L^p(0,T;X_0) : u' \in L^q(0,T;X_2)\}$$

with X_0, X_1, X_2 are Banach spaces such that X_0 is compactly embedded in X_1 and X_1 is continuously embedded in X_2. Then, the embedding of W into $L^p(0,T;X_1)$ is compact.

Remark 1.9. *The above lemma can be restated as follows (see [22]): we consider a sequence of functions $\{u_n\}$ of two variables t and x, with the time variable $t \in [0,T]$ and space variable x. Let X_0, X_1 and X_2 be Banach space. Assume that these conditions hold:*

- *The sequence $\{u_n\}_{n=1}^{\infty}$ is bounded uniformly in $L^p(0,T;X_0)$, with $1 < p < \infty$;*
- *time derivative $\{\partial_t u_n\}_{n=1}^{\infty}$ is bounded uniformly in $L^q(0,T;X_2)$, with $1 < q < \infty$;*
- *We have the embedding: X_0 is embedded compactly in X_1 and X_1 is continuous embedded in X_2.*

Then, the sequence $\{u_n\}$ admits a subsequence that converges strongly in $L^p(0,T,X_1)$.

Remark 1.10. *A common triple space for X_0, X_1 and X_2 are the Hilbert triple $H^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$. However, since the lemma only requires X_1 to be embedded continuously on X_2 and X_0 need not be the dual space of X_2, we can use other appropriate functional spaces, depending on the situation.*

1.4 Weak solutions of reaction-diffusion systems

In this section, we will recall the existence of bounded weak solution for system (2), which have been stated and proved in [10] for a more general setting.

Definition 1.16 ([10]). *We call (u_ϵ, v_ϵ) a weak solution of (2) on $(0,T)$ (with given $T > 0$), parameterized $\epsilon > 0$ if it satisfies the regularity conditions*

$$u_\epsilon \in C([0,T]; L^2(\Omega)) \text{ and } u_\epsilon \in L^\infty(0,T; L^\infty(\Omega)) \cap L^2(0,T; H^1(\Omega)); \quad (1.1)$$

$$v_\epsilon \in C([0,T]; L^2(\Gamma)) \text{ and } v_\epsilon \in L^\infty(0,T; L^\infty(\Gamma)) \cap L^2(0,T; H^1(\Gamma)); \quad (1.2)$$

and the following weak formulations hold

$$\int_0^T \int_\Omega (-u_\epsilon \varphi_t + d_u \nabla u_\epsilon \cdot \nabla \varphi) dxdt = \int_\Omega u_0 \varphi(0) dx - \frac{\alpha}{\epsilon} \int_0^T \int_\Gamma (u_\epsilon^\alpha - v_\epsilon^\beta) \varphi dSdt \quad (1.3)$$

$$\int_0^T \int_\Gamma (-v_\epsilon \psi_t + d_v \nabla_\Gamma v_\epsilon \cdot \nabla_\Gamma \psi) dSdt = \int_\Gamma v_0 \psi(0) dS + \frac{\beta}{\epsilon} \int_0^T \int_\Gamma (u_\epsilon^\alpha - v_\epsilon^\beta) \psi dSdt \quad (1.4)$$

for all non-negative test functions $\varphi \in C^1([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))$ and $\psi \in C^1([0,T]; L^2(\Gamma)) \cap L^2(0,T; H^1(\Gamma))$, with $\varphi(T) = 0$ and $\psi(T) = 0$.

Remark 1.11 (Abuse of notation). *In the above definition, the term u_ϵ appears on the integral of Γ is the trace of u_ϵ, follows by the trace theorem.*

The following theorem, which is a special case for Theorem 2.2 in [10]

Theorem 1.11 (Existence and uniqueness of weak solution [10]). *For each non-negative initial data $(u_0, v_0) \in L^\infty(\Omega) \times L^\infty(\Gamma)$, there exists an unique* *non-negative pair of functions* (u_ϵ, v_ϵ)*, corresponding to parameter* ϵ*, which the weak solution of system* (2) *(in the sense of Definition 1.16).*

Remark 1.12. *In the thesis, for convenience in calculation, we will use another weak formulation for* (u_ϵ, v_ϵ)*:*

$$\begin{cases} \langle \partial_t u_\epsilon, \varphi \rangle_\Omega + \int_\Omega d_u \nabla u_\epsilon \cdot \nabla \varphi dx = -\frac{\alpha}{\epsilon} \int_\Gamma (u_\epsilon^\alpha - v_\epsilon^\beta) \varphi dS \\ \langle \partial_t v_\epsilon, \psi \rangle_\Gamma + \int_\Gamma d_v \nabla_\Gamma v \cdot \nabla_\Gamma \psi dS = \frac{\beta}{\epsilon} \int_\Gamma (u_\epsilon^\alpha - v_\epsilon^\beta) \psi dS, \end{cases} \tag{1.5}$$

which holds for a.e. $t \in (0,T)$ *and for all test functions* $(\varphi, \psi) \in H^1(\Omega) \times H^1(\Gamma)$*, and satisfies the initial conditions* $u_\epsilon(x,0) = u_0(x)$ *on* $L^2(\Omega)$ *and* $v_\epsilon(x,0) = v_0(x)$ *on* $L^2(\Gamma)$*. The regularity condition becomes*

$$u_\epsilon \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^\infty(\Omega)) \text{ and } \partial_t u_\epsilon \in L^2(0,T;H^{-1}(\Omega)); \tag{1.6}$$

$$v_\epsilon \in L^2(0,T;H^1(\Gamma)) \cap L^\infty(0,T;L^\infty(\Gamma)) \text{ and } \partial_t v_\epsilon \in L^2(0,T;H^{-1}(\Gamma)). \tag{1.7}$$

The equivalence for weak formulations and defining the weak solution can be proved by using some theorems in Bochner space (for e.g., see [23, Chapter 3] and [16, Chapter 10]).

Remark 1.13. *The positivity of solution is preserved from the initial condition, due to the properties of rate functions called quasi-positive (see [12]). Besides, the positivity of* u_ϵ *still holds on the boundary, i.e. we have* $u_\epsilon \geq 0$ *for a.e. on* $\Gamma \times (0,T)$*, which can be proved by consider a non-negative sequence of function* $\{u_n\}$ *that approximates* u_ϵ *(e.g., mollifiers functions). The non-negative and smoothness to boundary of* $\{u_n\}$ *imply that* $\{u_n\}$ *is non-negative on the boundary, and the Trace theorem gives the conclusion.*

Remark 1.14. *From the first equation in the weak formulation* (1.5)*, we can show that* $u_\epsilon \in L^\infty(0,T;L^\infty(\Gamma))$ *for each fixed* ϵ*, by showing that*

$\|u_\epsilon(t)\|_{L^p(\Gamma)} \leq C$ and C does not depend on p. Then, $\|u_\epsilon(t)\|_{L^\infty(\Gamma)}$ is bounded (see [13, Proposition 1.16]). This property allows us to use some convergence theorems in the next chapter. A notice is that the upper bound for u_ϵ on $\Gamma \times (0,T)$ may vary on ϵ.

Remark 1.15. *In the following chapter, we will call the weak solution of system (2) by "the solution of system (2)", for short.*

Chapter 2

FAST REACTION LIMITS

In this chapter, we will study the convergence of solutions to the system (2) as $\epsilon \to 0$, i.e., the reaction rate constant $1/\epsilon$ tends to infinity. In detail, we will show that for arbitrary positive integers α, β, the sequence of weak solutions $\{(u_\epsilon, v_\epsilon)\}$ has a subsequence that converges strongly in $L^2(0,T; L^2(\Omega)\times L^2(\Gamma))$. About the limit of the convergence, we will describe later in Chapter 3.

2.1 The main theorem and outline of proof

The following lemma is the main result about the fast reaction limit of system (2) as the reaction rate tends to infinity.

Theorem 2.1 (Convergence of subsequence)**.** *Let* (u_ϵ, v_ϵ) *be the weak solution of system* (2) *with parameter* ϵ *and non-negative initial value* (u_0, v_0) $\in L^\infty(\Omega) \times L^\infty(\Gamma)$ *(from Theorem 1.11). Then, as* $\epsilon \to 0$*, the sequence* $\{(u_\epsilon, v_\epsilon)\}$ *has a subsequence that converges strongly in* $L^2(0,T;\ L^2(\Omega) \times L^2(\Gamma))$ *to* (w, z)*, where* $z = (w|_\Gamma)^{\alpha/\beta}$ *and* w *is a weak solution to* (3) *(we will define it in Chapter 3).*

Remark 2.1. *In the theorem, we only have the convergence for subsequence, not the whole sequence. To show that the whole sequence converges,* *it is sufficient to prove the uniqueness of the limit, was shown for the case $\alpha = \beta$ in [6]. However, for the general case, we do not have that.*

Now, we will sketch the proof of Theorem 2.1. The proof utilizes the Aubin–Lions Lemma on well-constructed product spaces.

Sketch of the proof of Theorem 2.1. By Lemma 2.1, we have the uniform boundedness of (u_ϵ, v_ϵ) in $L^\infty(0,T;L^\infty(\Omega) \times L^\infty(\Gamma))$. Moreover, according to Lemma 2.2, we have the gradient $(\nabla u_\epsilon, \nabla_\Gamma v_\epsilon)$ is also uniformly bounded in $L^2(0,T;L^2(\Omega) \times L^2(\Gamma))$. Therefore, combine these two lemmas, we have (u_ϵ, v_ϵ) in a bounded subset of $L^2(0,T;H^1(\Omega) \times H^1(\Gamma))$. On the other hand, Lemma 2.3 shows that there exist a functional space Z such that $Z \hookrightarrow L^2(\Omega) \times L^2(\Gamma) \hookrightarrow Z^*$ with compact and continuous embedding, respectively, and time derivative $\partial_t(u_\epsilon, v_\epsilon)$ is bounded (uniformly) in $L^2(0,T;Z^*)$. Then, in the view of Aubin–Lions Lemma (see Lemma 1.1 and its remark), there exists a subsequence of $\{(u_\epsilon, v_\epsilon)\}$ that it converges strongly in $L^2(0,T;L^2(\Omega) \times L^2(\Gamma))$ as $\epsilon \to 0$, and we denote its limit by (w,z). Finally, in Theorem 3.1, we show that $z = (w|_\Gamma)^{\alpha/\beta}$ and w is a weak solution of the limit problem (3). $\square$

We will present in detail the proof of these lemmas with the following structure. Section 2.2 shows the detailed proof for Lemma 2.1, then Lemma 2.2 about the boundedness of the gradient is presented in Section 2.3, and finally, the functional space for the time derivative is presented in Section 2.4. And, in Chapter 3, we will present the proof of Theorem 3.1, which discuss about the limit system and its weak solutions.

2.2 Uniform boundedness of the solution

In this section, we prove that the solution of the system (2) is uniformly bounded in $L^\infty(\Omega) \times L^\infty(\Gamma)$.

Lemma 2.1. *Let (u_ϵ, v_ϵ) be the weak solution of the system (2), correspond to parameter ϵ and the non-negative initial condition $(u_0, v_0) \in L^\infty(\Omega) \times L^\infty(\Gamma)$, we have the upper bound*

$$\|u_\epsilon\|_{L^\infty(0,T;L^\infty(\Omega))}, \|v_\epsilon\|_{L^\infty(0,T;L^\infty(\Gamma))} \leq M,$$

where M is a constant that does not depend on ϵ.

To prove this lemma, we will declare an L^p-energy functional that based on the structure of the system and this function is decreasing in time. Then, it leads to a prior estimation that holds for all p. Finally, due to the boundedness of initial condition, we can show an uniform estimation for L^p-norm of the solution. Therefore, we obtain the bounded for the solution in $L^\infty-$ norm.

Proof. We introduce the following entropy function

$$E_p[u_\epsilon, v_\epsilon](t) := \frac{1}{p\alpha^2 + \alpha} \int_\Omega u_\epsilon^{p\alpha+1}(t)dx + \frac{1}{p\beta^2 + \beta} \int_\Gamma v_\epsilon^{p\beta+1}(t)dS, \qquad (2.1)$$

where the parameter p is a positive integer. We first have the identity

$$\frac{d}{dt} \int_\Omega u_\epsilon^{p\alpha+1}(t)dx = (p\alpha + 1)\langle \partial_t u_\epsilon(t), u_\epsilon^{p\alpha}(t)\rangle_\Omega$$

for a.e. $t \in [0, T]$. Then, by choosing $u_\epsilon^{p\alpha}$ as the test function in the first equation of (1.5), we obtain

$$\begin{aligned}\frac{d}{dt} \int_\Omega \frac{u_\epsilon^{p\alpha+1}}{p\alpha^2 + \alpha} dx &= \frac{-d_u}{\alpha} \int_\Omega \nabla u_\epsilon \cdot \nabla (u_\epsilon^{p\alpha}) dx - \frac{1}{\epsilon} \int_\Gamma (u_\epsilon^\alpha - v_\epsilon^\beta) u_\epsilon^{p\alpha} dS \\ &= -d_u p \int_\Omega |\nabla u_\epsilon|^2 u_\epsilon^{p\alpha - 1}\, dx - \frac{1}{\epsilon} \int_\Gamma (u_\epsilon^\alpha - v_\epsilon^\beta) u_\epsilon^{p\alpha} dS\end{aligned}$$

By a similar argument, we have

$$\frac{d}{dt} \int_\Gamma \frac{v_\epsilon^{p\beta+1}}{p\beta^2 + \beta} dS = -d_v p \int_\Gamma |\nabla_\Gamma v_\epsilon|^2 v_\epsilon^{p\beta - 1}\, dS + \frac{1}{\epsilon} \int_\Gamma (u^\alpha - v^\beta) v_\epsilon^{p\beta} dS.$$

Combine these above, we obtain

$$\frac{d}{dt}E_p(t) = -\, d_u p \int_\Omega u_\epsilon^{p\alpha-1}|\nabla u_\epsilon|^2 dx - d_v p \int_\Gamma v_\epsilon^{p\beta-1}|\nabla_\Gamma v_\epsilon|^2 dS \\ + \frac{1}{\epsilon}\int_\Gamma (u_\epsilon^\alpha - v_\epsilon^\beta)(v_\epsilon^{p\beta} - u_\epsilon^{p\alpha}) dS. \tag{2.2}$$

Notice that

$$(u_\epsilon^\alpha - v_\epsilon^\beta)(v_\epsilon^{p\beta} - u_\epsilon^{p\alpha}) = (u_\epsilon^\alpha - v_\epsilon^\beta)(v_\epsilon^\beta - u_\epsilon^\alpha)\left(\sum_{i+j=p-1} u_\epsilon^{i\alpha} v_\epsilon^{j\beta}\right),$$

where p is a positive integer. Combine with the fact that (u_ϵ, v_ϵ) is non-negative, we have the estimate:

$$\frac{d}{dt}E_p(t) \le 0 \tag{2.3}$$

for all $p \in \mathbb{Z}^+$ and a.e. $t \in [0,T]$. Fixed a $t_0 \in [0,T]$, we have

$$E_p(t_0) \le E_p(0)$$

or equivalently,

$$\frac{1}{p\alpha^2+\alpha}\int_\Omega u_\epsilon^{p\alpha+1}(t_0)dx + \frac{1}{p\beta^2+\beta}\int_\Gamma v_\epsilon^{p\beta+1}(t_0)dS \\ \le \frac{1}{p\alpha^2+\alpha}\int_\Omega u_0^{p\alpha+1}dx + \frac{1}{p\beta^2+\beta}\int_\Gamma v_0^{p\beta+1}dS \tag{2.4}$$

for all $p \in \mathbb{Z}^+$. From (2.4), multiply both sides by p, we get

$$\frac{1}{\alpha^2+\alpha/p}\int_\Omega u_\epsilon^{p\alpha+1}(t_0)dx + \frac{1}{\beta^2+\beta/p}\int_\Gamma v_\epsilon^{p\beta+1}(t_0)dS \\ \le \frac{1}{\alpha^2+\alpha/p}\int_\Omega u_0^{p\alpha+1}dx + \frac{1}{\beta^2+\beta/p}\int_\Gamma v_0^{p\beta+1}dS,$$

which implies an estimate in which the coefficients of integrals do not depend on p

$$\frac{1}{2\alpha^2}\int_\Omega u_\epsilon^{p\alpha+1}(t_0)dx + \frac{1}{2\beta^2}\int_\Gamma v_\epsilon^{p\beta+1}(t_0)dS \le \int_\Omega u_0^{p\alpha+1}dx + \int_\Gamma v_0^{p\beta+1}dS.$$

Taking the root of power p on both sides of the above estimate, we have:

$$\left(\frac{1}{2\alpha^2}\int_\Omega u_0^{p\alpha+1}(t_0)dx + \frac{1}{2\beta^2}\int_\Gamma v_0^{p\beta+1}(t_0)dS\right)^{1/p} \le \left(\int_\Omega u_0^{p\alpha+1}dx + \int_\Gamma v_0^{p\beta+1}dS\right)^{1/p}. \tag{2.5}$$

The right hand side of (2.5) can be estimated uniformly by

$$\begin{aligned}\left(\int_\Omega u_0^{p\alpha+1}dx + \int_\Gamma v_0^{p\beta+1}dS\right)^{1/p} &\le (|\Omega|.\|u_0\|_{L^\infty(\Omega)}^{p\alpha+1} + |\Gamma|.\|v_0\|_{L^\infty(\Gamma)}^{p\beta+1})^{1/p}\\ &\le |\Omega|^{1/p}.\|u_0\|_{L^\infty(\Omega)}^{(p\alpha+1)/p} + |\Gamma|^{1/p}.\|v_0\|_{L^\infty(\Gamma)}^{(p\beta+1)/p}\\ &\le \max\{1,|\Omega|\}.\max\{1,\|u_0\|_{L^\infty(\Omega)}^{2\alpha}\}\\ &+\max\{1,|\Gamma|\}.\max\{1,\|v_0\|_{L^\infty(\Gamma)}^{2\beta}\}\\ &\le C_1(\alpha,\beta,|\Omega|,|\Gamma|,\|u_0\|_{L^\infty(\Omega)},\|v_0\|_{L^\infty(\Gamma)}),\end{aligned} \tag{2.6}$$

where the second inequality is deduced from the fact that $(a+b)^\theta \le a^\theta + b^\theta$ for all $a, b \ge 0$ and $\theta \in (0,1]$. Combine the estimations (2.5) and (2.6), we have the estimate

$$\left(\frac{1}{2\alpha^2}\int_\Omega u_\epsilon^{p\alpha+1}(t_0)dx\right)^{1/p} \le C_1$$

or

$$\left(\int_\Omega u_\epsilon^{p\alpha+1}(t_0)dx\right)^{1/p} \le C_2 \tag{2.7}$$

for all $p \in \mathbb{Z}^+$ and C_1, C_2 are constants that do not depend on ϵ. This implies

$$\|u_\epsilon(t_0)\|_{L^{p\alpha+1}(\Omega)} \le C_2^{\frac{p}{p\alpha+1}}.$$

Letting $p \to \infty$ yields

$$\|u_\epsilon(t_0)\|_{L^\infty(\Omega)} \leq C_2^{1/\alpha}.$$

Similarly, we also get

$$\|v_\epsilon(t_0)\|_{L^\infty(\Gamma)} \leq C_3^{1/\beta},$$

which end the proof. □

Remark 2.2. *With similar arguments, we also can prove that the solution also has a strictly positive lower bound for the case $\alpha = \beta$, which is independent of ϵ, and showing that $(\frac{1}{u_\epsilon}, \frac{1}{v_\epsilon})$ is bounded uniformly from above. The difference in the proof is that we substitute (u_ϵ, v_ϵ) by $(\frac{1}{u_\epsilon}, \frac{1}{v_\epsilon})$ in the entropy function (we also need to modify the coefficients of integrals), we add a small constant δ to u_ϵ and v_ϵ which avoid the blow-up during calculation and then take the limit $\delta \to 0$ by using the convergence theorems in Chapter 1.*

From the remark above, we have the following proposition about lower boundedness of the solution

Proposition 2.1. *Assume that $\alpha = \beta$ and the initial condition $(u_0, v_0) \in L^\infty(\Omega) \times L^\infty(\Gamma)$ is bounded from below by a positive constant m_0, that is, $u_0(x) \geq m_0$ a.e. $x \in \Omega$ and $v_0(x) \geq m_0$ a.e. $x \in \Gamma$. Then, the solution of (2) $u_\epsilon(x,t) \geq m$ for a.e. $(x,t) \in \Omega \times (0,T)$ and $v_\epsilon(x,t) \geq m$ for a.e. $(x,t) \in \Gamma \times (0,T)$, where the constant $m > 0$ and does not depend on ϵ.*

Proof. Similarly, we introduce another entropy function, with parameter $p \in \mathbb{Z}^+$ and a small constant $\delta > 0$

$$H_{p,\delta}[u_\epsilon, v_\epsilon](t) := \frac{\beta}{\alpha p - 1}\int_\Omega \frac{1}{(u_\epsilon + \delta)^{\alpha p - 1}} dx + \frac{\alpha}{\beta p - 1}\int_\Gamma \frac{1}{(v_\epsilon + \delta)^{\beta p - 1}} dS.$$

Since (u_ϵ, v_ϵ) is non-negative and the constants $\alpha, \beta, p \geq 1$, we have the positivity $H_{p,\delta}[u_\epsilon, v_\epsilon](t) > 0$. With a similar calculation for the time derivative of entropy function as Lemma 2.1, we can show that for a.e. $t \in [0,T]$:

$$\begin{aligned}\frac{d}{dt}H_{p,\delta}(t) = \beta \int_\Omega \nabla u_\epsilon \cdot \nabla (u_\epsilon+\delta)^{-\alpha p} dx + \alpha \int_\Gamma \nabla_\Gamma v_\epsilon \cdot \nabla_\Gamma (v_\epsilon+\delta)^{-\beta p} dS \\ + \frac{\alpha\beta}{\epsilon}\int_\Gamma (u_\epsilon^\alpha - v_\epsilon^\beta)((u_\epsilon+\delta)^{-\alpha p} - (v_\epsilon+\delta)^{-\beta p})dS.\end{aligned}$$

Fixed a $t_0 \in [0,T]$ and take the integration from 0 to t_0, we have:

$$\begin{aligned}H_{p,\delta}(t_0) - H_{p,\delta}(0) =& \beta \int_0^{t_0}\int_\Omega \nabla(u_\epsilon+\delta)\cdot\nabla(u_\epsilon+\delta)^{-\alpha p} dxdt \\ &+ \alpha \int_0^{t_0}\int_\Gamma \nabla_\Gamma(v_\epsilon+\delta)\cdot\nabla_\Gamma(v_\epsilon+\delta)^{-\beta p} dSdt \\ &+ \frac{\alpha\beta}{\epsilon}\int_0^{t_0}\int_\Gamma (u_\epsilon^\alpha - v_\epsilon^\beta)((u_\epsilon+\delta)^{-\alpha p} - (v_\epsilon+\delta)^{-\beta p})dSst \\ \leq& \frac{\alpha\beta}{\epsilon}\int_0^{t_0}\int_\Gamma (u_\epsilon^\alpha - v_\epsilon^\beta)((u_\epsilon+\delta)^{-\alpha p} - (v_\epsilon+\delta)^{-\beta p})dSdt.\end{aligned}$$

From the assumption $\alpha = \beta$, the right hand side of the above inequality is non-negative, which implies:

$$H_{p,\delta}(t_0) \leq H_{p,\delta}(0)$$

for all δ. Then, by using Monotone Convergence Theorem to take the limit of $\delta \to 0^+$, we get:

$$\begin{aligned}0 \leq \frac{\beta}{\alpha p - 1}\int_\Omega \frac{1}{u_\epsilon(t_0)^{\alpha p-1}}dx + \frac{\alpha}{\beta p - 1}\int_\Gamma \frac{1}{v_\epsilon(t_0)^{\beta p -1}}dS \\ \leq \frac{\beta}{\alpha p - 1}\int_\Omega \frac{1}{u_0^{\alpha p-1}}dx + \frac{\alpha}{\beta p - 1}\int_\Gamma \frac{1}{v_0^{\beta p-1}}dS.\end{aligned}$$

Since u_0, v_0 has a strictly positive lower bound, by using similar arguments in the proof of Lemma 2.1, we have:

$$\left(\int_\Omega \frac{1}{u_\epsilon^{\alpha p-1}(t_0)}dx\right)^{1/p}, \left(\int_\Gamma \frac{1}{v_\epsilon^{\beta p-1}(t_0)}dS\right)^{1/p}$$

bounded (uniformly in term of p and ϵ). And from the fact that (u_ϵ, v_ϵ) bounded uniformly, we have for any $p \geq 2$:

$$\|u_\epsilon^{-\alpha}(t_0)\|_{L^p(\Omega)} \leq M_2$$

and

$$\|v_\epsilon^{-\beta}(t_0)\|_{L^p(\Gamma)} \leq M_3$$

where the constant M_2, M_3 does not depend on ϵ and p. Then, with a similar argument at the end of proof for Lemma 2.1, we can show that $(\frac{1}{u_\epsilon}, \frac{1}{v_\epsilon})$ is bounded above or (u_ϵ, v_ϵ) has a lower bound $m > 0$, where the constant m doesn't depend on ϵ. □

Remark 2.3. *This property will be used in the next chapter, where we discuss on the convergence rate and we require the strictly positive of the solution (u_ϵ, v_ϵ). In the proof, we need to assume that the coefficient $\alpha = \beta$ to have the sign of the term $(u_\epsilon^\alpha - v_\epsilon^\beta)((u_\epsilon + \delta)^{-\alpha p} - (v_\epsilon + \delta)^{-\beta p})$. A question is, can we have the same result if we remove this assumption, which seems to be quite complicated since it is quite difficult to have any estimation for this term to use the convergence theorems.*

2.3 Boundedness of gradient operator

Next, we will prove that the (spatial) gradients of the solution are uniformly bounded.

Lemma 2.2. *With (u_ϵ, v_ϵ) as the solution of (2), we have the estimate:*

$$\|\nabla u_\epsilon\|_{L^2(0,T;L^2(\Omega))}, \|\nabla_\Gamma v_\epsilon\|_{L^2(0,T;L^2(\Gamma))} \leq M_D$$

with a constant M_D does not depend on ϵ.

Proof. We consider the logarithm entropy function (see [10]):

$$E[u_\epsilon, v_\epsilon](t) = \int_\Omega u_\epsilon(\log u_\epsilon - 1)dx + \int_\Gamma v_\epsilon(\log v_\epsilon - 1)dS, \tag{2.8}$$

where $\log(x)$ is the natural logarithm of the positive number x. Formally, we have the dissipation of entropy function $D(t) = \frac{-d}{dt}E(t) = \langle -\partial_t u_\epsilon, \log(u_\epsilon)\rangle_\Omega + \langle -\partial_t v_\epsilon, \log(v_\epsilon)\rangle_\Gamma$ and then choose $\log(u_\epsilon)$ (and $\log(v_\epsilon)$, respectively) as the test functions. However, we don't have a lower bound for u_ϵ and v_ϵ, in generally, which means that $\log(u_\epsilon)$ and $\log(v_\epsilon)$ do not belong to $L^2(\Omega)$ and $L^2(\Gamma)$, respectively. A naive idea is adding the assumption that the solution (u_ϵ, v_ϵ) is strictly positive. Fortunately, there is a way to avoid this assumption by using a similar technique in [5] (we will present the theorem if we have the strictly positive assumption in the remark below). Indeed, we replace u_ϵ (and v_ϵ) by $u_\epsilon + \delta$ (and $v_\epsilon + \delta$), with δ is a small parameter and taking the limit $\delta \to 0$.

Again, we introduce the modified entropy function:

$$\begin{aligned} E_\delta[u_\epsilon, v_\epsilon](t) = &\int_\Omega (u_\epsilon + \delta)(\log(u_\epsilon + \delta) - 1)dx \\ &+ \int_\Gamma (v_\epsilon + \delta)(\log(v_\epsilon + \delta) - 1)dS. \end{aligned} \tag{2.9}$$

With a similar calculation for the time derivative entropy function, we have the dissipation function:

$$\begin{aligned} D_\delta(t) &= -\frac{d}{dt}E_\delta(t) \\ &= \langle \partial_t(u_\epsilon + \delta), -\log(u_\epsilon + \delta)\rangle_\Omega \\ &\quad + \langle \partial_t(v_\epsilon + \delta), -\log(v_\epsilon + \delta)\rangle_\Gamma \\ &= \langle \partial_t(v_\epsilon), -\log(u_\epsilon + \delta)\rangle_\Omega + \langle \partial_t(v_\epsilon), -\log(v_\epsilon + \delta)\rangle_\Gamma, \end{aligned}$$

for a.e. $t \in [0, T]$ and δ is a small, positive constant. Then, by choosing the logarithm of solution $(-\log(u_\epsilon+\delta), -\log(v_\epsilon+\delta))$ as the test function in weak formulation (1.5), we have

$$\begin{aligned}
D_\delta(t) =& - d_u\int_\Omega \nabla u_\epsilon \cdot \nabla(\log(u_\epsilon+\delta))dx + \frac{\alpha}{\epsilon}\int_\Gamma (u_\epsilon^\alpha - v_\epsilon^\beta)\log(u_\epsilon+\delta)dS \\
&- d_v\int_\Gamma \nabla_\Gamma v_\epsilon \cdot \nabla_\Gamma(\log(v_\epsilon+\delta))dS - \frac{\beta}{\epsilon}\int_\Gamma (u_\epsilon^\alpha - v_\epsilon^\beta)\log(v_\epsilon+\delta)dS \\
=& d_u\int_\Omega \frac{|\nabla u_\epsilon|^2}{u_\epsilon+\delta}dx + d_v\int_\Gamma \frac{|\nabla_\Gamma v_\epsilon|^2}{v_\epsilon+\delta}dS + \frac{1}{\epsilon}\int_\Gamma (u_\epsilon^\alpha - v_\epsilon^\beta)\log\frac{(u_\epsilon+\delta)^\alpha}{(v_\epsilon+\delta)^\beta}dS.
\end{aligned}$$

Then, by integrating $D_\delta(t)$ from 0 to T, we get

$$\begin{aligned}
E_\delta(0) - E_\delta(T) &= \int_0^T D_\delta(t)dt \\
&= \int_0^T \left(d_u\int_\Omega \frac{|\nabla u_\epsilon|^2}{u_\epsilon+\delta}dx + d_v\int_\Gamma \frac{|\nabla_\Gamma v_\epsilon|^2}{v_\epsilon+\delta}dS\right)dt \\
&+ \int_0^T \frac{1}{\epsilon}\int_\Gamma (u_\epsilon^\alpha - v_\epsilon^\beta)\log\frac{(u_\epsilon+\delta)^\alpha}{(v_\epsilon+\delta)^\beta}dSdt.
\end{aligned}$$

We can rewrite the above equation to:

$$\begin{aligned}
\int_0^T \left(d_u\int_\Omega \frac{|\nabla u_\epsilon|^2}{u_\epsilon+\delta}dx + d_v\int_\Gamma \frac{|\nabla_\Gamma v_\epsilon|^2}{v_\epsilon+\delta}dS\right)dt &= E_\delta(0) - E_\delta(T) \\
&+ \frac{1}{\epsilon}\int_0^T\int_\Gamma (v_\epsilon^\beta - u_\epsilon^\alpha)\log\frac{(u_\epsilon+\delta)^\alpha}{(v_\epsilon+\delta)^\beta}dS
\end{aligned}$$

From the fact that the initial condition (u_0, v_0) is bounded, we have $E_\delta(0)$ can be bounded uniformly (choose $\delta < 1$). Moreover, $E_\delta(T)$ has a lower bound without depending on δ. Therefore, $E_\delta(0) - E_\delta(T)$ is bounded and we have the estimate:

$$\begin{aligned}
d_u\int_0^T\int_\Omega \frac{|\nabla u_\epsilon|^2}{u_\epsilon+\delta}dxdt + d_v\int_0^T\int_\Gamma \frac{|\nabla_\Gamma v_\epsilon|^2}{v_\epsilon+\delta}dSdt \\
\le C + \frac{1}{\epsilon}\int_0^T\int_\Gamma (v_\epsilon^\beta - u_\epsilon^\alpha)\log\frac{(u_\epsilon+\delta)^\alpha}{(v_\epsilon+\delta)^\beta}dS, \quad (2.10)
\end{aligned}$$

with C is a constant that does not depend on δ and also ϵ. Now, similar to [5], we can use the convergence theorem to take the limit $\delta \to 0^+$. Indeed, the left hand side can be taken limit $\delta \to 0^+$ by using Monotone Convergence Theorem (Theorem 1.3)

$$\lim_{\delta\to 0}\left(d_u\int_0^T\int_\Omega \frac{|\nabla u_\epsilon|^2}{u_\epsilon+\delta}dx + d_v\int_0^T\int_\Gamma \frac{|\nabla_\Gamma v_\epsilon|^2}{v_\epsilon+\delta}dS\,dt\right)$$
$$= d_u\int_0^T\int_\Omega \frac{|\nabla u_\epsilon|^2}{u_\epsilon}dxdt + d_v\int_0^T\int_\Gamma \frac{|\nabla_\Gamma v_\epsilon|^2}{v_\epsilon}dSdt$$

Now, we only need to work with the term containing logarithm. Since the solution (u_ϵ, v_ϵ) is non-negative and bounded, this term has an upper bound (for each fixed ϵ):

$$(v_\epsilon^\beta - u_\epsilon^\alpha)\log(u_\epsilon+\delta)^\alpha = \alpha v_\epsilon^\beta \log(u_\epsilon+\delta) - \alpha u_\epsilon^\alpha \log(u_\epsilon+\delta) \le C,$$

where the constant C does not depend on δ. Similarly, we have

$$(-v_\epsilon^\beta + u_\epsilon^\alpha)\log(v_\epsilon+\delta)^\beta \le C,$$

which implies that $(v_\epsilon^\beta - u_\epsilon^\alpha)\log\frac{(u_\epsilon+\delta)^\alpha}{(v_\epsilon+\delta)^\beta}$ has an upper bound. Then, we can use Fatou's lemma for a function that has upper bounded (see Theorem 1.2 and its remark)

$$\limsup_{\delta\to 0}\int_0^T\int_\Gamma (v_\epsilon^\beta - u_\epsilon^\alpha)\log\frac{(u_\epsilon+\delta)^\alpha}{(v_\epsilon+\delta)^\beta}dS$$
$$\le \int_0^T\int_\Gamma \limsup_{\delta\to 0}(v_\epsilon^\beta - u_\epsilon^\alpha)\log\frac{(u_\epsilon+\delta)^\alpha}{(v_\epsilon+\delta)^\beta}dS$$
$$= \int_0^T\int_\Gamma (v_\epsilon^\beta - u_\epsilon^\alpha)\log\frac{(u_\epsilon)^\alpha}{(v_\epsilon)^\beta}dS \le 0.$$

Combine these all and take the limit $\delta \to 0^+$ of (2.10), we obtain the uniform estimation

$$d_u\int_0^T\int_\Omega \frac{|\nabla u_\epsilon|^2}{u_\epsilon}dxdt + d_v\int_0^T\int_\Gamma \frac{|\nabla_\Gamma v_\epsilon|^2}{v_\epsilon}dSdt \le C,$$

where C is a constant that does not depend on ϵ. Since $d_u, d_v > 0$ and the solution (u_ϵ, v_ϵ) is non-negative, we obtain

$$\int_0^T \int_\Omega \frac{|\nabla u_\epsilon|^2}{u_\epsilon} dxdt \le C,$$
$$\int_0^T \int_\Gamma \frac{|\nabla_\Gamma v_\epsilon|^2}{v_\epsilon} dSdt \le C,$$

where constant C doesn't depend on ϵ.

Finally, from the fact that (u_ϵ, v_ϵ) is bounded uniformly in Lemma 2.1, we show that the gradient is also bounded uniformly:

$$\begin{aligned}
\|\nabla u_\epsilon\|^2_{L^2(0,T;L^2(\Omega))} &= \int_0^T \int_\Omega |\nabla u_\epsilon|^2 dxdt \\
&= \int_0^T \int_\Omega \frac{|\nabla u_\epsilon|^2}{u_\epsilon}(u_\epsilon) dxdt \\
&\le \int_0^T \|u_\epsilon(t)\|_{L^\infty(\Omega)} \int_\Omega \frac{|\nabla u_\epsilon|^2}{u_\epsilon} dxdt \\
&\le \|u_\epsilon\|_{L^\infty(0,T;L^\infty(\Omega))} \int_0^T \int_\Omega \frac{|\nabla u_\epsilon|^2}{u_\epsilon} dxdt \\
&\le M_D
\end{aligned}$$

with a constant M_D is independent to ϵ. By a similar argument, we also get

$$\|\nabla_\Gamma v_\epsilon\|^2_{L^2(0,T;L^2(\Gamma))} \le M_D,$$

which concludes the proof. □

Remark 2.4. *As mentioned above, we can still use the entropy function* (2.8) *without modification if we add an assumption that the solution* (u_ϵ, v_ϵ) *is strictly positive. Moreover, we can do it in a more simple way by choosing* $(u_\epsilon^\alpha, v_\epsilon^\beta)$ *as the test function and using the fact that* u_ϵ, v_ϵ *is non-negative and bounded uniformly. Indeed, we start with the time derivative of entropy function* (2.2) *in the proof of Lemma 2.1, and choose the parameter* $p = 1$

to get

$$d_u \int_\Omega u_\epsilon^{\alpha-1} |\nabla u_\epsilon|^2 dx \;+\; d_v \int_\Gamma v_\epsilon^{\beta-1} |\nabla_\Gamma v_\epsilon|^2 dS + \frac{d}{dt} E_1(t) \\ = \frac{1}{\epsilon} \int_\Gamma (u_\epsilon^\alpha - v_\epsilon^\beta)(v_\epsilon^\beta - u_\epsilon^\alpha) dS$$

for a.e. $t \in [0, T]$ *and entropy function* E_1 *is defined by*

$$E_1(t) := \frac{1}{\alpha^2 + \alpha} \int_\Omega u_\epsilon^{\alpha+1}(t) dx + \frac{1}{\beta^2 + \beta} \int_\Gamma v_\epsilon^{\beta+1}(t) dS.$$

Then, by integrating both sides from 0 *to* T *and using the fact that the right-hand side is non-positive, we have the estimatation*

$$d_u \int_0^T \int_\Omega u_\epsilon^{\alpha-1} |\nabla u_\epsilon|^2 dx dt \;+\; d_v \int_0^T \int_\Gamma v_\epsilon^{\beta-1} |\nabla_\Gamma v_\epsilon|^2 dS dt \leq E_1(0) - E_1(T). \tag{2.11}$$

The lower boundedness of (u_ϵ, v_ϵ) *allows us to estimate the left hand side by*

$$d_u \int_0^T \int_\Omega u_\epsilon^{\alpha-1} |\nabla u_\epsilon|^2 dx dt \;+\; d_v \int_0^T \int_\Gamma v_\epsilon^{\beta-1} |\nabla_\Gamma v_\epsilon|^2 dS dt \\ \geq d_u m^{\alpha-1} \|\nabla u_\epsilon\|_{L^2(0,T;L^2(\Omega))} + d_v m^{\beta-1} \|\nabla_\Gamma v_\epsilon\|_{L^2(0,T;L^2(\Gamma))},$$

with constant $m > 0$ *is the lower bound of* (u_ϵ, v_ϵ), *that is, for a.e.* $t \in (0, T)$, $u_\epsilon(x, t) \geq m$ *on* Ω *and* $v_\epsilon(x, t) \geq m$ *on* Γ. *On the other hand, the right-hand side of* (2.11) *is bounded since* (u_ϵ, v_ϵ) *is bounded. Therefore, we have the conclusion*

$$\|\nabla u_\epsilon\|_{L^2(0,T;L^2(\Omega))}, \|\nabla_\Gamma v_\epsilon\|_{L^2(0,T;L^2(\Gamma))} \leq M_D.$$

In this case, we can prove that the gradient is bounded without using the assumption that the solution has a strictly lower bound. However, in some cases, we will need this condition.

2.4 Functional space for time derivative

We finish this chapter by showing that there exists a suitable functional space for the time derivative of solution $\partial_t(u_\epsilon, v_\epsilon)$, where the Aubin–Lions lemma is applicable. Therefore, combine with Lemma 2.1 and Lemma 2.2, the sequence of solution $\{(u_\epsilon, v_\epsilon)\}$ has a subseqeunce that converges strongly in $L^2(0,T;L^2(\Omega)\times L^2(\Gamma))$.

Lemma 2.3. *There exists a Banach space Z such that the embedding of Z into $L^2(\Omega)\times L^2(\Gamma)$ is compact and $\partial_t(u_\epsilon, v_\epsilon)$ is bounded (uniformly with respect to ϵ) in the space $L^2(0,T;Z^*)$.*

Proof. First, we recall a weak formulation, which have been mentioned in Chapter 1:

$$\langle \partial_t u_\epsilon, \varphi\rangle_\Omega + \int_\Omega d_u \nabla u_\epsilon \cdot \nabla\varphi dx = -\frac{\alpha}{\epsilon}\int_\Gamma (u_\epsilon^\alpha - v_\epsilon^\beta)\varphi dS, \tag{2.12}$$

$$\langle \partial_t v_\epsilon, \psi\rangle_\Gamma + \int_\Gamma d_v \nabla_\Gamma v_\epsilon \cdot \nabla_\Gamma\psi dS = \frac{\beta}{\epsilon}\int_\Gamma (u_\epsilon^\alpha - v_\epsilon^\beta)\psi dS. \tag{2.13}$$

Since our target is to estimate the time derivative by a constant without dependence on ϵ, we would like to remove the right hand sides of these equality by adding the two equations above. To do it, we define a new functional space for the test function

$$Z := \{(\phi, \phi|_\Gamma) : \phi \in H^1(\Omega) \text{ and } \phi|_\Gamma \in H^1(\Gamma)\}, \tag{2.14}$$

and inherited the norm and inner product of $H^1(\Omega)\times H^1(\Gamma)$

$$\|(\phi,\phi|_\Gamma)\|_Z := \|\phi\|_{H^1(\Omega)} + \|\phi|_\Gamma\|_{H^1(\Gamma)}, \tag{2.15}$$

$$\Big((\phi_1,\phi_1|_\Gamma),(\phi_2,\phi_2|_\Gamma)\Big)_Z := (\phi_1,\phi_2)_{H^1(\Omega)} + (\phi_1|_\Gamma, \phi_2|_\Gamma)_{H^1(\Gamma)}. \tag{2.16}$$

The functional space Z can be shown that it is a closed subspace of $H^1(\Omega)\times$

$H^1(\Gamma)$, so it is a Hilbert space. More detail about this functional space can be seen at [24] and its references. Besides, we have that the functional space Z is compactly embedded in $L^2(\Omega)\times L^2(\Gamma)$. So, we only need to prove that the time derivative is bounded in $L^2(0,T;Z^*)$. Choose $(\varphi,\varphi|_\Gamma)\in Z$ arbitrary and multiply (2.12) by β, (2.13) by α and take the sum of these and take the integration from 0 to T, we have:

$$\int_0^T\left(\int_\Omega \alpha\langle\partial_t u_\epsilon,\varphi\rangle_\Omega dx+\int_\Gamma \beta\langle\partial_t v_\epsilon,\varphi|_\Gamma\rangle_\Gamma dS\right)dt$$
$$=-\int_0^T\left(\int_\Omega \beta d_u\nabla u_\epsilon\cdot\nabla\varphi dx+\int_\Gamma \alpha d_v\nabla_\Gamma v_\epsilon\cdot\nabla_\Gamma\varphi|_\Gamma dS\right)dt. \tag{2.17}$$

Setting $V:=L^2(0,T;Z)$, the dual of V is V^*, which can be identified by $L^2(0,T;Z^*)$. So, the left hand side of equation (2.17) can be rewritten as the pairing between V^* and V:

$$\int_0^T\left(\int_\Omega \alpha\langle\partial_t u_\epsilon,\varphi\rangle_\Omega dx+\int_\Gamma \beta\langle\partial_t v_\epsilon,\varphi|_\Gamma\rangle_\Gamma dS\right)dt$$
$$=\langle\partial_t(\alpha u_\epsilon,\beta v_\epsilon);(\varphi,\varphi|_\Gamma)\rangle_{V^*\times V}. \tag{2.18}$$

Besides, the right hand side of (2.17) can be bounded in the norm of V due to the Lemma 2.2, which give the estimate:

$$\begin{aligned}&\int_0^T\int_\Omega \beta d_u\nabla u_\epsilon\cdot\nabla\varphi dx+\int_\Gamma \alpha d_v\nabla_\Gamma v_\epsilon\cdot\nabla_\Gamma(\varphi|_\Gamma)dSdt\\
\le&\,\beta d_u\|\nabla u_\epsilon\|_{L^2(0,T;L^2(\Omega))}\|\nabla\varphi\|_{L^2(0,T;L^2(\Omega))}\\
&+\alpha d_v\|\nabla_\Gamma v_\epsilon\|_{L^2(0,T;L^2(\Gamma))}\|\nabla_\Gamma(\varphi|_\Gamma)\|_{L^2(0,T;L^2(\Gamma))}\\
\le&\,C(\|\varphi\|_{L^2(0,T;H^1(\Omega))}+\|\varphi|_\Gamma\|_{L^2(0,T;H^1(\Gamma))})\\
\le&\,C\,\|(\varphi,\varphi|_\Gamma)\|_V,\end{aligned}$$

where the first inequality given by Holder inequality. In the estimate, C is a constant that does not depend on ϵ. Combine with (2.17) and (2.18), we have:

$$\langle \partial_t(\alpha u_\epsilon, \beta v_\epsilon); (\varphi, \varphi|_\Gamma)\rangle_{V^* \times V} \leq C\|(\varphi, \varphi|_\Gamma)\|_V \tag{2.19}$$

for all $(\varphi, \varphi|_\Gamma) \in V$. Therefore, the time derivative is bounded in functional space V^*:

$$\|\partial_t(u_\epsilon, v_\epsilon)\|_{V^*} \leq C.$$

Hence, the proof is complete. □

Chapter 3

LIMIT SYSTEM AND CONVERGENCE RATE

In this chapter, we will discuss two problems: the first one is describing the limit of the (sub)sequence $\{(u_\epsilon, v_\epsilon)\}$ as $\epsilon \to 0$, what we have shown in the previous chapter, and the second one, where we will have a result about the convergence rate of the sequence in the case $\alpha = \beta$, which is a new result in this area.

3.1 Limit system

In this section, we will go in detail about the limit of convergence in the last chapter. As mentioned in the Introduction part, formally, we expect that the limit system is the heat equation with nonlinear dynamical boundary condition (3).

However, we only have the weak solution of (2) and the convergence only in the space $L^2(0, T; L^2(\Omega) \times L^2(\Gamma))$. So, we only expect that the limit is still the solution of (3) but in a weaker sense. Denote (w, z) the limit of a subsequence of $\{(u_\epsilon, v_\epsilon)\}$ that converges in $L^2(0, T; L^2(\Omega) \times L^2(\Gamma))$, we will show that w is a weak solution of the limit system (we will define the weak solution in below). To do that, we first remind that in Chapter 2, we have shown that the sequence $\{(u_\epsilon, v_\epsilon)\}$ have

- The existence of a subsequence that converges strongly in $L^2(0,T; L^2(\Omega) \times L^2(\Gamma))$;
- The uniform boundedness of u_ϵ in $L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^\infty(\Omega))$, which implies we have the same for u_ϵ^α;
- Both sequences v_ϵ and v_ϵ^β are uniformly bounded in $L^2(0,T;H^1(\Gamma)) \cap L^\infty(0,T;L^\infty(\Gamma))$.

Due to Theorem 1.8, from the uniform boundedness of a sequence of solution in certain functional spaces, we have the weak convergences of subsequences of u_ϵ^α and v_ϵ^β in the corresponding spaces. Therefore, combine with the setting of (w, z) above, there exists a subsequence $\{(u_{\epsilon_i}, v_{\epsilon_i})\}_{i=1}^\infty$ of $\{(u_\epsilon, v_\epsilon)\}$ (where $\epsilon_i \to 0$ as $i \to \infty$) that satisfies

- The sequence $\{(u_{\epsilon_i}, v_{\epsilon_i})\}$ converges strongly to (w, z) as $i \to \infty$, in $L^2(0,T;L^2(\Omega) \times L^2(\Gamma))$;
- The sequence $\{u_{\epsilon_i}\}$, and also $\{u_{\epsilon_i}^\alpha\}$ converge weakly to w and w^α, respectively in $L^2(0,T;H^1(\Omega))$ as $i \to \infty$;
- The sequence $\{v_{\epsilon_i}\}$ and $\{v_{\epsilon_i}^\beta\}$ converge weakly to z and z^β, respectively in $L^2(0,T;H^1(\Gamma))$ as $i \to \infty$.

So, to work with the limit (w, z), we can consider that it is the limit of $\{(u_{\epsilon_i}, v_{\epsilon_i})\}$ as $i \to 0$. Now, we will give the definition of weak solution of the limit system (3).

Definition 3.1. *We call w a weak solution of problem* (3) *if it satisfies the regular condition*

$$w \in C([0,T];L^2(\Omega)) \textit{ and } w \in L^\infty(0,T;L^\infty(\Omega)) \cap L^2(0,T;H^1(\Omega));$$

and it satisfies the following weak formulation

$$\begin{aligned}&\int_0^T\int_\Omega -w\varphi_t+\nabla w\cdot\nabla\varphi dxdt\\&+\frac{\alpha}{\beta}\int_0^T\int_\Gamma -(w|_\Gamma)^{\alpha/\beta}(\varphi|_\Gamma)_t+\nabla_\Gamma(w|_\Gamma^{\alpha/\beta})\cdot\nabla_\Gamma(\varphi|_\Gamma)dSdt \qquad (3.1)\\&\qquad\qquad=\int_\Omega u_0\varphi(0)dx+\frac{\alpha}{\beta}\int_\Gamma v_0\varphi|_\Gamma(0)dS\end{aligned}$$

where the test function $(\varphi,\varphi|_\Gamma)\in C^1([0,T],H^1(\Omega)\times H^1(\Gamma))$ *and* $\varphi(T)=0$ *(which means* $\varphi|_\Gamma(T)=0$*, also). Moreover, the trace function* $z^\beta(t)=w^\alpha|_\Gamma(t)$ *(*$0<t<T$*) is existed and satisfies the following regularity condition*

$$z\in C([0,T];L^2(\Gamma)) \text{ and } w\in L^\infty(0,T;L^\infty(\Gamma))\cap L^2(0,T;H^1(\Gamma)).$$

Remark 3.1. *The space of test functions can be rewritten by using the functional space* Z*, which is defined in Lemma 2.3:*

$$(\varphi,\varphi|_\Gamma)\in C^1([0,T];Z).$$

Theorem 3.1 (Limit system)**.** *Denote* (w,z) *is the limit of a subsequence* $\{(u_\epsilon,v_\epsilon)\}$ *converges in* $L^2(0,T;L^2(\Omega)\times L^2(\Gamma))$ *as* $\epsilon\to 0$*, then* w *is a weak solution of* (3) *(in the sense of Definition 3.1). Moreover* $(w|_\Gamma)^\alpha(t)=z^\beta(t)$ *for a.e.* $t\in(0,T)$*, so we can rewrite the limit by* $(w,(w|_\Gamma)^{\alpha/\beta})$*.*

Proof. The proof is similar to the case $\alpha=\beta=1$ in the article [6]. First, we proof the identity $w^\alpha|_\Gamma(t)=z^\beta(t)$ by starting again with equation (2.2), choose $p=1$ and take the integration from 0 to T, the uniform boundedness of (u_ϵ,v_ϵ) implies

$$\frac{1}{\epsilon}\|u_\epsilon^\alpha-v_\epsilon^\beta\|^2_{L^2(\Gamma\times(0,T))}\le C.$$

where C is independent of ϵ, which means $u_\epsilon^\alpha-v_\epsilon^\beta\to 0$ strongly in $L^2(\Gamma\times(0,T))$. Then, we have the weak convergence $u_\epsilon^\alpha-v_\epsilon^\beta\to 0$ in the same functional space, or in other words

$$\lim_{\epsilon\to 0}\int_{\Gamma}(u_{\epsilon}^{\alpha}-v_{\epsilon}^{\beta})\eta dS=0$$

for all $\eta\in L^2(\Gamma\times(0,T))$. Now, we restrict to the subsequence $\{(u_{\epsilon_i},v_{\epsilon_i})\}$, i.e., we have:

$$\lim_{i\to\infty}\int_{\Gamma}(u_{\epsilon_i}^{\alpha}-v_{\epsilon_i}^{\beta})\eta dS=0. \tag{3.2}$$

On the other hand, we have the weak convergence

$$\begin{cases} u_{\epsilon_i}^{\alpha}\rightharpoonup w^{\alpha}\in L^2(0,T;H^1(\Omega)),\\ v_{\epsilon_i}^{\beta}\rightharpoonup z^{\beta}\in L^2(0,T;H^1(\Gamma)).\end{cases}$$

Due to Trace Theorem, it implies

$$\begin{cases} u_{\epsilon_i}^{\alpha}\rightharpoonup w^{\alpha}\in L^2(0,T;L^2(\Gamma)),\\ v_{\epsilon_i}^{\beta}\rightharpoonup z^{\beta}\in L^2(0,T;L^2(\Gamma)).\end{cases}$$

So, for any $\eta\in L^2(0,T;L^2(\Gamma))$

$$\begin{cases} \lim_{i\to\infty}(u_{\epsilon_i}^{\alpha}-w^{\alpha},\eta)_{L^2(0,T;L^2(\Gamma))}=0,\\ \lim_{i\to\infty}(v_{\epsilon_i}^{\beta}-z^{\beta},\eta)_{L^2(0,T;L^2(\Gamma))}=0.\end{cases}$$

Combine these and (3.2), we can deduce

$$(w^{\alpha}-z^{\beta},\eta)_{L^2(0,T;L^2(\Gamma))}=0$$

for all $\eta\in L^2(0,T;L^2(\Gamma))$, which give the conclusion

$$w^{\alpha}=z^{\beta}$$

for a.e on $\Gamma\times(0,T)$. So, we can rewrite the limit (w,z) by $(w,(w|_{\Gamma})^{\alpha/\beta})$.

Next, we will complete the proof of theorem by showing that w satisfies the weak formulation (3.1). First, we recall the weak formulation (1.3) and (1.4), and choose the test function $(\varphi, \varphi|_\Gamma) \in C([0,T]; Z)$

$$\int_0^T \int_\Omega (-u_\epsilon \varphi_t + d_u \nabla u_\epsilon \cdot \nabla \varphi) dx dt = \int_\Omega u_0 \varphi(0) dx - \frac{\alpha}{\epsilon} \int_0^T \int_\Gamma (u_\epsilon^\alpha - v_\epsilon^\beta) \varphi dS dt \tag{3.3}$$

$$\int_0^T \int_\Gamma (-v_\epsilon (\varphi|_\Gamma)_t + d_v \nabla_\Gamma v_\epsilon \cdot \nabla_\Gamma (\varphi|_\Gamma)) dS dt = \int_\Gamma v_0 \varphi|_\Gamma(0) dS + \frac{\beta}{\epsilon} \int_0^T \int_\Gamma (u_\epsilon^\alpha - v_\epsilon^\beta) \varphi|_\Gamma dS dt \tag{3.4}$$

Then, multiply (3.3) by β, (3.4) by α, and take the sum, we obtain

$$\beta \int_0^T \int_\Omega -u_\epsilon \varphi_t + d_u \nabla u_\epsilon \cdot \nabla \varphi dx dt + \alpha \int_0^T \int_\Gamma -v_\epsilon (\varphi|_\Gamma)_t + d_v \nabla_\Gamma v_\epsilon \cdot \nabla_\Gamma (\varphi|_\Gamma) dS dt = \beta \int_\Omega u_0 \varphi(0) dx + \alpha \int_\Gamma v_0 \varphi|_\Gamma(0) dS. \tag{3.5}$$

Consider the subsequence $\{(u_{\epsilon_i}, v_{\epsilon_i})\}$ to the equation (3.5) and take the limit $i \to \infty$. The convergence of the sequence $\{(u_{\epsilon_i}, v_{\epsilon_i})\}_{i=1}^\infty$ to $(w, (w|_\Gamma)^{\alpha/\beta})$ gives the weak formulation (3.1), which conclude this proof. □

Remark 3.2. *The regularity condition $(w, z) \in C([0,T]; L^2(\Omega) \times L^2(\Gamma))$ in fact is consequence of the regularities $(w, z) \in L^2(0,T; Z)$ and time derivative $\partial_t (w, z) \in L^2(0,T; Z^*)$, with Z defined in the Section 2.4 (see Theorem 1.6 and its remarks). To show that $\partial_t (w, z) \in L^2(0,T; Z^*)$, we go back to the uniform boundedness of time derivative in Lemma 2.3, then we have a subsequence of $\{(u_\epsilon, v_\epsilon)\}$ converges weakly in this functional space. Com-*

bine with the fact that $\{(u_{\epsilon_i}, v_{\epsilon_i})\}$ *converges weakly to* (w,z) *in* $L^2(0,T;Z)$, *we can show that a subsequence of* $\{\partial_t(u_{\epsilon_i}, v_{\epsilon_i})\}$ *converges weakly to* $\partial_t(w,z)$ *in* $L^2(0,T;Z^*)$ *(for e.g., see [14, Section 7.5]).*

Remark 3.3. *Moreover, we can rewrite the weak formulation to a form similar to* (1.5)

$$\langle \partial_t(w,(w|_\Gamma)^{\alpha/\beta});(\varphi,\varphi|_\Gamma)\rangle_{V^*\times V} + d_u\int_\Omega \nabla w\cdot\nabla\varphi dx + d_v\int_\Gamma \nabla_\Gamma(w|_\Gamma)^{\alpha/\beta}\cdot\nabla_\Gamma(\varphi|_\Gamma)dS = 0 \quad (3.6)$$

and test function $(\varphi,\varphi|_\Gamma)\in Z$.

Remark 3.4. *Consequently, the limit system* (3) *has a weak solution, and to the best of our knowledge, there is not any results related to this type of system.*

As mentioned in the last chapter, we have noticed that the convergence holds for the whole sequence if the limit is unique (e.g., see [11]). In this case, up to our knowledge, we do not have the uniqueness of the solution for (3). However, we have it hold for the case $\alpha=\beta$, shown in [6].

Theorem 3.2 ([6])**.** *The limit system in the case* $\alpha=\beta$ *(see system* (3.7)*) has a unique weak solution, with initial condition* $(u_0,v_0)\in L^2(\Omega)\times L^2(\Gamma)$.

3.2 Convergence rate

In this section, we will investigate the convergence rate. We will consider this situation since we only have the uniqueness when $\alpha=\beta$. We will use a similar technique in [7]. In this article, the authors define functions that represent the differences between the solution of the original system and limit system's solution, assuming that these systems have classical solutions with the best regularities (the solutions are smooth and bounded).

In here, we assume that the initial condition belongs to functional space Z in the last chapter, which means

$$u_0 \in H^1(\Omega)\,, v_0 \in H^1(\Gamma) \text{ and } v_0 = u_0|_\Gamma.$$

This kind of condition is called compatibility condition, and it leads the limit system to a heat equation with Wentzell boundary condition

$$\begin{cases} \partial_t w - d_u \Delta w = 0, & x \in \Omega, t > 0 \\ d_u \nabla w \cdot \nu = -\partial_t w + d_v \Delta_\Gamma w, & x \in \Gamma, t > 0 \\ w(x,0) = u_0(x), & x \in \Omega, t > 0 \\ w|_\Gamma(x,0) = v_0(x), & x \in \Gamma, t > 0, \end{cases} \tag{3.7}$$

where $(u_0, v_0) \in Z$, defined as above. We can use many results related to this type of equation. Indeed, the equation (3.7) has been proved that it has a unique solution with "nice" regularity that we require (see [24, Theorem 1]).

Proposition 3.1 ([24]). *If the initial condition* $(u_0, v_0) \in Z$*, then the system* (3.7) *has an unique solution, satisfies the following regularity.*

$$w \in C^1([0,\infty), H^1(\Omega)) \cap C^1((0,\infty), H^2(\Omega)) \cap C((0,\infty), H^3(\Omega))$$

and

$$w|_\Gamma \in C^1([0,\infty), H^1(\Gamma)) \cap C^1((0,\infty), H^2(\Gamma)) \cap C((0,\infty), H^3(\Gamma)).$$

So, the solution of (3.7) w satisfies the following weak formulation

$$\langle w_t, \varphi \rangle_\Omega + \langle (w|_\Gamma)_t, \varphi|_\Gamma \rangle_\Gamma + d_u \int_\Omega \nabla w \cdot \nabla \varphi dx \\ + d_v \int_\Gamma \nabla_\Gamma (w|_\Gamma) \cdot \nabla_\Gamma (\varphi|_\Gamma) dS = 0 \tag{3.8}$$

for all $\varphi \in H^1(\Omega)$ and $\varphi|_\Gamma \in H^1(\Gamma)$ and a.e. $t \in (0,T)$.

On the other hand, when $\alpha = \beta$,the original system has the form

$$\begin{cases} u_t - d_u \Delta u = 0, & x \in \Omega, t > 0, \\ d_u \nabla u \cdot \nu = -\frac{\alpha}{\epsilon}(u^\alpha - v^\alpha), & x \in \Gamma, t > 0, \\ \partial_t v - d_v \Delta_\Gamma v = \frac{\alpha}{\epsilon}(u^\alpha - v^\alpha), & x \in \Gamma, t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \Omega, \\ v(x,0) = v_0(x) = (u_0)_{|\Gamma}(x) \geq 0, & x \in \Gamma, \end{cases} \tag{3.9}$$

and we use the following weak formulation:

$$\langle \partial_t u_\epsilon, \varphi \rangle_\Omega + \int_\Omega d_u \nabla u_\epsilon \cdot \nabla \varphi dx = -\frac{\alpha}{\epsilon} \int_\Gamma (u_\epsilon^\alpha - v_\epsilon^\alpha) \varphi dS, \tag{3.10}$$

$$\langle \partial_t v_\epsilon, \psi \rangle_\Gamma + \int_\Gamma d_v \nabla_\Gamma v_\epsilon \cdot \nabla_\Gamma \psi dS = \frac{\alpha}{\epsilon} \int_\Gamma (u_\epsilon^\alpha - v_\epsilon^\alpha) \psi dS, \tag{3.11}$$

for a.e. $t \in [0,T]$ and for all $(\varphi, \psi) \in H^1(\Omega) \times H^1(\Gamma)$.

We are interested in the convergence rate for the system (3.9) converges to equation (3.7) as $\epsilon \to 0$. First, fix a small ϵ, we set

$$\begin{aligned} U(x,t) &:= u_\epsilon(x,t) - w(x,t) \quad \text{on } \Omega \times [0,T], \\ V(x,t) &:= v_\epsilon(x,t) - w(x,t) \quad \text{on } \Gamma \times [0,T]. \end{aligned}$$

Remark that we have: $U - V = u - v$ a.e. on $\Gamma \times (0,T)$.

We have the time derivative

$$\frac{d}{dt}\|U(t)\|^2_{L^2(\Omega)} = \langle \partial_t u_\epsilon, u_\epsilon - w \rangle_\Omega - \langle \partial_t w, u_\epsilon - w \rangle_\Omega$$

for a.e. $t \in (0,T)$. Similarly, we have

$$\frac{d}{dt}\|V(t)\|^2_{L^2(\Gamma)} = \langle \partial_t v_\epsilon, v_\epsilon - w \rangle_\Gamma - \langle \partial_t w, v_\epsilon - w \rangle_\Gamma.$$

Combine these, we get

$$\begin{aligned}\frac{d}{dt}(\|U(t)\|^2_{L^2(\Omega)} + \|V(t)\|^2_{L^2(\Gamma)}) =& \langle \partial_t u_\epsilon, u_\epsilon - w\rangle_\Omega - \langle \partial_t w, u_\epsilon - w\rangle_\Omega \\ &+ \langle \partial_t v_\epsilon, v_\epsilon - w\rangle_\Gamma - \langle \partial_t w, v_\epsilon - w\rangle_\Gamma \\ =& \langle \partial_t u_\epsilon, u_\epsilon - w\rangle_\Omega + \langle \partial_t v_\epsilon, v_\epsilon - w\rangle_\Gamma \\ &- (\langle \partial_t w, u_\epsilon - w\rangle_\Omega + \langle \partial_t w, u_\epsilon - w\rangle_\Gamma) \\ &+ \langle \partial_t w, u_\epsilon - v_\epsilon\rangle_\Gamma.\end{aligned}$$

Then, choosing the appropriate test function for weak formulation (3.8), (3.10) and (3.11), we obtain

$$\begin{aligned}\langle \partial_t u_\epsilon, u_\epsilon - w\rangle_\Omega =& - d_u \int_\Omega \nabla u_\epsilon \cdot \nabla(u_\epsilon - w) dx \\ &- \frac{\alpha}{\epsilon}\int_\Gamma (u_\epsilon^\alpha - v_\epsilon^\alpha)(u_\epsilon - w) dS,\end{aligned}$$

$$\begin{aligned}\langle \partial_t v_\epsilon, v_\epsilon - w\rangle_\Gamma =& - d_v \int_\Gamma \nabla_\Gamma v_\epsilon \cdot \nabla_\Gamma(v_\epsilon - w) dS \\ &+ \frac{\alpha}{\epsilon}\int_\Gamma (u_\epsilon^\alpha - v_\epsilon^\alpha)(v_\epsilon - w) dS,\end{aligned}$$

and

$$\begin{aligned}\langle \partial_t w, u_\epsilon - w\rangle_\Omega + \langle \partial_t w, u_\epsilon - w\rangle_\Gamma =& - d_u \int_\Omega \nabla w \cdot \nabla(u_\epsilon - w) dx \\ &- d_v \int_\Gamma \nabla_\Gamma(w|_\Gamma) \cdot \nabla_\Gamma(u_\epsilon - w) dS.\end{aligned}$$

Combine these, we have:

$$\begin{aligned}\frac{d}{dt}(\|U(t)\|^2_{L^2(\Omega)} + \|V(t)\|^2_{L^2(\Gamma)}) =& - d_u\|\nabla(u_\epsilon - w)\|^2_{L^2(\Omega)} - d_v\|\nabla(v_\epsilon - w)\|^2_{L^2(\Gamma)} \\ &- \frac{\alpha}{\epsilon}\int_\Gamma (u_\epsilon^\alpha - v_\epsilon^\alpha)(u_\epsilon - v_\epsilon) dS \\ &+ \langle \partial_t w, u_\epsilon - v_\epsilon\rangle_\Gamma.\end{aligned}$$

The first and second terms of the right-hand side are non-positive, so we only need to work with the other two. For the integral term, we have:

$$
\begin{aligned}
-\frac{\alpha}{\epsilon}\int_\Gamma (u_\epsilon^\alpha - v_\epsilon^\alpha)(u_\epsilon - v_\epsilon)dS &= -\frac{\alpha}{\epsilon}\int_\Gamma (\sum_{i=0}^{\alpha-1} u_\epsilon^i v_\epsilon^{\alpha-1-i})(u_\epsilon - v_\epsilon)(u_\epsilon - v_\epsilon)dS \\
&= -\frac{\alpha}{\epsilon}\int_\Gamma (\sum_{i=0}^{\alpha-1} u_\epsilon^i v_\epsilon^{\alpha-1-i})|U-V|^2 dS.
\end{aligned}
$$

We use the assumption that the initial condition is strictly positive. Then, due to Proposition 2.1, v_ϵ has a lower bound $m > 0$ that does not depend on ϵ. So, we have the estimate:

$$
\sum_{i=0}^{\alpha-1} u_\epsilon^i v_\epsilon^{\alpha-1-i} \geq v_\epsilon^{\alpha-1} \geq m^{\alpha-1} > 0 \quad a.e. \, (0,T)\times\Gamma,
$$

which implies:

$$
-\frac{\alpha}{\epsilon}\int_\Gamma (u_\epsilon^\alpha - v_\epsilon^\alpha)(u_\epsilon - v_\epsilon)dS \leq \frac{-c_1}{\epsilon}\|U-V\|^2_{L^2(\Gamma)},
$$

With c_1 is a constant does not depend on ϵ. On the other hand, the regularity of w allows us to rewrite

$$
\langle \partial_t w, u_\epsilon - v_\epsilon\rangle_\Gamma = (\partial_t w, u_\epsilon - v_\epsilon)_{L^2(\Gamma)} \leq c_2\|U-V\|_{L^2(\Gamma)},
$$

Moreover, c_2 is a constant not dependent on ϵ. The above inequality holds since w_t is bounded in $L^2(\Gamma)$, $u_\epsilon - v_\epsilon = U - V$ on Γ and Holder's inequality. Combine these above, we deduce

$$
\begin{aligned}
\frac{d}{dt}(\|U(t)\|^2_{L^2(\Omega)} + \|V(t)\|^2_{L^2(\Gamma)}) &\leq \frac{-c_1}{\epsilon}\|U-V\|^2_{L^2(\Gamma)} + c_2\|U-V\|_{L^2(\Gamma)} \\
&\leq \frac{-c_1}{\epsilon}\|U-V\|^2_{L^2(\Gamma)} + \frac{c_1}{2\epsilon}\|U-V\|^2_{L^2(\Gamma)} + \frac{2c_2^2}{c_1}\epsilon \\
&\leq c_3\epsilon,
\end{aligned}
$$

with $c_3 = \frac{2c_2^2}{c_1}$. So for any fixed $t_0 \in [0, T]$, take the integration from 0 to t_0 and using the fact that $U(0) = V(0) = 0$

$$\|U(t_0)\|^2_{L^2(\Omega)} + \|V(t_0)\|^2_{L^2(\Gamma)} \leq c_3 t_0 \epsilon.$$

From the argument, we have the following theorem:

Theorem 3.3 (Convergence rate). *Assume that initial condition* $(u_0, v_0) \in Z$ *is strictly positive, the solution of* (3.9) $\{(u_\epsilon, v_\epsilon)\}$ *converges to the solution of* (3.7) *as* $\epsilon \to 0$ *with convergence rate*

$$\|u(t) - w(t)\|_{L^2(\Omega)} + \|v(t) - w|_\Gamma(t)\|_{L^2(\Gamma)} \leq c\sqrt{\epsilon}.$$

Remark 3.5. *Notice that the positivity of the initial condition is only necessary if* $\alpha > 1$. *In the linear case* $\alpha = 1$, *the constant* $c_1 = 1$ *intimately, then we can remove that condition.*

Remark 3.6. *In Section 3.1, we have shown that* $u_\epsilon^\alpha - v_\epsilon^\beta$ *converges strongly to 0 with the rate* $\epsilon^{1/2}$ *also. Therefore, it seems that the convergence rate of* $\{(u_\epsilon, v_\epsilon)\}$ *generally cannot exceed this rate.*

CONCLUSION AND DISCUSSION

In this thesis, we have introduced the fast reaction limits for the nonlinear bulk-surface reaction-diffusion problems. In detail, we have given a result related to the open question in article [6], which is the fast reaction limit problem for the bulk-surface reaction-diffusion system, modeled by the chemical reaction (1). We have shown that the solution of the original system (2) admits a subsequence that converges strongly to the solution of (3), in the functional space $L^2(0,T;L^2(\Omega)\times L^2(\Gamma))$. Moreover, we have a result about the convergence rate, with some technical assumptions.

Due to the lack of time, there are still many open questions related to this problem. Here are some example:

- Can we have a better convergence, for example in the case $\alpha=\beta$, the authors of [6] have shown that the convergence holds in the space $L^2(0,T;H^1(\Omega)\times H^1(\Gamma))$? This is proved by using convergence to equilibrium, which we also have for this situation.
- Expansion for the rate function: Here, we work with $F(u,v)=u^\alpha-v^\beta$, based on the mass action law. What happens if we replace it with a more general function, for example, a polynomial?
- Adding more components in the chemical reaction (1): What can we say if adding one more component in the reaction, for example

$$U \underset{k_1}{\overset{k_1}{\leftrightarrows}} u \underset{k_2}{\overset{k_2}{\leftrightarrows}} v$$

with U are on Ω while u and v are located on Γ? The question here is what can we conclude about the fast reaction limit if $k_2 > 0$ is fixed and k_1 tends to infinity. The well-posedness of the system have been proved in [9]

- Can we remove any assumption in the convergence rate question?

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