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Nguyen Quang Huy

A SPACE – TIME FINITE ELEMENT METHOD FOR AN ADVECTION – DIFFUSION PROBLEM WITH A MOVING INTERFACE

MASTER'S THESIS IN MATHEMATICS

Hanoi – 2024

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Major: **Applied Mathematics**

Number: **8 46 01 12**

Supervisor:

Prof. Dr. habil. Đinh Nho Hào

Hanoi – 2024

BỘ GIÁO DỤC VÀ ĐÀO TẠO

VIỆN HÀN LÂM KHOA HỌC VÀ CÔNG NGHỆ VIỆT NAM

HỌC VIỆN KHOA HỌC VÀ CÔNG NGHỆ

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Ngành: Toán ứng dụng

Mã số: 8 46 01 12

NGƯỜI HƯỚNG DẪN KHOA HỌC:

GS.TSKH. Đinh Nho Hào

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Declaration

I hereby declare that the work presented in the Thesis titled **A space-time finite element method for an advection-diffusion problem with a moving interface** is an original piece of research conducted under the supervision of Prof. Dr. habil. Đinh Nho Hào. The matter embodied in the Thesis has not appeared in any other research.

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Nguyen Quang Huy

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Introduction

Advection-diffusion problems with moving subdomains play a significant role in various fields of engineering and physical phenomena that involve moving multiple-component systems, as in mass transport [1], heat transfer [2], electromagnetics [3], or heat induction [4]. The Thesis aims to present the numerical analysis of an interface-fitted space-time finite element method [5] for a boundary value problem for an advection-diffusion equation with a moving interface and an inverse source problem for this problem.

Relevant literature

The main difficulty of solving interface problems is the non-sufficient smoothness of the solution across the interface, which results in sub-optimal convergence orders of classical finite element methods [6]. In the last 50 years, many studies have focused on handling the issue, forming two major approaches: interface-unfitted and interface-fitted methods. The former approach includes some examples such as the extended finite element method (XFEM) [7], the immersed finite element method (IFEM) [8], and the multiscale finite element method (MsFEM) [9]. These methods approximate discontinuous quantities by modifying the local finite element basis functions on interface elements instead of using a triangulation that fits the interface. When the interface evolves, this approach is efficient since it allows fixed interface-independent simplicial triangulations [10, 11, 12].

In contrast, interface-fitted methods prevent the interface from cutting through an element arbitrarily or resolve the interface approximately [13, 14]. However, as opposed to the success of unfitted approaches, interface-fitted methods have received little attention for solving time-dependent problems with moving subdomains. The reason is that the re-meshing procedure at each time step introduces additional errors in interpolating two consecutive meshes, which can exceed the feasible effort. The method is then applied to an inverse source problem with observations inside the space domain.

Related to our problem setting, Bellassoued and Yamamoto [15] studied an inverse source problem for a parabolic transmission equation. The authors established a conditional stability of determining the spatial component of the source from a single measurement on a fixed subdomain. Recently, also invoked a partial interior observation, Chen et al. [16] simultaneously reconstructed the initial value and the spatial part of the source. They ended up with a conditional stability result and an iterative thresholding algorithm to solve the inverse problem. In [17], Zhang et al. studied a distributed optimal control problem (a special case of the inverse source problem) for a parabolic interface system. The authors presented an error analysis of the finite element discretization of the problem and obtained the optimal order error estimates of the control, state, and adjoint. However, note that inverse source problems for time-dependent equations with moving interfaces have not been studied so far.

Contributions

The Thesis presents an interface-fitted space-time finite element method [5] for an advection-diffusion equation with a moving interface. This method resolves a disadvantage of interface-fitted methods and allows us to cope with geometrically complicated interfaces. We establish two new optimal order priori error estimates for the method, supplementing the results in [5].

Next, we focus on an inverse source problem for the advection-diffusion equation with a moving interface under non-negative constraints. It is an ill-posed problem. We first regularize the problem by using the Tikhonov method and then study the existence and stability of the regularized source with respect to the noise. Second, we propose a strategy for discretizing our problem. We discretize the regularized state and adjoint with element-wise linear finite elements associated with an unstructured mesh [5]. On the other hand, we employ the variational approach [18] for the regularized source. We arrive at the optimal order priori error estimates of the regularized source, state, and adjoint.

From these estimates, we suggest a condition for the strong convergence of the discrete regularized source to the continuous unregularized one and the corresponding convergence rate, following the idea of Hào et al. [19] for elliptic inverse source problems. To the best of our knowledge, this type of convergence rate for inverse source problems for an advection-diffusion problem with a moving interface is new.

Outline

The Thesis comprises three chapters, excluding introduction, conclusion, and bibliography. Chapter 1 provides basic functional spaces and the background related to bounded linear operators. Chapter 2 is devoted to the Galerkin finite element discretization of the advection-diffusion equation with moving subdomains, in which we derive the optimal order prior error estimates in various norms. Chapter 3 studies an inverse source problem for the equation in chapter 2 from a partial interior observation. We present the Tikhonov regularization, the finite element discretization errors in two norms, and a condition for the convergence of the discrete regularized source to the continuous unregularized one. Lastly, we will give some perspectives and comments about future work.

Chapter 1

Background

This chapter presents basic functional spaces and some topics related to bounded linear operators. All the contents are cited from [20], [21], [22], and [23]. We denote by $C > 0$ a generic constant depending on the space-time domain Q_T, the coefficient κ and the operator S, but independent of the function u. Their different values in different contexts are allowed.

1.1 Functional spaces

Definition 1.1. (Normed space) Let U be a real vector space. A function $\|\cdot\|_{\mathrm{U}} : \mathrm{U} \to \mathbb{R}$ is called a norm on U if it satisfies the following conditions:

a) $\|u\|_{\mathrm{U}} = 0$ if and only if $u = 0$ for all $u \in \mathrm{U}$.

b) $\|\eta u\|_{\mathrm{U}} = |\eta| \, \|u\|_{\mathrm{U}}$ for all $\eta \in \mathbb{R}$ and $u \in \mathrm{U}$.

c) $\|u + v\|_{\mathrm{U}} \leq \|u\|_{\mathrm{U}} + \|v\|_{\mathrm{U}}$ for all $u, v \in \mathrm{U}$.

The vector space U equipped with the norm $\|\cdot\|_{\mathrm{U}}$ is called the normed space.

Definition 1.2. (Inner product space) Let U be a real vector space. A function $(\cdot, \cdot)_{\mathrm{U}} : \mathrm{U} \times \mathrm{U} \to \mathbb{R}$ is called an inner product on U if it satisfies the following conditions:

a) $(u,u)_{\mathrm{U}} \geq 0$ for all $u \in \mathrm{U}$ and $(u,u)_{\mathrm{U}} = 0$ if and only if $u = 0$.

b) $(u,v)_{\mathrm{U}} = (v,u)_{\mathrm{U}}$ for all $u, v \in \mathrm{U}$.

c) $(u+v,w)_{\mathrm{U}} = (u,w)_{\mathrm{U}} + (v,w)_{\mathrm{U}}$ for all $u, v, w \in \mathrm{U}$.

d) $(\eta u, v)_{\mathrm{U}} = \eta\,(u,v)_{\mathrm{U}}$ for all $\eta \in \mathbb{R}$ and $u, v \in \mathrm{U}$.

The vector space U endowed with the inner product $(\cdot,\cdot)_{\mathrm{U}}$ is called the inner product space.

Every inner product space U is a normed space with respect to the induced norm $\|u\|_{\mathrm{U}} = \sqrt{(u,u)_{\mathrm{U}}}$ for all $u \in \mathrm{U}$.

Definition 1.3. (Convergence and the Cauchy sequence) Let U be a normed space. We say that the sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathrm{U}$ converges to $u \in \mathrm{U}$ if $\lim\limits_{n\to\infty} \|u_n - u\|_{\mathrm{U}} = 0$, denoted by $\lim\limits_{n\to\infty} u_n = u$ or $u_n \to u$ in U.

We say that the sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathrm{U}$ is the Cauchy sequence if for all $\varepsilon > 0$, there exists $M = M(\varepsilon) \in \mathbb{N}$ such that $\|u_m - u_n\|_{\mathrm{U}} < \varepsilon$ for all $m, n > M$.

Definition 1.4. (Complete space) A normed space U is said to be complete if all Cauchy sequences in U converge.

Definition 1.5. (Banach space and Hilbert space) A complete normed space is called a Banach space. A complete inner product space with respect to the induced norm is called a Hilbert space.

1.1.1. Sobolev spaces

In this section, we consider Ω to be an open set in $\mathbb{R}^d$ $(d = 1, 2, 3)$.

Definition 1.6. (Lebesgue space) Let $q \in [1, \infty)$. The space $\mathrm{L}^q(\Omega)$ consists of all Lebesgue measurable functions u, defined at almost everywhere in Ω such that

$$\|u\|_{\mathrm{L}^q(\Omega)} := \left(\int\limits_\Omega |u(\boldsymbol{x})|^q \, \mathrm{d}\boldsymbol{x} \right)^{\frac{1}{q}} < \infty.$$

The space $\mathrm{L}^\infty(\Omega)$ includes all Lebesgue measurable functions u, defined at almost everywhere in Ω that satisfing

$$\|u\|_{\mathrm{L}^\infty(\Omega)} := \operatorname*{ess\,sup}_{\boldsymbol{x} \in \Omega} |u(\boldsymbol{x})| < \infty.$$

Definition 1.7. (Integer order Sobolev space) Let $s \in \mathbb{N}, q \in [1, \infty]$. We define the space $\mathrm{W}^{s,q}(\Omega)$ as

$$\mathrm{W}^{s,q}(\Omega) := \{u \in \mathrm{L}^q(\Omega) \mid \partial^{\boldsymbol{\alpha}} u \in \mathrm{L}^q(\Omega) \text{ for all } 0 \le |\boldsymbol{\alpha}| \le s\},$$

where $\partial^{\boldsymbol{\alpha}} u$ denotes the $\boldsymbol{\alpha}$-order weak derivative of u, endowed with the norm

$$\|u\|_{\mathrm{W}^{s,q}(\Omega)} := \left(\sum_{0 \le |\boldsymbol{\alpha}| \le s} \|\partial^{\boldsymbol{\alpha}} u\|^q_{\mathrm{L}^q(\Omega)} \right)^{\frac{1}{q}} \qquad \forall u \in \mathrm{W}^{s,q}(\Omega).$$

Definition 1.8. (Fractional order Sobolev space) Let $s > 0, s \notin \mathbb{N}$ and $q \in [1, \infty)$. The space $\mathrm{W}^{s,q}(\Omega)$ consists of all functions $u \in \mathrm{W}^{\lfloor s \rfloor, q}(\Omega)$ such that

$$\sum_{|\boldsymbol{\alpha}| = \lfloor s \rfloor} \int\limits_\Omega \int\limits_\Omega \frac{|\partial^{\boldsymbol{\alpha}} u(\boldsymbol{x}) - \partial^{\boldsymbol{\alpha}} u(\boldsymbol{y})|^q}{|\boldsymbol{x} - \boldsymbol{y}|^{d + q(s - \lfloor s \rfloor)}} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\boldsymbol{y} < \infty,$$

equipped with the norm

$$\|u\|^q_{\mathrm{W}^{s,q}(\Omega)} := \|u\|^q_{\mathrm{W}^{\lfloor s \rfloor, q}(\Omega)} + \sum_{|\boldsymbol{\alpha}| = \lfloor s \rfloor} \int\limits_\Omega \int\limits_\Omega \frac{|\partial^{\boldsymbol{\alpha}} u(\boldsymbol{x}) - \partial^{\boldsymbol{\alpha}} u(\boldsymbol{y})|^q}{|\boldsymbol{x} - \boldsymbol{y}|^{d + q(s - \lfloor s \rfloor)}} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\boldsymbol{y},$$

for all $u \in \mathrm{W}^{s,q}(\Omega)$.

Theorem 1.9. *Let* $s \geq 0$. *The space* $\mathrm{W}^{s,q}(\Omega)$ *furnished with the norm* $\|\cdot\|_{\mathrm{W}^{s,q}(\Omega)}$ *is a Banach space for all* $q \in [1,\infty]$. *In particular, the space* $\mathrm{W}^{s,2}(\Omega)$ *is a Hilbert space, denoted by* $\mathrm{H}^{s}(\Omega)$. *The norm on this space is denoted by* $\|\cdot\|_{\mathrm{H}^{s}(\Omega)}$.

Theorem 1.10. *(Embedding and compact embedding) Let* $s > 0, q \in [1,\infty]$ *and* Ω *be a Lipschitz domain in* $\mathbb{R}^d$ $(d = 1,2,3)$.

a) If $sq > d$ *then we have the embedding* $\mathrm{W}^{s,q}(\Omega) \hookrightarrow \mathrm{L}^{\infty}(\Omega)$ *and the compact embedding* $\mathrm{W}^{s,q}(\Omega) \hookrightarrow \mathrm{C}\left(\overline{\Omega}\right)$.

b) If $sq \leq d$ *then for all* $q' \in \left[1, \frac{qd}{d-sq}\right)$, *we have the compact embedding* $\mathrm{W}^{s,q}(\Omega) \hookrightarrow \mathrm{L}^{q'}(\Omega)$.

1.1.2. Anisotropic Sobolev spaces

In this section, we consider Ω to be an open set in $\mathbb{R}^d$ $(d = 1,2,3)$ and $T > 0$ to be a positive number. Denote by $Q_T := \Omega \times (0,T)$ a space-time domain.

Definition 1.11. (t-anisotropic Sobolev space) Let $l,k \in \mathbb{N}$. We define the space $\mathrm{H}^{l,k}(Q_T)$ as

$$\mathrm{H}^{l,k}(Q_T) := \left\{u \in \mathrm{L}^2(Q_T) \mid \partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} \partial_t^r u \in \mathrm{L}^2(Q_T) \text{ for all } 0 \leq |\boldsymbol{\alpha}| \leq l,\ r = 0,1,\ldots,k\right\},$$

where $\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} u$ and $\partial_t^r u$ are the weak derivatives with respect to $\boldsymbol{x}$ and t of u, respectively, endowed with the norm

$$\|u\|_{\mathrm{H}^{l,k}(Q_T)} := \left(\sum_{0\leq|\boldsymbol{\alpha}|\leq l,\ 0\leq r\leq k} \|\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} \partial_t^r u\|_{\mathrm{L}^2(Q_T)}^2\right)^{\frac{1}{2}} \qquad \forall u \in \mathrm{H}^{l,k}(Q_T).$$

Let $\mathrm{H}_0^{1,0}(Q_T)$ be the closure of the space $\mathrm{C}_0^1(Q_T)$ with respect to the norm $\|\cdot\|_{\mathrm{H}^{1,0}(Q_T)}$. For convenience, we use the compact notation $\mathrm{Y} := \mathrm{H}_0^{1,0}(Q_T)$, furnished with an equivalent norm

$$\|u\|_{\mathrm{Y}}^2 := \int_0^T \int_\Omega \kappa \, |\nabla u|^2 \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t,$$

where $\kappa = \kappa(\boldsymbol{x}, t)$ will be introduced in chapter 2. The equivalence results from the Poincaré–Steklov inequality

$$\|u\|_{\mathrm{L}^2(Q_T)} \leq C \, \|\nabla u\|_{\mathbf{L}^2(Q_T)} \qquad \forall u \in \mathrm{Y}, \tag{1.1}$$

we refer to [20]. The dual space of Y is denoted by Y' (see Definition 1.18), and the duality pairing between Y' and Y is denoted by $\langle \cdot, \cdot \rangle$. Let us introduce the spaces

$$\mathrm{X} := \{u \in \mathrm{Y} \mid \partial_t u \in \mathrm{Y}'\}, \tag{1.2}$$

and

$$\mathrm{X}_t := \{u \in \mathrm{X} \mid u(\cdot, t) = 0\},$$

with $t \in \{0, T\}$, equipped with the norm $\|u\|_{\mathrm{X}}^2 := \|u\|_{\mathrm{Y}}^2 + \|\partial_t u\|_{\mathrm{Y}'}^2$.

Theorem 1.12. *(Time trace and integration by parts) Let X be the space defined in (1.2). Then, the following statements hold true*

a) The space X is embedded into the space $\mathrm{C}\left([0, T], \mathrm{L}^2(\Omega)\right)$.

b) The trace operator $u \in \mathrm{X} \to u(\cdot, t) \in \mathrm{L}^2(\Omega)$ is bounded for almost every $t \in [0, T]$. The following inequality holds

$$\sup_{t \in [0,T]} \|u(\cdot, t)\|_{\mathrm{L}^2(\Omega)} \leq C \, \|u\|_{\mathrm{X}} \qquad \forall u \in \mathrm{X}. \tag{1.3}$$

c) For all $u, v \in \mathrm{X}$*, we have*

$$\langle \partial_t u, v \rangle + \langle \partial_t v, u \rangle = \int\limits_{\Omega} u\left(\boldsymbol{x}, T\right) v\left(\boldsymbol{x}, T\right) \, \mathrm{d}\boldsymbol{x} - \int\limits_{\Omega} u\left(\boldsymbol{x}, 0\right) v\left(\boldsymbol{x}, 0\right) \, \mathrm{d}\boldsymbol{x}.$$

Consider the case Q_T is separated into two subdomains Q_1 and Q_2 by the space-time interface $\Gamma^* := \partial Q_1 \cap \partial Q_2$. In this scenario, the space $\mathrm{H}^{1,0}\left(Q_1 \cup Q_2\right)$ is important, since in it, the trace operators $\gamma_i : \mathrm{H}^{1,0}\left(Q_i\right) \to \mathrm{L}^2\left(\Gamma^*\right) \ (i = 1, 2)$ are well-defined under mild assumptions on Γ^* [24].

Let us present the Stein extension operators [23], which are crucial for handling functions with global low but local high regularity. For any fixed $s \geq 0$, a function $u \in \mathrm{H}^s\left(Q_1 \cup Q_2\right)$, and $i = 1, 2$, denote by $u_i := u_{|\,Q_i} \in \mathrm{H}^s\left(Q_i\right)$ the restriction of u to the subdomain Q_i.

Theorem 1.13. *Assume that* Γ^* *is a Lipschitz continuous hypersurface in* $\mathbb{R}^{d+1}$*, then there exists smooth extensions* $\mathrm{E}_i : \mathrm{H}^s\left(Q_i\right) \to \mathrm{H}^s\left(Q_T\right)$ *such that*

$$\mathrm{E}_i\, u = u_i \quad \textit{in } Q_i, \quad \left\|\mathrm{E}_i\, u\right\|_{\mathrm{H}^s(Q_T)} \leq C \left\|u_i\right\|_{\mathrm{H}^s(Q_i)} \qquad (i = 1, 2). \tag{1.4}$$

1.2 Bounded linear operators

In this section, we consider U and W to be two normed spaces.

Definition 1.14. (Linear operator and linear functional) An operator $S : \mathcal{D}\left(S\right) \subset \mathrm{U} \to \mathrm{W}$ is said to be linear if it satisfies the following conditions:

a) $S\left(u + v\right) = S\left(u\right) + S\left(v\right)$ for all $u, v \in \mathcal{D}\left(S\right)$.

b) $S\left(\eta u\right) = \eta S\left(u\right)$ for all $\eta \in \mathbb{R}$ and $u \in \mathcal{D}\left(S\right)$.

When $\mathrm{W} = \mathbb{R}$, we call S a linear functional.

Definition 1.15. (Continuous operator) An operator $S : \mathcal{D}(S) \subset \mathrm{U} \to \mathrm{W}$ is said to be continuous if for any sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{D}(S)$ such that there exists $u \in \mathcal{D}(S)$ satisfying $u_n \to u$ in U, we have $S(u_n) \to S(u)$ in W.

Definition 1.16. (Bounded operator) An operator $S : \mathcal{D}(S) \subset \mathrm{U} \to \mathrm{W}$ is said to be bounded if there exists a constant $C = C(S) > 0$ such that $\|S(u)\|_{\mathrm{W}} \leq C \|u\|_{\mathrm{U}}$ for all $u \in \mathrm{U}$.

Theorem 1.17. *A linear operator is continuous if and only if it is bounded.*

Definition 1.18. (Dual space) Let U be a normed space. A dual space of U, denoted by U', consists of all bounded linear functionals $S : \mathcal{D}(S) \subset \mathrm{U} \to \mathbb{R}$, furnished with the norm

$$\|S\|_{\mathrm{U}'} := \sup_{u\in\mathrm{U}} \frac{|S(u)|}{\|u\|_{\mathrm{U}}} = \sup_{u\in\mathrm{U}} \frac{\left|\langle S, u\rangle_{\mathrm{U}',\mathrm{U}}\right|}{\|u\|_{\mathrm{U}}} \qquad \forall S \in \mathrm{U}'.$$

Definition 1.19. (Adjoint operator) Let U, W be two Banach spaces and $S : \mathcal{D}(S) \subset \mathrm{U} \to \mathrm{W}$ be a bounded linear operator. An adjoint operator of S is a bounded linear operator $S^* : \mathrm{W}' \to \mathrm{U}'$ such that

$$\langle S^*(w'), u\rangle_{\mathrm{U}',\mathrm{U}} = \langle w', S(u)\rangle_{\mathrm{W}',\mathrm{W}} \qquad \forall (u, w') \in \mathrm{U} \times \mathrm{W}'.$$

1.2.1. Weak convergence in Banach spaces

Definition 1.20. (Weak convergence) Let U be a normed space. We say that the sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathrm{U}$ converges weakly to $u \in \mathrm{U}$, written as $u_n \rightharpoonup u$ in U, if $S(u_n) \to S(u)$ for all $S \in \mathrm{U}'$.

Theorem 1.21. *Let* U *be a normed space. Then, every closed and convex subset* E *of* U *is weakly sequentially closed, which means for any sequence* $\{u_n\}_{n\in\mathbb{N}} \subset \mathrm{E}$ *such that* $u_n \rightharpoonup u$ *in* U*, we imply* $u \in \mathrm{E}$.

Theorem 1.22. *(Weak convergence in a reflexive space) Let* U *be a reflexive Banach space. Then, every bounded, closed and convex subset* E *of* U *is weakly sequentially compact, that is, for any sequence* $\{u_n\}_{n\in\mathbb{N}} \subset \mathrm{E}$*, we can extract a subsequence* $\{u_{n_k}\}_{k\in\mathbb{N}}$ *such that there exists* $u \in \mathrm{E}$ *satisfying* $u_n \rightharpoonup u$ *in* U.

Theorem 1.23. *(Weak convergence with convex continuous functional) Let* U *be a Banach space. Then, every convex continuous functional* $S : \mathcal{D}(S) \subset \mathrm{U} \to \mathbb{R}$ *is weakly lower semicontinuous, which means for any sequence* $\{u_n\}_{n\in\mathbb{N}} \subset \mathrm{U}$ *such that there exists* $u \in \mathrm{U}$ *satisfying* $u_n \rightharpoonup u$ *in* U*, we have* $\liminf_{n\to\infty} S(u_n) \geq S(u)$.

1.2.2. Differentiability in Banach spaces

Consider U, W as two Banach spaces and E as an open subset of U.

Definition 1.24. (Gâteaux derivative) An operator $S : \mathrm{E} \to \mathrm{W}$ is said to have a directional derivative in the direction $h \in \mathrm{U}$ at an element $u \in \mathrm{E}$, written as $DS(u,h)$, if there exists the limit

$$\lim_{\tau\to 0} \frac{S(u+\tau h) - S(u)}{\tau} =: DS(u,h).$$

If the operator $h \to DS(u,h)$ is a bounded linear operator, then we say that S is Gâteaux differentiable at $u \in \mathrm{E}$ with the Gâteaux differential $DS(u,h)$ and the Gâteaux derivative S' given by $DS(u,h) =: S'(u)(h)$.

Definition 1.25. (Frechét derivative) A continuous operator $S : \mathrm{E} \to \mathrm{W}$ is said to be Frechét differentiable at an element $u \in \mathrm{E}$ if there esists a bounded linear operator $DS(u) : \mathrm{U} \to \mathrm{W}$ such that

$$\lim_{h \to 0} \frac{\|S(u+h) - S(u) - DS(u)(h)\|_{\mathrm{W}}}{\|h\|_{\mathrm{U}}} = 0.$$

The term $DS(u)(h)$ is refered as the Frechét differential of S at $u \in \mathrm{E}$ with the variation $h \in \mathrm{U}$, and $DS(u)$ is the Frechét derivative of S at $u \in \mathrm{E}$, denoted by $S'(u)$.

Clearly, if an operator is Frechét differentiable operator, then it is Gâteaux differentiable. Moreover, in that case, these two derivatives coincide.

Chapter 2

An advection-diffusion equation with a moving interface

This chapter presents the interface-fitted space-time finite element method [5] for solving an advection-diffusion equation with a moving interface. This problem appears in various fields of engineering and physical phenomenon that involve moving multiple-component systems, such as mass transport [1], heat transfer [2], electromagnetics [3], or heat induction [4]. The unknown U in (2.1) may represent the concentration of the pollutant or the electron transported at a velocity $\mathbf{v}$ owing to the advection and diffusion effect.

Let Ω be a Lipschitz domain in $\mathbb{R}^d$ $(d = 1, 2)$ with the boundary $\partial\Omega$. The domain Ω is splitted into two time-dependent subdomains $\Omega_1(t)$ and $\Omega_2(t)$ by an interface $\Gamma(t)$, for all $t \in [0, T]$ with $T > 0$. The interface $\Gamma(t)$ is transported by a velocity field $\mathbf{v} = \mathbf{v}(\boldsymbol{x}, t) \in \mathrm{C}\left([0, T], \mathbf{C}^2(\Omega)\right)$ that satisfying $\nabla \cdot \mathbf{v}(\boldsymbol{x}, t) = 0$ for all $(\boldsymbol{x}, t) \in \Omega \times [0, T]$ [25]. We denote by $Q_T := \Omega \times (0, T)$ the space-time domain and

$$Q_i := \bigcup_{t \in (0,T)} \Omega_i(t) \times \{t\} \qquad (i = 1, 2)$$

two subdomains seperated by the space-time interface $\Gamma^* := \bigcup_{t\in(0,T)} \Gamma(t) \times \{t\}$. Assume that Γ^* is a C^2-regular hypersurface in $\mathbb{R}^{d+1}$ and $\Gamma(t) \cap \partial\Omega = \varnothing$ for all $t \in [0,T]$. Consider the following problem

$$\begin{cases} \partial_t U + \mathbf{v}\cdot\nabla U - \nabla\cdot(\kappa\nabla U) = F & \text{in } Q_T, \\ [U] = 0 & \text{on } \Gamma^*, \\ [\kappa\nabla U\cdot\mathbf{n}] = 0 & \text{on } \Gamma^*, \\ U = 0 & \text{on } \partial\Omega\times(0,T), \\ U(\cdot,0) = U_0 & \text{in } \Omega, \end{cases} \tag{2.1}$$

where F is the source term, U_0 is the initial value, and $\mathbf{n}$ stands for the unit normal at $\Gamma(t)$ pointing from $\Omega_1(t)$ into $\Omega_2(t)$. The notation $[U] = U_{1\,|\,\Gamma(t)} - U_{2\,|\,\Gamma(t)}$ denotes the jump of U across $\Gamma(t)$, with $U_{i\,|\,\Gamma(t)}$ the limiting value from $\Omega_i(t)$ of U $(i = 1,2)$.

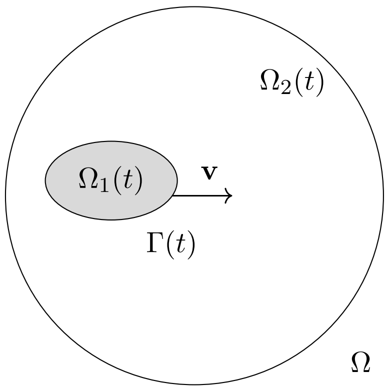

Figure 2.1: The interface $\Gamma(t)$, which envolves by a velocity $\mathbf{v}$, devides the domain Ω into two subdomains $\Omega_1(t)$ and $\Omega_2(t)$, consider the case $d = 1$ [5].

For simplicity, let us assume that the diffusion coefficient κ is a positive constant on each subdomain

$$\kappa = \begin{cases} \kappa_1 > 0 & \text{in} \quad \Omega_1(t), \\ \kappa_2 > 0 & \text{in} \quad \Omega_2(t), \end{cases} \qquad t \in [0,T]\,.$$

The general setting of the subdomain-wise continuous uniformly positive coefficient $\kappa \in \mathrm{L}^{\infty}(Q_T)$ can be treated similarly. In this chapter, the constant $C > 0$ depends on the space-time domain Q_T, the position of the space-time interface Γ^*, the norm $\|\mathbf{v}\|_{\mathbf{L}^{\infty}(Q_T)}$, and the coefficient κ, but is independent of the function u, the function $\overline{u}$, and the mesh size h. Their different values in different contexts are allowed.

2.1 Variational formulation

In this section, we recall from [5] the variational formulation of the problem (2.1) and its well-posedness. Let $F \in \mathrm{Y}'$ and $U_0 \in \mathrm{H}_0^1(\Omega)$. Denote by $u_0 \in \mathrm{X}$ an extension of $U_0 \in \mathrm{H}_0^1(\Omega)$. We define the solution of the problem (2.1) as the function $U = \overline{u} + u_0 \in \mathrm{X}$ such that $\overline{u} \in \mathrm{X}_0$ solves

$$a(\overline{u}, \varphi) = \langle F, \varphi \rangle - a(u_0, \varphi) \qquad \forall \varphi \in \mathrm{Y}, \tag{2.2}$$

where the bilinear form $a : \mathrm{X} \times \mathrm{Y} \to \mathbb{R}$ is given by

$$a(u, \varphi) := \langle \partial_t u, \varphi \rangle + \int_0^T \int_\Omega (\mathbf{v} \cdot \nabla u)\, \varphi + \kappa \nabla u \cdot \nabla \varphi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

Lemma 2.1. *There exists a constant $C > 0$ that satisfies*

$$\sup_{\varphi \in \mathrm{Y} \setminus \{0\}} \frac{a(\overline{u}, \varphi)}{\|\varphi\|_{\mathrm{Y}}} \geq C \|\overline{u}\|_{\mathrm{X}} \qquad \forall \overline{u} \in \mathrm{X}_0 .$$

Lemma 2.2. *If $a(\overline{u}, \varphi) = 0$ for all $\overline{u} \in \mathrm{X}_0$ then $\varphi = 0$.*

The well-posedness of the problem (2.2) results from Lemmas 2.1 and 2.2, according to the Banach-Nečas-Babuška theorem [26].

Theorem 2.3. *Let $F \in \mathrm{Y}'$ and $U_0 \in \mathrm{H}_0^1(\Omega)$. Then, the problem (2.2) admits a unique solution $\overline{u} \in \mathrm{X}_0$ such that $\|\overline{u}\|_{\mathrm{X}} \leq C\left(\|F\|_{\mathrm{Y}'} + \|u_0\|_{\mathrm{X}}\right)$.*

Remark 2.4. If $F \in \mathrm{L}^2(Q_T)$ then we have a priori estimate

$$\|\overline{u}\|_{\mathrm{X}} \leq C\left(\|F\|_{\mathrm{L}^2(Q_T)} + \|u_0\|_{\mathrm{X}}\right), \tag{2.3}$$

since from the inequality (1.1), it holds that

$$\|F\|_{\mathrm{Y}'} = \sup_{\varphi \in \mathrm{Y}\setminus\{0\}} \frac{\langle F, \varphi \rangle}{\|\varphi\|_{\mathrm{Y}}} \leq \sup_{\varphi \in \mathrm{Y}\setminus\{0\}} \frac{\|F\|_{\mathrm{L}^2(Q_T)} \|\varphi\|_{\mathrm{L}^2(Q_T)}}{\|\varphi\|_{\mathrm{Y}}} \leq C \|F\|_{\mathrm{L}^2(Q_T)}.$$

Therefore, we imply $\|U\|_{\mathrm{X}} \leq C\left(\|F\|_{\mathrm{L}^2(Q_T)} + \|u_0\|_{\mathrm{X}}\right)$.

Regarding an additional regularity of the solution $U \in \mathrm{X}$ of the problem (2.1), let us introduce the following assumption:

Assumption 2.1. *For $F \in \mathrm{Y}'$ and $U_0 \in \mathrm{H}_0^1(\Omega)$, assume that the solution $U \in \mathrm{X}$ of the problem (2.1) satisfies $U \in \mathrm{H}^1(Q_T) \cap \mathrm{H}^s(Q_1 \cup Q_2)$ with a given $s > \frac{d+3}{2}$ and there exists a constant $C > 0$ independent of F and U_0 such that*

$$\|U\|_{\mathrm{H}^1(Q_T)} \leq C\left(\|F\|_{\mathrm{Y}'} + \|u_0\|_{\mathrm{X}}\right). \tag{2.4}$$

In this work, we assume that the assumption 2.1 is satisfied.

2.2 Finite element discretization

Assume that Ω is a polyhedron in $\mathbb{R}^d$. The domain Q_T is divided into shape-regular simplicial finite elements by an interface-fitted triangulation $\mathcal{T}_h$, where the mesh size $h \in (0, h_*)$ for a given $h_* > 0$ [27]. Hence, every triangle or tetrahedron $K \in \mathcal{T}_h$ falls into one of the following scenarios:

1. $K \subset \overline{Q_1}$;
2. $K \subset \overline{Q_2}$;

3. $K \cap Q_1 \neq \varnothing$ and $K \cap Q_2 \neq \varnothing$, then $d+1$ vertices of K lie on Γ^*.

Moreover, suppose that $\mathcal{T}_h$ is quasi-uniform. We denote by Γ_h^* the linear approximation of Γ^*, consisting of all edges (or faces) with the nodes lying on Γ^*. The discrete interface Γ_h^* seperates Q_T into two subdomains $Q_{1,h}$ and $Q_{2,h}$, which are approximated counterparts of Q_1 and Q_2, respectively.

2.2.1. Interface-fitted space-time method

We discretize the problem (2.2) by using the interface-fitted space-time finite element method, based on the work [5]. For simplicity, we assume that $U_0 = 0$, which means $u_0 = 0$ in (2.2). Let Y_h be the finite element space of continuous element-wise linear functions on $\mathcal{T}_h$ with zero values on $\partial\Omega \times (0,T)$. We define $\mathrm{X}_{h,0} = \{\varphi_h \in \mathrm{Y}_h \mid \varphi_h = 0 \text{ on } \Omega \times \{0\}\}$. Obviously, $\mathrm{Y}_h \subset \mathrm{Y}$ and $\mathrm{X}_{h,0} \subset \mathrm{X}_0$. Consider the discrete problem: Find $\overline{u}_h \in \mathrm{X}_{h,0}$ that satisfies

$$a_h\left(\overline{u}_h, \varphi_h\right) = \langle F, \varphi_h \rangle \qquad \forall \varphi_h \in \mathrm{Y}_h, \tag{2.5}$$

with the bilinear form $a_h : \mathrm{X}_0 \times \mathrm{Y} \to \mathbb{R}$ given by

$$a_h\left(u, \varphi\right) = \langle \partial_t u, \varphi \rangle + \int_0^T \int_\Omega \left(\mathbf{v} \cdot \nabla u\right) \varphi + \kappa_h \nabla u \cdot \nabla \varphi \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t,$$

where κ_h approximates κ by means of

$$\kappa_h := \begin{cases} \kappa_1 > 0 & \text{in} \quad Q_{1,h}, \\ \kappa_2 > 0 & \text{in} \quad Q_{2,h}. \end{cases}$$

Regarding the numerical analysis, we introduce the seminorm

$$\|\!|v|\!\|^2 := \sum_{i=1}^{2} \int_{Q_{i,h}} \kappa_i \left|\nabla v\right|^2 \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \qquad \forall v \in \mathrm{H}^{1,0}\left(Q_{1,h} \cup Q_{2,h}\right).$$

Note that in case $v \in \mathrm{Y} \subset \mathrm{H}^{1,0}\left(Q_{1,h} \cup Q_{2,h}\right)$, the right-hand side becomes

$$\sum_{i=1}^{2} \int_{Q_{i,h}} \kappa_i \left|\nabla v\right|^2 \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t = \int_0^T \int_\Omega \kappa_h \left|\nabla v\right|^2 \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t,$$

which defines an equivalent norm in Y, also denoted by $|||v|||$. Since this norm involves the coefficient κ_h, it is more favorable for studying discrete problems than the norm $\|v\|_{\mathrm{Y}}$. On the space $\mathrm{H}^1\left(Q_{1,h} \cup Q_{2,h}\right)$, we introduce the norm

$$|||v|||_*^2 := |||v|||^2 + |||z_h(v)|||^2 \qquad \forall v \in \mathrm{H}^1\left(Q_{1,h} \cup Q_{2,h}\right),$$

where $z_h(v) \in \mathrm{Y}_h$ is a unique solution of the problem

$$\int_0^T \int_\Omega \kappa_h \nabla z_h(v) \cdot \nabla \psi_h \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t = \sum_{i=1}^{2} \int_{Q_{i,h}} (\partial_t v) \psi_h \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \qquad \forall \psi_h \in \mathrm{Y}_h .$$

Lemma 2.5. *There exists a constant $C > 0$ such that*

$$\sup_{\varphi_h \in \mathrm{Y}_h \setminus \{0\}} \frac{a_h\left(\overline{u}_h, \varphi_h\right)}{|||\varphi_h|||} \geq C |||\overline{u}_h|||_* \qquad \forall \overline{u}_h \in \mathrm{X}_{h,0} . \tag{2.6}$$

Using the discrete Banach-Nečas-Babuška theorem [26], we conclude that the problem (2.5) is uniquely solvable.

2.2.2. Auxiliary results

In this section, we provide some auxiliary findings. We first present a result regarding the mismatch between each space-time subdomain Q_i and its approximated counterpart $Q_{i,h}$, for $i = 1, 2$. Define by

$$S_h^1 := Q_{1,h} \setminus \overline{Q_1} = Q_2 \setminus \overline{Q_{2,h}}, \qquad S_h^2 := Q_{2,h} \setminus \overline{Q_2} = Q_1 \setminus \overline{Q_{1,h}},$$

and $S_h = S_h^1 \cup S_h^2$ (see Figure 2.2). We denote $\mathcal{T}_h^* = \{K \in \mathcal{T}_h \mid K \cap \Gamma^* \neq \varnothing\}$ the set of all interface elements.

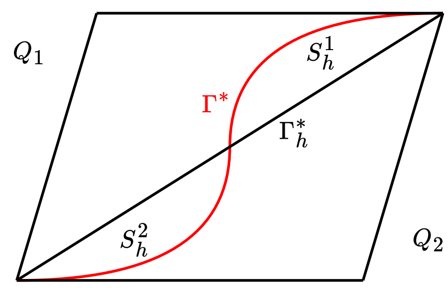

Figure 2.2: The mismatch region $S_h = S_h^1 \cup S_h^2$ lies between the space-time interface Γ^* and the discrete one Γ_h^*, consider the case $d = 1$ [5].

Lemma 2.6. *Assume that Γ^* is a C^2-continuous hypersurface in $\mathbb{R}^{d+1}$ $(d = 1, 2)$ and $\mathcal{T}_h$ is a quasi-uniform mesh. Then, for each $K \in \mathcal{T}_h^*$, we have*

$$|K \cap S_h| \leq Ch^{d+2}. \tag{2.7}$$

It holds for the cardinality of the set $\mathcal{T}_h^*$ that

$$\sum_{K \in \mathcal{T}_h^*} 1 \leq Ch^{-d}. \tag{2.8}$$

Proof. When $d = 1$, the proof of the first inequality can be found in [27]. We obtain the second one by combining this inequality with [28]. All the arguments can be extended to the case $d = 2$ without essential changes. $\square$

We continue by studying the approximability of the Lagrangian interpolant. Let $u \in \mathrm{H}^1(Q_T) \cap \mathrm{H}^2(Q_1 \cup Q_2)$. For $u \in \mathrm{H}^2(Q_1 \cup Q_2)$, the Sobolev embedding [20] follows that $u \in \mathrm{C}\left(\overline{Q_1}\right) \cap \mathrm{C}\left(\overline{Q_2}\right)$. In addition, if $u \in \mathrm{H}^1(Q_T)$, then $\gamma_1 u - \gamma_2 u = 0$, which implies $u \in \mathrm{C}\left(\overline{Q_T}\right)$. Let $I_h : \mathrm{C}\left(\overline{Q_T}\right) \to \mathrm{X}_{h,0}$ be the nodal interpolation operator. When $d = 1$, the interpolation estimate was addressed by Chen and Zou [27]. However, the order was nearly optimal up to the factor $|\log h|$, where h is the mesh size. This paper imposes an additional condition on u and follows their approach to recover an optimal order estimate. We end up with the following result:

Lemma 2.7. *For* $u \in \mathrm{H}^1(Q_T) \cap \mathrm{H}^s(Q_1 \cup Q_2)$ *with* $s > \frac{d+3}{2}$*, the interpolation operator* I_h *satisfies the following inequality*

$$\|u - I_h u\|_{\mathrm{L}^2(Q_T)} + h \|\mathrm{D}(u - I_h u)\|_{\mathbf{L}^2(Q_T)} \leq Ch^2 \|u\|_{\mathrm{H}^s(Q_1 \cup Q_2)}, \tag{2.9}$$

where $\mathrm{D} := (\nabla, \partial_t)^\top$ *denotes the space-time gradient operator.*

Proof. Let us focus on $\|u - I_h u\|_{\mathrm{L}^2(Q_T)}$, since $\|\mathrm{D}(u - I_h u)\|_{\mathbf{L}^2(Q_T)}$ can be esimated similarly. The idea is first to estimate the interpolation error on each element $K \in \mathcal{T}_h$, then sum over all elements to obtain the desired result. Under the assumption, we have $u \in \mathrm{H}^2(K)$ on any $K \notin \mathcal{T}_h^*$. The classical interpolation theory [20] yields

$$\|u - I_h u\|_{\mathrm{L}^2(K)} \leq Ch^2 \|u\|_{\mathrm{H}^2(K)}. \tag{2.10}$$

Next, consider an arbitrary element $K \in \mathcal{T}_h^*$. Without loss of generality, suppose that $K \cap S_h \subset Q_1$ and $K \setminus S_h \subset Q_2$. For $u \in \mathrm{H}^s(Q_1 \cup Q_2)$ with $s > \frac{d+3}{2}$, note that $\mathrm{E}_i\, u \in \mathrm{H}^s(Q_T) \subset \mathrm{W}^{1,\infty}(Q_T)$ $(i = 1, 2)$ [20], with E_1 and E_2 the extension operators in (1.4). The inequality (2.7) and classical interpolation theories give us

$$\begin{aligned}
&\|u - I_h u\|^2_{\mathrm{L}^2(K)} \\
&= \|\mathrm{E}_1\, u - I_h(\mathrm{E}_1\, u)\|^2_{\mathrm{L}^2(K \cap S_h)} + \|\mathrm{E}_2\, u - I_h(\mathrm{E}_2\, u)\|^2_{\mathrm{L}^2(K \setminus S_h)} \\
&\leq |K \cap S_h|\, \|\mathrm{E}_1\, u - I_h(\mathrm{E}_1\, u)\|^2_{\mathrm{L}^\infty(K \cap S_h)} + \|\mathrm{E}_2\, u - I_h(\mathrm{E}_2\, u)\|^2_{\mathrm{L}^2(K \setminus S_h)} \\
&\leq Ch^{d+2} \|\mathrm{E}_1\, u - I_h(\mathrm{E}_1\, u)\|^2_{\mathrm{L}^\infty(K)} + \|\mathrm{E}_2\, u - I_h(\mathrm{E}_2\, u)\|^2_{\mathrm{L}^2(K)} \\
&\leq Ch^{d+4} \|\mathrm{D}(\mathrm{E}_1\, u)\|^2_{\mathbf{L}^\infty(K)} + Ch^4 \left\|\mathrm{D}^2(\mathrm{E}_2\, u)\right\|^2_{\mathbf{L}^2(K)}.
\end{aligned}$$

We sum over all $K \in \mathcal{T}_h^*$ and use the inequality (2.8), the Sobolev embedding $\mathbf{H}^{s-1}(Q_T) \hookrightarrow \mathbf{L}^\infty(Q_T)$ for $s > \frac{d+3}{2}$ [20], and the extension operators (1.4) again to get

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}_h^*} \|u - I_h u\|^2_{\mathrm{L}^2(K)} \le \\
&\le Ch^{d+4} \max\left\{\|\mathrm{D}(\mathrm{E}_1 u)\|^2_{\mathbf{L}^\infty(Q_T)}, \|\mathrm{D}(\mathrm{E}_2 u)\|^2_{\mathbf{L}^\infty(Q_T)}\right\} \left(\sum_{K\in\mathcal{T}_h^*} 1\right) \\
&\qquad\qquad + Ch^4 \left(\left\|\mathrm{D}^2(\mathrm{E}_1 u)\right\|^2_{\mathbf{L}^2(Q_T)} + \left\|\mathrm{D}^2(\mathrm{E}_2 u)\right\|^2_{\mathbf{L}^2(Q_T)}\right) \\
&\le Ch^4 \max\left\{\|\mathrm{D}(\mathrm{E}_1 u)\|^2_{\mathbf{H}^{s-1}(Q_T)}, \|\mathrm{D}(\mathrm{E}_2 u)\|^2_{\mathbf{H}^{s-1}(Q_T)}\right\} \\
&\qquad\qquad + Ch^4 \|u\|^2_{\mathrm{H}^2(Q_1\cup Q_2)} \\
&\le Ch^4 \|u\|^2_{\mathrm{H}^s(Q_1\cup Q_2)} .
\end{aligned}
\tag{2.11}
$$

Together with (2.10), we imply $\|u - I_h u\|_{\mathrm{L}^2(Q_T)} \le Ch^2 \|u\|_{\mathrm{H}^s(Q_1\cup Q_2)}$. Using the same arguments, one obtains $\|\mathrm{D}(u - I_h u)\|_{\mathbf{L}^2(Q_T)} \le Ch \|u\|_{\mathrm{H}^s(Q_1\cup Q_2)}$. The proof is complete. □

Please note that the mismatch between $\mathcal{T}_h$ and Q_T at the interface leads to the non-conformal property of $a_h(\cdot,\cdot)$. In particular, we have the following lemma:

Lemma 2.8. *Let $\overline{u} \in \mathrm{X}_0$ and $\overline{u}_h \in \mathrm{X}_{h,0}$ be the solutions of the problems* (2.2) *and* (2.5)*, respectively. There holds the following equality*

$$
a_h(\overline{u} - \overline{u}_h, \varphi_h) = \int_{S_h} (\kappa_h - \kappa) \nabla\overline{u} \cdot \nabla\varphi_h \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \qquad \forall \varphi_h \in \mathrm{Y}_h . \tag{2.12}
$$

Proof. For $\overline{u} \in \mathrm{X}_0, \overline{u}_h \in \mathrm{X}_{h,0}$ and $\varphi_h \in \mathrm{Y}_h$, we invoke the equations (2.2) and (2.5) to have

$$a_h(\overline{u},\varphi_h) = \langle\partial_t\overline{u},\varphi_h\rangle + \int_0^T\int_\Omega (\mathbf{v}\cdot\nabla\overline{u})\,\varphi_h + \kappa_h\nabla\overline{u}\cdot\nabla\varphi_h\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$= a(\overline{u},\varphi_h) + \int_0^T\int_\Omega (\kappa_h-\kappa)\,\nabla\overline{u}\cdot\nabla\varphi_h\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$= \langle F,\varphi_h\rangle + \int_{S_h} (\kappa_h-\kappa)\,\nabla\overline{u}\cdot\nabla\varphi_h\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$= a_h(\overline{u}_h,\varphi_h) + \int_{S_h} (\kappa_h-\kappa)\,\nabla\overline{u}\cdot\nabla\varphi_h\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,$$

observed that $\kappa_h-\kappa$ vanishes everywhere outside of S_h. □

2.2.3. A priori error estimates

Now, we estimate the error $\overline{u}-\overline{u}_h$ in various norms, where $\overline{u}\in\mathrm{X}_0$ and $\overline{u}_h\in\mathrm{X}_{h,0}$ be the solutions of the problems (2.2) and (2.5), respectively. The following result looks at the error $\overline{u}-\overline{u}_h$ in the norm $|\!|\!|\cdot|\!|\!|_*$ (please see [5] for more details).

Lemma 2.9. *Let $\overline{u}\in\mathrm{X}_0$ and $\overline{u}_h\in\mathrm{X}_{h,0}$ be the solutions of the problems (2.2) and (2.5), respectively. Assume that Assumption 2.1 is satisfied. Then, we have the following estimate*

$$|\!|\!|\overline{u}-\overline{u}_h|\!|\!|_* \le Ch\,\|\overline{u}\|_{\mathrm{H}^s(Q_1\cup Q_2)}\,. \tag{2.13}$$

We continue by estimating the state error in the $\mathrm{L}^2(\Omega)$-norm at $t=T$. Following the duality argument, let us define the space

$$\mathrm{V}_T := \left\{\psi\in\mathrm{X}\mid \psi(\cdot,T) = \gamma_T\,\|(\overline{u}-\overline{u}_h)(\cdot,T)\|^{-1}_{\mathrm{L}^2(\Omega)}\,(\overline{u}-\overline{u}_h)(\cdot,T)\ \text{in}\ \Omega\right\},$$

where $\gamma_T > 0$ is a sufficient large number, $\overline{u} \in \mathrm{X}_0$ and $\overline{u}_h \in \mathrm{X}_{h,0}$ are the solutions of the problems (2.2) and (2.5), respectively. Assume that there exists $y \in \mathrm{V}_T$ that solves the problem

$$-\langle \partial_t y, \phi \rangle + \int_0^T \int_\Omega -(\mathbf{v} \cdot \nabla y)\, \phi + \kappa \nabla y \cdot \nabla \phi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = 0 \qquad \forall \phi \in \mathrm{Y}\,. \tag{2.14}$$

Under the assumption 2.1, one has $y \in \mathrm{H}^1(Q_T) \cap \mathrm{H}^s(Q_1 \cup Q_2)$ with $s > \frac{d+3}{2}$. We further assume that it satisfies

$$\|y\|_{\mathrm{H}^s(Q_1 \cup Q_2)} \le C, \tag{2.15}$$

where the constant $C > 0$ is independent of $\overline{u}$ and $\overline{u}_h$.

Theorem 2.10. *Let $\overline{u} \in \mathrm{X}_0$ and $\overline{u}_h \in \mathrm{X}_{h,0}$ be the solutions of the problems (2.2) and (2.5), respectively. Assume that Assumption 2.1 is satisfied and the problem (2.14) admits a solution $y \in \mathrm{V}_T$ that satisfies (2.15). Then, we have*

$$\|(\overline{u} - \overline{u}_h)(\cdot, T)\|_{\mathrm{L}^2(\Omega)} \le Ch^2 \|\overline{u}\|_{\mathrm{H}^s(Q_1 \cup Q_2)}\,. \tag{2.16}$$

Proof. We choose $\phi = \overline{u} - \overline{u}_h \in \mathrm{X}_0$ in (2.14), then employ the integration by parts formula to get

$$\begin{aligned}
&\gamma_T \|(\overline{u} - \overline{u}_h)(\cdot, T)\|_{\mathrm{L}^2(\Omega)} \\
&= \int_\Omega (\overline{u} - \overline{u}_h)(\boldsymbol{x}, 0)\, y(\boldsymbol{x}, 0)\,\mathrm{d}\boldsymbol{x} + \langle \partial_t (\overline{u} - \overline{u}_h), y \rangle \\
&\qquad + \int_0^T \int_\Omega -(\mathbf{v} \cdot \nabla y)(\overline{u} - \overline{u}_h) + \kappa \nabla y \cdot \nabla (\overline{u} - \overline{u}_h)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&= \langle \partial_t (\overline{u} - \overline{u}_h), y \rangle \\
&\qquad + \int_0^T \int_\Omega -(\mathbf{v} \cdot \nabla y)(\overline{u} - \overline{u}_h) + \kappa \nabla (\overline{u} - \overline{u}_h) \cdot \nabla y \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,
\end{aligned}$$

using the fact that $(\overline{u}-\overline{u}_h)(\cdot,0) = 0$ in Ω in the last step. To handle the advection part on the right-hand side, we invoke the property $\nabla\cdot\mathbf{v}(\boldsymbol{x},t) = 0$ for all $(\boldsymbol{x},t) \in \Omega\times[0,T]$, the divergence theorem, and the homogeneous Dirichlet boundary condition. We have

$$\begin{aligned}\int_0^T\int_\Omega (\mathbf{v}\cdot\nabla y)(\overline{u}-\overline{u}_h)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t &= \int_0^T\int_\Omega \nabla\cdot\left(y\left(\overline{u}-\overline{u}_h\right)\mathbf{v}\right)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\ &\quad-\int_0^T\int_\Omega \left(\mathbf{v}\cdot\nabla\left(\overline{u}-\overline{u}_h\right)\right)y + y\left(\overline{u}-\overline{u}_h\right)(\nabla\cdot\mathbf{v})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\ &= \int_0^T\int_{\partial\Omega} y\left(\overline{u}-\overline{u}_h\right)\mathbf{v}\cdot\mathbf{n}_\Omega\,\mathrm{d}s\,\mathrm{d}t - \int_0^T\int_\Omega \left(\mathbf{v}\cdot\nabla\left(\overline{u}-\overline{u}_h\right)\right)y\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\ &= -\int_0^T\int_\Omega \left(\mathbf{v}\cdot\nabla\left(\overline{u}-\overline{u}_h\right)\right)y\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t, \end{aligned} \tag{2.17}$$

with $\mathbf{n}_\Omega$ the outward normal to $\partial\Omega$. Hence, we get

$$\begin{aligned}\gamma_T\left\|\left(\overline{u}-\overline{u}_h\right)(\cdot,T)\right\|_{\mathrm{L}^2(\Omega)} &= \left\langle\partial_t\left(\overline{u}-\overline{u}_h\right),y\right\rangle\\ &\quad+\int_0^T\int_\Omega \left(\mathbf{v}\cdot\nabla\left(\overline{u}-\overline{u}_h\right)\right)y + \kappa\nabla\left(\overline{u}-\overline{u}_h\right)\cdot\nabla y\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\ &= a_h\left(\overline{u}-\overline{u}_h,y\right) + \int_{S_h}(\kappa-\kappa_h)\nabla\left(\overline{u}-\overline{u}_h\right)\cdot\nabla y\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,\end{aligned}$$

noticing that $\kappa-\kappa_h = 0$ outside of S_h. Since $y\in\mathrm{H}^1(Q_T)\cap\mathrm{H}^s(Q_1\cup Q_2)$ for $s>\frac{d+3}{2}$, we are able to invoke the interpolation $I_h y$. We denote $e := y - I_h y$ for convenience, and choose $\varphi_h = I_h y\in\mathrm{X}_{h,0}$ in (2.12) to obtain

$$\begin{aligned}&\gamma_T\left\|\left(\overline{u}-\overline{u}_h\right)(\cdot,T)\right\|_{\mathrm{L}^2(\Omega)} =\\ &= a_h\left(\overline{u}-\overline{u}_h,y\right) + \int_{S_h}(\kappa-\kappa_h)\nabla\left(\overline{u}-\overline{u}_h\right)\cdot\nabla y\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\end{aligned}$$

$$= a_h\left(\overline{u}-\overline{u}_h, e\right) + a_h\left(\overline{u}-\overline{u}_h, I_h y\right) + \int\limits_{S_h} \left(\kappa-\kappa_h\right) \nabla\left(\overline{u}-\overline{u}_h\right) \cdot \nabla y \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

$$= a_h\left(\overline{u}-\overline{u}_h, e\right)$$
$$+ \int\limits_{S_h} \left(\kappa_h-\kappa\right) \nabla\overline{u} \cdot \nabla\left(I_h y\right) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int\limits_{S_h} \left(\kappa-\kappa_h\right) \nabla\left(\overline{u}-\overline{u}_h\right) \cdot \nabla y \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

$$= a_h\left(\overline{u}-\overline{u}_h, e\right)$$
$$+ \int\limits_{S_h} \left(\kappa-\kappa_h\right) \left(\nabla\overline{u} \cdot \nabla e - \nabla\overline{u} \cdot \nabla y + \nabla\left(\overline{u}-\overline{u}_h\right) \cdot \nabla y\right) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t. \qquad (2.18)$$

We first estimate the discrete bilinear term. In doing so, we integrate by parts again, note that $\overline{u}-\overline{u}_h \in \mathrm{X}_0$ and $e \in \mathrm{H}^1\left(Q_T\right)$, and apply the inequalities (2.9), (1.1), (2.13), and (2.15). One has

$$a_h\left(\overline{u}-\overline{u}_h, e\right) =$$
$$= \int\limits_{\Omega} \left(\overline{u}-\overline{u}_h\right)\left(\boldsymbol{x}, T\right) e\left(\boldsymbol{x}, T\right) \,\mathrm{d}\boldsymbol{x} + \int\limits_0^T \int\limits_{\Omega} -\left(\overline{u}-\overline{u}_h\right)\left(\partial_t e\right) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$+ \int\limits_0^T \int\limits_{\Omega} \left(\mathbf{v} \cdot \nabla\left(\overline{u}-\overline{u}_h\right)\right) e + \kappa_h \nabla\left(\overline{u}-\overline{u}_h\right) \cdot \nabla e \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$\leq \left\|\left(\overline{u}-\overline{u}_h\right)(\cdot, T)\right\|_{\mathrm{L}^2(\Omega)} \left\|e(\cdot, T)\right\|_{\mathrm{L}^2(\Omega)} + \left\|\overline{u}-\overline{u}_h\right\|_{\mathrm{L}^2(Q_T)} \left\|\mathrm{D}\, e\right\|_{\mathbf{L}^2(Q_T)}$$
$$+ C \left\|\nabla\left(\overline{u}-\overline{u}_h\right)\right\|_{\mathbf{L}^2(Q_T)} \left(\|e\|_{\mathrm{L}^2(Q_T)} + \left\|\mathrm{D}\, e\right\|_{\mathbf{L}^2(Q_T)}\right)$$
$$\leq \left\|\left(\overline{u}-\overline{u}_h\right)(\cdot, T)\right\|_{\mathrm{L}^2(\Omega)} \left\|e(\cdot, T)\right\|_{\mathrm{L}^2(\Omega)}$$
$$+ C \left\|\nabla\left(\overline{u}-\overline{u}_h\right)\right\|_{\mathbf{L}^2(Q_T)} \left(\|e\|_{\mathrm{L}^2(Q_T)} + \left\|\mathrm{D}\, e\right\|_{\mathbf{L}^2(Q_T)}\right)$$
$$\leq \left\|\left(\overline{u}-\overline{u}_h\right)(\cdot, T)\right\|_{\mathrm{L}^2(\Omega)} \left\|e(\cdot, T)\right\|_{\mathrm{L}^2(\Omega)} + Ch^2 \left\|\overline{u}\right\|_{\mathrm{H}^s(Q_1 \cup Q_2)}.$$

By using the trace inequality (1.3), the inequality (1.1), the estimate (2.9), and the inequality (2.15), we observe that

$$\|e(\cdot,T)\|_{\mathrm{L}^2(\Omega)} \leq C\|e\|_{\mathrm{X}} \leq C\|e\|_{\mathrm{H}^1(Q_T)} \leq Ch\|y\|_{\mathrm{H}^s(Q_1\cup Q_2)} \leq Ch, \quad (2.19)$$

which yields

$$a_h(\overline{u}-\overline{u}_h, e) \leq Ch\|(\overline{u}-\overline{u}_h)(\cdot,T)\|_{\mathrm{L}^2(\Omega)} + Ch^2\|\overline{u}\|_{\mathrm{H}^s(Q_1\cup Q_2)}. \quad (2.20)$$

Next, consider the second integral on the right-hand side of (2.18), denoted by I for short. By using the Cauchy-Schwarz inequality, together with (2.9) and (2.13), we bound I by

$$\begin{aligned} I &:= \int_{S_h} (\kappa-\kappa_h)\left(\nabla\overline{u}\cdot\nabla e - \nabla\overline{u}\cdot\nabla y + \nabla(\overline{u}-\overline{u}_h)\cdot\nabla y\right) \mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ &\leq C\left(\|\nabla\overline{u}\|_{\mathbf{L}^2(S_h)}\|\mathrm{D}\,e\|_{\mathbf{L}^2(Q_T)} + \|\nabla\overline{u}\|_{\mathbf{L}^2(S_h)}\|\nabla y\|_{\mathbf{L}^2(S_h)}\right) \\ &\qquad + C\|\nabla(\overline{u}-\overline{u}_h)\|_{\mathbf{L}^2(Q_T)}\|\nabla y\|_{\mathbf{L}^2(S_h)} \\ &\leq C\left(\|\nabla\overline{u}\|_{\mathbf{L}^2(S_h)} h\|y\|_{\mathrm{H}^s(Q_1\cup Q_2)} + \|\nabla\overline{u}\|_{\mathbf{L}^2(S_h)}\|\nabla y\|_{\mathbf{L}^2(S_h)}\right) \\ &\qquad + Ch\|\overline{u}\|_{\mathrm{H}^s(Q_1\cup Q_2)}\|\nabla y\|_{\mathbf{L}^2(S_h)}. \end{aligned}$$

We follow the arguments of (2.11) to estimate $\|\nabla\overline{u}\|_{\mathbf{L}^2(S_h)}$ and $\|\nabla y\|_{\mathbf{L}^2(S_h)}$. Take $\|\nabla\overline{u}\|_{\mathbf{L}^2(S_h)}$ for instance. Under Assumption 2.1, we have $\overline{u}\in\mathrm{H}^1(Q_T)\cap\mathrm{H}^s(Q_1\cup Q_2)$ with $s>\frac{d+3}{2}$, and hence

$$\begin{aligned} \|\nabla\overline{u}\|^2_{\mathbf{L}^2(S_h)} = \sum_{K\in\mathcal{T}_h^*}\|\nabla\overline{u}\|^2_{\mathbf{L}^2(K\cap S_h)} &\leq \sum_{K\in\mathcal{T}_h^*}|K\cap S_h|\,\|\nabla\overline{u}\|^2_{\mathbf{L}^\infty(K\cap S_h)} \\ &\leq Ch^2\|\overline{u}\|^2_{\mathrm{H}^s(Q_1\cup Q_2)}, \end{aligned}$$

which means $\|\nabla\overline{u}\|_{\mathbf{L}^2(S_h)} \leq Ch\|\overline{u}\|_{\mathrm{H}^s(Q_1\cup Q_2)}$.

Similarly, for $y\in\mathrm{H}^1(Q_T)\cap\mathrm{H}^s(Q_1\cup Q_2)$ with $s>\frac{d+3}{2}$, we can prove that $\|\nabla y\|_{\mathbf{L}^2(S_h)} \leq Ch\|y\|_{\mathrm{H}^s(Q_1\cup Q_2)} \leq Ch$ by using the inequality (2.15).

Therefore, we obtain

$$I \le Ch^2 \left\| \overline{u} \right\|_{\mathrm{H}^s(Q_1 \cup Q_2)}. \tag{2.21}$$

We substitute (2.20) and (2.21) into (2.18) to arrive at

$$(\gamma_T - Ch) \left\| (\overline{u} - \overline{u}_h)(\cdot, T) \right\|_{\mathrm{L}^2(\Omega)} \le Ch^2 \left\| \overline{u} \right\|_{\mathrm{H}^s(Q_1 \cup Q_2)}.$$

Since $h \in (0, h_*)$ for a given $h_* > 0$, the proof is finished by choosing γ_T such that $\gamma_T \ge Ch_* + 1$. □

Using the inequality (2.16), we are now able to estimate $\left\| \overline{u} - \overline{u}_h \right\|_{\mathrm{L}^2(Q_T)}$, where $\overline{u} \in \mathrm{X}_0$ and $\overline{u}_h \in \mathrm{X}_{h,0}$ be the solutions of the problems (2.2) and (2.5), respectively. In the following lemma, assume that there exists a solution $z \in \mathrm{X}$ to the problem

$$\begin{aligned} &- \langle \partial_t z, \phi \rangle + \int_0^T \int_\Omega - (\mathbf{v} \cdot \nabla z)\, \phi + \kappa \nabla z \cdot \nabla \phi \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t = \\ &= \left\| \overline{u} - \overline{u}_h \right\|_{\mathrm{L}^2(Q_T)}^{-1} \int_0^T \int_\Omega (\overline{u} - \overline{u}_h)\, \phi \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \qquad \forall \phi \in \mathrm{Y}, \end{aligned} \tag{2.22}$$

with $z(\cdot, T) \in \mathrm{H}_0^1(\Omega)$. The assumption 2.1 yields $z \in \mathrm{H}^1(Q_T) \cap \mathrm{H}^s(Q_1 \cup Q_2)$ with $s > \frac{d+3}{2}$. Moreover, assume that

$$\left\| z \right\|_{\mathrm{H}^s(Q_1 \cup Q_2)} \le C, \tag{2.23}$$

for a constant $C > 0$ independent of $\overline{u}$ and $\overline{u}_h$.

Theorem 2.11. *Let $\overline{u} \in \mathrm{X}_0$ and $\overline{u}_h \in \mathrm{X}_{h,0}$ be the solutions of the problems (2.2) and (2.5), respectively. Assume that the assumption of lemma 2.10 is satisfied and there exists a solution $z \in \mathrm{X}$ of the problem (2.22) that satisfies (2.23). Then, there holds the following estimate*

$$\left\| \overline{u} - \overline{u}_h \right\|_{\mathrm{L}^2(Q_T)} \le Ch^2 \left\| \overline{u} \right\|_{\mathrm{H}^s(Q_1 \cup Q_2)}. \tag{2.24}$$

Proof. We choose $\phi = \overline{u} - \overline{u}_h \in \mathrm{X}_0$ in (2.22), then apply the integration by parts formula, the arguments of (2.17), and the equality (2.12) to arrive at

$$\begin{aligned}
\|\overline{u} - \overline{u}_h\|_{\mathrm{L}^2(Q_T)} &= -\left\langle \partial_t z, \overline{u} - \overline{u}_h \right\rangle \\
&\quad + \int_0^T \int_\Omega -\left(\mathbf{v} \cdot \nabla z\right)\left(\overline{u} - \overline{u}_h\right) + \kappa \nabla z \cdot \nabla\left(\overline{u} - \overline{u}_h\right) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&= a_h\left(\overline{u} - \overline{u}_h, z\right) + \int_{S_h} \left(\kappa - \kappa_h\right) \nabla\left(\overline{u} - \overline{u}_h\right) \cdot \nabla z \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&\quad - \int_\Omega \left(\overline{u} - \overline{u}_h\right)\left(\boldsymbol{x}, T\right) z\left(\boldsymbol{x}, T\right) \,\mathrm{d}\boldsymbol{x} \\
&= a_h\left(\overline{u} - \overline{u}_h, e'\right) + a_h\left(\overline{u} - \overline{u}_h, I_h z\right) \\
&\quad + \int_{S_h} \left(\kappa - \kappa_h\right) \nabla\left(\overline{u} - \overline{u}_h\right) \cdot \nabla z \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_\Omega \left(\overline{u} - \overline{u}_h\right)\left(\boldsymbol{x}, T\right) z\left(\boldsymbol{x}, T\right) \,\mathrm{d}\boldsymbol{x} \\
&= a_h\left(\overline{u} - \overline{u}_h, e'\right) + \int_{S_h} \left(\kappa_h - \kappa\right)\left(\nabla \overline{u} \cdot \nabla\left(I_h z\right) - \nabla\left(\overline{u} - \overline{u}_h\right) \cdot \nabla z\right) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&\quad - \int_\Omega \left(\overline{u} - \overline{u}_h\right)\left(\boldsymbol{x}, T\right) z\left(\boldsymbol{x}, T\right) \,\mathrm{d}\boldsymbol{x},
\end{aligned} \tag{2.25}$$

where we use the initial condition $\left(\overline{u} - \overline{u}_h\right)(\cdot, 0) = 0$ in Ω in the second step and denote $e' := z - I_h z$. Here, we can employ the interpolation $I_h z$ since $z \in \mathrm{H}^1\left(Q_T\right) \cap \mathrm{H}^s\left(Q_1 \cup Q_2\right)$ with $s > \frac{d+3}{2}$. For the term $a_h\left(\overline{u} - \overline{u}_h, e'\right)$, we integrate by parts again with $\overline{u} - \overline{u}_h \in \mathrm{X}_0$ and $e' \in \mathrm{H}^1\left(Q_T\right)$, and employ the inequalities (2.9), (1.1), (2.13), and (2.23) to obtain

$$\begin{aligned}
& a_h\left(\overline{u} - \overline{u}_h, e'\right) = \\
&= \int_\Omega \left(\overline{u} - \overline{u}_h\right)\left(\boldsymbol{x}, T\right) e'\left(\boldsymbol{x}, T\right) \,\mathrm{d}\boldsymbol{x} + \int_0^T \int_\Omega -\left(\overline{u} - \overline{u}_h\right)\left(\partial_t e'\right) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t
\end{aligned}$$

$$+\int_0^T\int_\Omega \left(\mathbf{v}\cdot\nabla\left(\overline{u}-\overline{u}_h\right)\right)e'+\kappa_h\nabla\left(\overline{u}-\overline{u}_h\right)\cdot\nabla e'\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

$$\leq \int_\Omega \left(\overline{u}-\overline{u}_h\right)(\boldsymbol{x},T)\,e'(\boldsymbol{x},T)\,\mathrm{d}\boldsymbol{x}+\left\|\overline{u}-\overline{u}_h\right\|_{\mathrm{L}^2(Q_T)}\left\|\mathrm{D}\,e'\right\|_{\mathbf{L}^2(Q_T)}$$

$$+C\left\|\nabla\left(\overline{u}-\overline{u}_h\right)\right\|_{\mathbf{L}^2(Q_T)}\left(\left\|e'\right\|_{\mathrm{L}^2(Q_T)}+\left\|\mathrm{D}\,e'\right\|_{\mathbf{L}^2(Q_T)}\right)$$

$$\leq \int_\Omega \left(\overline{u}-\overline{u}_h\right)(\boldsymbol{x},T)\,e'(\boldsymbol{x},T)\,\mathrm{d}\boldsymbol{x}$$

$$+C\left\|\nabla\left(\overline{u}-\overline{u}_h\right)\right\|_{\mathbf{L}^2(Q_T)}\left(\left\|e'\right\|_{\mathrm{L}^2(Q_T)}+\left\|\mathrm{D}\,e'\right\|_{\mathbf{L}^2(Q_T)}\right)$$

$$\leq \int_\Omega \left(\overline{u}-\overline{u}_h\right)(\boldsymbol{x},T)\,e'(\boldsymbol{x},T)\,\mathrm{d}\boldsymbol{x}+Ch^2\left\|\overline{u}\right\|_{\mathrm{H}^s(Q_1\cup Q_2)}. \tag{2.26}$$

On the other hand, by following the technique as in (2.21), one can show that

$$J:=\int_{S_h}\left(\kappa_h-\kappa\right)\left(\nabla\overline{u}\cdot\nabla\left(I_hz\right)-\nabla\left(\overline{u}-\overline{u}_h\right)\cdot\nabla z\right)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

$$=\int_{S_h}\left(\kappa_h-\kappa\right)\left(\nabla\overline{u}\cdot\nabla z-\nabla\overline{u}\cdot\nabla e'-\nabla\left(\overline{u}-\overline{u}_h\right)\cdot\nabla z\right)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

$$\leq C\left(\left\|\nabla\overline{u}\right\|_{\mathbf{L}^2(S_h)}\left\|\nabla z\right\|_{\mathbf{L}^2(S_h)}+\left\|\nabla\overline{u}\right\|_{\mathbf{L}^2(S_h)}\left\|\nabla e'\right\|_{\mathbf{L}^2(Q_T)}\right)$$

$$+C\left\|\nabla\left(\overline{u}-\overline{u}_h\right)\right\|_{\mathbf{L}^2(Q_T)}\left\|\nabla z\right\|_{\mathbf{L}^2(S_h)}$$

$$\leq Ch^2\left\|\overline{u}\right\|_{\mathrm{H}^s(Q_1\cup Q_2)}. \tag{2.27}$$

By substituting (2.26) and (2.27) into (2.25), we imply

$$\left\|\overline{u}-\overline{u}_h\right\|_{\mathrm{L}^2(Q_T)}\leq Ch^2\left\|\overline{u}\right\|_{\mathrm{H}^s(Q_1\cup Q_2)}$$

$$+\int_\Omega \left(\overline{u}-\overline{u}_h\right)(\boldsymbol{x},T)\,e'(\boldsymbol{x},T)\,\mathrm{d}\boldsymbol{x}-\int_\Omega \left(\overline{u}-\overline{u}_h\right)(\boldsymbol{x},T)\,z(\boldsymbol{x},T)\,\mathrm{d}\boldsymbol{x}$$

$$=Ch^2\left\|\overline{u}\right\|_{\mathrm{H}^s(Q_1\cup Q_2)}-\int_\Omega \left(\overline{u}-\overline{u}_h\right)(\boldsymbol{x},T)\left(I_hz\right)(\boldsymbol{x},T)\,\mathrm{d}\boldsymbol{x}$$

$$\leq Ch^2 \|\overline{u}\|_{\mathrm{H}^s(Q_1\cup Q_2)} + \|(\overline{u}-\overline{u}_h)(\cdot,T)\|_{\mathrm{L}^2(\Omega)} \|(I_h z)(\cdot,T)\|_{\mathrm{L}^2(\Omega)}$$
$$\leq Ch^2 \|\overline{u}\|_{\mathrm{H}^s(Q_1\cup Q_2)} + Ch^2 \|\overline{u}\|_{\mathrm{H}^s(Q_1\cup Q_2)} \|(I_h z)(\cdot,T)\|_{\mathrm{L}^2(\Omega)},$$

invoking the estimate (2.16) in the final line.

The last step is to estimate $\|(I_h z)(\cdot,T)\|_{\mathrm{L}^2(\Omega)}$. We apply the technique as in (2.19), the inequality (1.1) with $I_h z \in \mathrm{X}_{h,0}$ and the H^1-seminorm stability of the interpolation operator I_h [20]. One gets

$$\begin{aligned}\|(I_h z)(\cdot,T)\|^2_{\mathrm{L}^2(\Omega)} \leq C\|I_h z\|^2_{\mathrm{X}} &\leq C\left(\|I_h z\|^2_{\mathrm{L}^2(Q_T)} + \|\mathrm{D}\,(I_h z)\|^2_{\mathbf{L}^2(Q_T)}\right)\\ &\leq C\left(\|\nabla\,(I_h z)\|^2_{\mathbf{L}^2(Q_T)} + \|\mathrm{D}\,(I_h z)\|^2_{\mathbf{L}^2(Q_T)}\right)\\ &\leq C\,\|\mathrm{D}\,(I_h z)\|^2_{\mathbf{L}^2(Q_T)} \leq C\,\|\mathrm{D}\,z\|^2_{\mathbf{L}^2(Q_T)}.\end{aligned}$$

The proof is finished by employing a priori estimate (2.4) for the problem (2.22) (after changing the time and the velocity field directions). We have

$$\|\mathrm{D}\,z\|_{\mathbf{L}^2(Q_T)} \leq \|z\|_{\mathrm{H}^1(Q_T)} \leq C\left\|\|\overline{u}-\overline{u}_h\|^{-1}_{\mathrm{L}^2(Q_T)}(\overline{u}-\overline{u}_h)\right\|_{\mathrm{L}^2(Q_T)} = C.$$

The proof is complete. □

Chapter 3

An inverse source problem for the advection-diffusion equation with a moving interface

The aim of this chapter is to study an inverse source problem for the advection-diffusion equation with a moving interface: *Assume that in the problem* (2.1)*, the initial value* U_0 *is zero and the source term* F *has the form* $F(\boldsymbol{x},t) = \ell(\boldsymbol{x},t) f(\boldsymbol{x},t) + g(\boldsymbol{x},t)$ *for all* $(\boldsymbol{x},t) \in Q_T$*, where* $\ell \in \mathrm{L}^\infty(Q_T)$ *and* $g \in \mathrm{L}^2(Q_T)$ *are given. Moreover, assume that there exists a constant* $L > 0$ *such that* $\ell \geq L$ *at almost everywhere in* Q_T*. Let* U *be the solution of this problem. Determine* $f \in \mathrm{L}^2(Q_T)$*, given a partial interior data* $U_d := U_{|\omega_T}$ *and a priori information* $f \geq 0$ *at almost everywhere in* Q_T*.*

Since $U_0 = 0$, we see that $u_0 = 0$ and the solution $\overline{u} \in \mathrm{X}_0$ of the problem (2.2) can be splitted as follows $\overline{u} = u + u^*$, where $u \in X_0$ is the solution to the variational problem

$$a(u,\varphi) = (\ell f, \varphi)_{\mathrm{L}^2(Q_T)} \qquad \forall \varphi \in \mathrm{Y}, \tag{3.1}$$

and $u^* \in \mathrm{X}_0$ solves the variational problem

$$a(u^*, \varphi) = (g, \varphi)_{\mathrm{L}^2(Q_T)} \qquad \forall \varphi \in \mathrm{Y}. \tag{3.2}$$

The function u^* is uniquely determined. We aim to identify $f \in F_+$ in (3.1) from the partial interior data $U_d \in \mathrm{L}^2(\omega_T)$ in the subdomain ω_T, where the admissible set is defined by

$$F_+ := \left\{ f \in \mathrm{L}^2(Q_T) \mid f \geq 0 \text{ at almost everywhere in } Q_T \right\}. \tag{3.3}$$

This set is non-empty, close and convex. Our inverse source problem reads as the following operator equation with a priori information

$$Af = z_d, \qquad f \in F_+, \tag{3.4}$$

where $z_d := U_d - u^*_{|\omega_T} \in \mathrm{L}^2(\omega_T)$ is the exact data and A is the bounded linear operator, defined by

$$A : \mathrm{L}^2(Q_T) \to \mathrm{L}^2(\omega_T),$$
$$f \mapsto u(f)_{|\omega_T}.$$

Here, we use the notation $u(f)$ to emphasize the dependence of u in (3.1) on f. For avoiding ambiguity, we interpret $z_d \equiv 0$ in $Q_T \setminus \overline{\omega_T}$ so that it is well-defined in $\mathrm{L}^2(Q_T)$.

In this chapter, $C > 0$ is a constant dependent on the space-time domain Q_T, the position of the space-time interface Γ^*, the norm $\|\mathbf{v}\|_{\mathbf{L}^\infty(Q_T)}$, the function ℓ, the function g, and the coefficient κ, but does not depend on the parameter λ, the noise level ε, the regularized state u^ε_λ and adjoint p^ε_λ, the sources f_+ and f^ε_λ, and the mesh size h. Their different values in different contexts are allowed.

3.1 The ill-posedness of the problem

Let us discuss the concept of solutions to the problem (3.4). Firstly, this problem may not have solutions since z_d can be outside the restricted range $A(F_+)$. On the other hand, we can construct examples in which two solutions of the problem (3.1) coincide in the subdomain Q_1 but behave differently in the subdomain Q_2. Therefore, in general, the operator A is not injective, and the problem (3.4) may have many solutions. As a result, it is essential to recall from [29] the following definition:

Definition 3.1. Let F_+ be the admissible set in (3.3). An element $f_+ \in F_+$ is called the F_+-best approximated solution of the problem (3.4) if among all $f \in F_+$ that solve this problem, it has the minimal $\mathrm{L}^2(Q_T)$-norm

$$\|f_+\|_{\mathrm{L}^2(Q_T)} \leq \|f\|_{\mathrm{L}^2(Q_T)}.$$

Clearly, $f_+ \in F_+$ is uniquely determined. In some contexts, we also call it the continuous unregularized source. Regarding the ill-posedness of the problem (3.4), note that despite A being a linear operator, this problem is nonlinear, owing to the presence of the inequality constraint. Therefore, the ill-posedness criterion for linear problems [30] is not applicable. Instead, we invoke the local ill-posedness concepts in [31] for nonlinear problems.

Definition 3.2. Let $f \in F_+$ be a solution of the problem (3.4). The problem (3.4) is said to be locally well-posed at $f \in F_+$ if there exists a closed ball $\overline{B}_r(f) \subset \mathrm{L}^2(Q_T)$ with the center $f \in F_+$ and radius $r > 0$ such that for every sequence $\{f_n\}_{n\in\mathbb{N}} \subset F_+ \cap \overline{B}_r(f)$, if $\lim_{n\to\infty} \|Af_n - Af\|_{\mathrm{L}^2(\omega_T)} = 0$ then $\lim_{n\to\infty} \|f_n - f\|_{\mathrm{L}^2(Q_T)} = 0$. Otherwise, the problem (3.4) is said to be locally ill-posed at $f \in F_+$.

The compact embedding $\mathrm{H}^{1,0}(Q_T) \hookrightarrow \mathrm{L}^2(Q_T)$ [20] implies A is a compact operator. Together with the arguments in [32], we conclude that the problem (3.4) is locally ill-posed at every point in F_+.

3.2 Tikhonov regularization

The ill-posedness of the problem (3.4) means its approximated solution does not depend continuously on the data. Hence, regularization is required to overcome this challenge and derive a stable solution. In this work, we employ the Tikhonov regularization: Approximate the problem (3.4) by the problem

$$\min_{f\in F_+} J_\lambda^\varepsilon(f) := \frac{1}{2}\|u(f) - z_d^\varepsilon\|^2_{\mathrm{L}^2(\omega_T)} + \frac{\lambda}{2}\|f\|^2_{\mathrm{L}^2(Q_T)}, \quad \text{subject to (3.1)}, \tag{3.5}$$

where $z_d^\varepsilon := U_d^\varepsilon - u^*_{|\omega_T} \in \mathrm{L}^2(\omega_T)$ is the noise data and $\lambda > 0$ denotes the regularization parameter. Similar to the problem (3.4), we treat z_d^ε as an element of $\mathrm{L}^2(Q_T)$ by interpreting $z_d^\varepsilon \equiv 0$ out side of ω_T. Here, given a noise level $\varepsilon > 0$, we define $U_d^\varepsilon \in \mathrm{L}^2(\omega_T)$ the imprecise observation of $U_d \in \mathrm{L}^2(\omega_T)$ that satisfies $\|U_d - U_d^\varepsilon\|_{\mathrm{L}^2(\omega_T)} \le \varepsilon$.

Theorem 3.3. *The regularized problem (3.5) has a unique solution $f_\lambda^\varepsilon \in F_+$.*

Proof. Clearly, the set $\mathcal{F}_+ := \{f \in F_+ \mid \text{the problem (3.1) is well-posed}\}$ is non-empty. Together with $J_\lambda^\varepsilon(f) \ge 0$ on $\mathcal{F}_+$, we deduce that $j := \inf_{f\in\mathcal{F}_+} J_\lambda^\varepsilon(f)$ is finite. Hence, there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_+$ such that

$$\lim_{n\to\infty} J_\lambda^\varepsilon(f_n) = j.$$

The inequality $\|f_n\|^2_{\mathrm{L}^2(Q_T)} \leq \frac{2}{\lambda} J^\varepsilon_\lambda(f_n)$ for all $n \in \mathbb{N}$ implies that the sequence $\{f_n\}_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^2(Q_T)$, which allows us to extract a (not relabeled) weakly convergent subsequence $\{f_n\}_{n\in\mathbb{N}}$ such that $f_n \rightharpoonup f^\varepsilon_\lambda$ in $\mathrm{L}^2(Q_T)$ with $f^\varepsilon_\lambda \in \mathrm{L}^2(Q_T)$. Moreover, there exists a sufficiently large $r > 0$ such that

$$\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_+ \cap \overline{B}_r,$$

where $\overline{B}_r$ denotes a closed ball with the radius $r > 0$ in $\mathrm{L}^2(Q_T)$. Since $\mathcal{F}_+ \cap \overline{B}_r$ is a closed, bounded, and convex subset of $\mathrm{L}^2(Q_T)$, it is weakly sequentially compact [22]. This gives us $f^\varepsilon_\lambda \in \mathcal{F}_+$.

Consider the variational problem: Find $u_n := u(f_n) \in \mathrm{X}_0$ that satisfies

$$a(u_n, \varphi) = (\ell f_n, \varphi)_{\mathrm{L}^2(Q_T)} \qquad \forall \varphi \in \mathrm{Y}\,. \tag{3.6}$$

This problem is well-posed. A priori estimate (2.3) says

$$\|u_n\|_{\mathrm{X}} \leq C\,\|f_n\|_{\mathrm{L}^2(Q_T)} \qquad \forall n \in \mathbb{N},$$

which means the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X. Hence, there exists $u^\varepsilon_\lambda \in \mathrm{X}$ and a (not relabeled) weakly convergent subsequence $\{u_n\}_{n\in\mathbb{N}}$ such that $u_n \rightharpoonup u^\varepsilon_\lambda$ in X. Therefore, for any $\varphi \in \mathrm{Y}$, we have

$$\lim_{n\to\infty}\left[\langle \partial_t u_n, \varphi\rangle + \int_0^T\int_\Omega (\mathbf{v}\cdot\nabla u_n)\,\varphi + \kappa\nabla u_n\cdot\nabla\varphi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\right] =$$
$$= \langle \partial_t u^\varepsilon_\lambda, \varphi\rangle + \int_0^T\int_\Omega (\mathbf{v}\cdot\nabla u^\varepsilon_\lambda)\,\varphi + \kappa\nabla u^\varepsilon_\lambda\cdot\nabla\varphi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

By passing the limit into (3.6), we arrive at

$$a(u^\varepsilon_\lambda, \varphi) = (\ell f^\varepsilon_\lambda, \varphi)_{\mathrm{L}^2(Q_T)} \qquad \forall \varphi \in \mathrm{Y}\,. \tag{3.7}$$

To conclude that $u_\lambda^\varepsilon = u(f_\lambda^\varepsilon)$, one needs to prove $u_\lambda^\varepsilon \in \mathrm{X}_0$. In (3.6), we choose $\varphi \in \mathrm{C}\left([0,T], \mathrm{H}_0^1(\Omega)\right)$ with $\varphi(\cdot,T) = 0$ in Ω and integrate by parts to get

$$-\langle \partial_t u_n, \varphi\rangle + \int_0^T \int_\Omega (\mathbf{v}\cdot\nabla u_n)\,\varphi + \kappa\nabla u_n\cdot\nabla\varphi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = (\ell f_n, \varphi)_{\mathrm{L}^2(Q_T)}, \quad (3.8)$$

since $u_n(\cdot,0) = 0$ in Ω. We take $n\to\infty$ to obtain

$$-\langle \partial_t u_\lambda^\varepsilon, \varphi\rangle + \int_0^T \int_\Omega (\mathbf{v}\cdot\nabla u_\lambda^\varepsilon)\,\varphi + \kappa\nabla u_\lambda^\varepsilon\cdot\nabla\varphi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = (\ell f_\lambda^\varepsilon, \varphi)_{\mathrm{L}^2(Q_T)}.$$

On the other hand, by applying the technique in (3.8), we rewrite (3.7) as follows

$$\begin{aligned}-\langle \partial_t u_\lambda^\varepsilon, \varphi\rangle + \int_0^T \int_\Omega (\mathbf{v}\cdot\nabla u_\lambda^\varepsilon)\,\varphi + \kappa\nabla u_\lambda^\varepsilon\cdot\nabla\varphi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t &= \\ = (\ell f_\lambda^\varepsilon, \varphi)_{\mathrm{L}^2(Q_T)} + \int_\Omega u_\lambda^\varepsilon(\boldsymbol{x},0)\,\varphi(\boldsymbol{x},0)\,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

From the last two equations, we imply $u_\lambda^\varepsilon(\cdot,0) = 0$ in Ω, and hence $u_\lambda^\varepsilon = u(f_\lambda^\varepsilon)$. Therefore, we get

$$\begin{aligned} j = \liminf_{n\to\infty} J_\lambda^\varepsilon(f_n) &= \liminf_{n\to\infty} \frac{1}{2}\|u(f_n) - z_d^\varepsilon\|_{\mathrm{L}^2(\omega_T)}^2 + \liminf_{n\to\infty}\frac{\lambda}{2}\|f_n\|_{\mathrm{L}^2(Q_T)}^2 \\ &\ge \frac{1}{2}\|u_\lambda^\varepsilon - z_d^\varepsilon\|_{\mathrm{L}^2(\omega_T)}^2 + \frac{\lambda}{2}\|f_\lambda^\varepsilon\|_{\mathrm{L}^2(Q_T)}^2 \\ &= J_\lambda^\varepsilon(f_\lambda^\varepsilon), \end{aligned}$$

which indicates that $f_\lambda^\varepsilon \in \mathcal{F}_+$ is a minimizer. The uniqueness follows from the strict convexity of the functional J_λ^ε. The proof is complete. □

Next, we derive the optimality conditions of the regularized problem (3.5). In doing so, let us introduce the following adjoint problem: Identify $p(f) \in$

X_T such that

$$a'\left(p\left(f\right),\phi\right)=\left(\chi_{\omega_T}\left(u\left(f\right)-z_d^{\varepsilon}\right),\phi\right)_{\mathrm{L}^2(Q_T)}\qquad\forall\phi\in\mathrm{Y},\tag{3.9}$$

where the bilinear form $a':\mathrm{X}\times\mathrm{Y}\to\mathbb{R}$ is defined by

$$a'\left(p,\phi\right)=-\left\langle\partial_t p,\phi\right\rangle+\int_0^T\int_\Omega-\left(\mathbf{v}\cdot\nabla p\right)\phi+\kappa\nabla p\cdot\nabla\phi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,$$

and χ_{ω_T} is the characteristic function of the subdomain ω_T. By changing the time and the velocity field directions, and applying [5] with $\chi_{\omega_T}\left(u\left(f\right)-z_d^{\varepsilon}\right)\in\mathrm{L}^2\left(Q_T\right)$, we conclude the well-posedness of this problem.

Theorem 3.4. *The unique solution $f_\lambda^{\varepsilon}\in F_+$ of the problem (3.5), together with the corresponding state $u_\lambda^{\varepsilon}\in\mathrm{X}_0$ and adjoint $p_\lambda^{\varepsilon}\in\mathrm{X}_T$, satisfies the following optimality conditions*

$$a\left(u_\lambda^{\varepsilon},\varphi\right)=\left(\ell f_\lambda^{\varepsilon},\varphi\right)_{\mathrm{L}^2(Q_T)}\qquad\forall\varphi\in\mathrm{Y},\tag{3.10}$$

and

$$a'\left(p_\lambda^{\varepsilon},\phi\right)=\left(\chi_{\omega_T}\left(u_\lambda^{\varepsilon}-z_d^{\varepsilon}\right),\phi\right)_{\mathrm{L}^2(Q_T)}\qquad\forall\phi\in\mathrm{Y},\tag{3.11}$$

and the variational inequality

$$\left(\ell p_\lambda^{\varepsilon}+\lambda f_\lambda^{\varepsilon},f-f_\lambda^{\varepsilon}\right)_{\mathrm{L}^2(Q_T)}\geq0\qquad\forall f\in F_+.\tag{3.12}$$

Proof. Following the classical arguments [22], we show that the functional J_λ^{ε} defined by (3.5) is Fréchet differentiable and its gradient $\nabla J_\lambda^{\varepsilon}\left(f\right)$ at $f\in F_+$ has the form

$$\nabla J_\lambda^{\varepsilon}\left(f\right)=\ell p\left(f\right)+\lambda f,$$

with $p(f) \in \mathrm{X}_T$ solves the problem (3.9). Indeed, take a small variation $\delta f \in \mathrm{L}^2(Q_T)$ of $f \in F_+$, we have

$$\begin{aligned} & J_\lambda^\varepsilon(f+\delta f) - J_\lambda^\varepsilon(f) = \\ & = \frac{1}{2}\|u(f+\delta f) - z_d^\varepsilon\|^2_{\mathrm{L}^2(\omega_T)} - \frac{1}{2}\|u(f) - z_d^\varepsilon\|^2_{\mathrm{L}^2(\omega_T)} \\ & \qquad\qquad + \frac{\lambda}{2}\|f+\delta f\|^2_{\mathrm{L}^2(Q_T)} - \frac{\lambda}{2}\|f\|^2_{\mathrm{L}^2(Q_T)} \\ & = \frac{1}{2}\|u(f+\delta f) - u(f)\|^2_{\mathrm{L}^2(\omega_T)} + (u(f+\delta f) - u(f), u(f) - z_d^\varepsilon)_{\mathrm{L}^2(\omega_T)} \\ & \qquad\qquad + \frac{\lambda}{2}\|\delta f\|^2_{\mathrm{L}^2(Q_T)} + \lambda(f,\delta f)_{\mathrm{L}^2(Q_T)} \\ & = \frac{1}{2}\|u(\delta f)\|^2_{\mathrm{L}^2(\omega_T)} + (u(\delta f), u(f) - z_d^\varepsilon)_{\mathrm{L}^2(\omega_T)} \\ & \qquad\qquad + \frac{\lambda}{2}\|\delta f\|^2_{\mathrm{L}^2(Q_T)} + \lambda(f,\delta f)_{\mathrm{L}^2(Q_T)}. \end{aligned}$$

Here, we know that $u(\delta f) \in \mathrm{X}_0$ is the solution of the problem

$$a(u(\delta f), \varphi) = (\ell\delta f, \varphi)_{\mathrm{L}^2(Q_T)} \qquad \forall \varphi \in \mathrm{Y}. \tag{3.13}$$

Owing to the inequality (1.1) and a priori estimate (2.4), one has

$$\|u(\delta f)\|_{\mathrm{L}^2(\omega_T)} < C\|u(\delta f)\|_{\mathrm{Y}} < C\|u(\delta f)\|_{\mathrm{X}} \le C\|\delta f\|_{\mathrm{L}^2(Q_T)},$$

which implies $\|u(\delta f)\|_{\mathrm{L}^2(\omega_T)} = o\left(\|\delta f\|_{\mathrm{L}^2(Q_T)}\right)$ as $\|\delta f\|_{\mathrm{L}^2(Q_T)} \to 0$. Hence

$$\begin{aligned} & J_\lambda^\varepsilon(f+\delta f) - J_\lambda^\varepsilon(f) = \\ & = (u(\delta f), u(f) - z_d^\varepsilon)_{\mathrm{L}^2(\omega_T)} + \lambda(f,\delta f)_{\mathrm{L}^2(Q_T)} + o\left(\|\delta f\|^2_{\mathrm{L}^2(Q_T)}\right). \end{aligned} \tag{3.14}$$

To derive the functional gradient, we rewrite the first term on the right-hand side of (3.14) as a scalar product in the solution space. Let $p(f) \in \mathrm{X}_T$ be the solution of (3.9), we choose $\phi = u(\delta f) \in \mathrm{X}_0$ in (3.9) to arrive at

$$(u(\delta f), u(f) - z_d^\varepsilon)_{\mathrm{L}^2(\omega_T)} = (u(\delta f), \chi_{\omega_T}(u(f) - z_d^\varepsilon))_{\mathrm{L}^2(Q_T)}$$

$$= -\left\langle \partial_t p\left(f\right), u\left(\delta f\right)\right\rangle + \int_0^T \int_\Omega -\left(\mathbf{v}\cdot\nabla p\left(f\right)\right) u\left(\delta f\right) + \kappa \nabla p\left(f\right)\cdot\nabla u\left(\delta f\right)\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t. \quad (3.15)$$

By integrating by parts, we can rewrite $\left\langle \partial_t p\left(f\right), u\left(\delta f\right)\right\rangle$ as

$$\begin{aligned}\left\langle \partial_t p\left(f\right), u\left(\delta f\right)\right\rangle &= \int_\Omega p\left(f\right)\left(\boldsymbol{x}, T\right) u\left(\delta f\right)\left(\boldsymbol{x}, T\right)\, \mathrm{d}\boldsymbol{x} \\ &\quad - \int_\Omega p\left(f\right)\left(\boldsymbol{x}, 0\right) u\left(\delta f\right)\left(\boldsymbol{x}, 0\right)\, \mathrm{d}\boldsymbol{x} - \left\langle \partial_t u\left(\delta f\right), p\left(f\right)\right\rangle \\ &= -\left\langle \partial_t u\left(\delta f\right), p\left(f\right)\right\rangle,\end{aligned}$$

using $p\left(f\right) \in \mathrm{X}_T$ and $u\left(\delta f\right) \in \mathrm{X}_0$ in the final step. To handle the advection part on the right-hand side of (3.15), we invoke the technique as in (2.17). We get

$$\int_0^T \int_\Omega \left(\mathbf{v}\cdot\nabla p\left(f\right)\right) u\left(\delta f\right)\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t = -\int_0^T \int_\Omega \left(\mathbf{v}\cdot\nabla u\left(\delta f\right)\right) p\left(f\right)\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t.$$

Hence, (3.15) becomes

$$\begin{aligned}&\left(u\left(\delta f\right), u\left(f\right) - z_d^{\varepsilon}\right)_{\mathrm{L}^2(\omega_T)} = \\ &= \left\langle \partial_t u\left(\delta f\right), p\left(f\right)\right\rangle \\ &\qquad + \int_0^T \int_\Omega \left(\mathbf{v}\cdot\nabla u\left(\delta f\right)\right) p\left(f\right) + \kappa \nabla u\left(\delta f\right)\cdot\nabla p\left(f\right)\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t \\ &= a\left(u\left(\delta f\right), p\left(f\right)\right) = \left(\ell p\left(f\right), \delta f\right)_{\mathrm{L}^2(Q_T)}. \end{aligned} \quad (3.16)$$

Here, we choose $\varphi = p\left(f\right) \in \mathrm{X}_T$ in (3.13) to obtain the first equality. By substituting into (3.14), we can conclude the Fréchet differentiability of the functional J_λ^ε, together with its gradient. □

We end this section with some convergence properties of the Tikhonov regularization. We start by showing that the solution $f_\lambda^\varepsilon \in F_+$ of the problem (3.5) is stable with respect to the noise in the observation $z_d^\varepsilon \in \mathrm{L}^2(Q_T)$. The following theorem is the constrained variant of the results presented in [33] and [34].

Theorem 3.5. *For a fixed $\lambda > 0$, let $\{z_n\}_{n\in\mathbb{N}} \subset \mathrm{L}^2(\omega_T)$ be the sequence that converges strongly to z_d^ε in $\mathrm{L}^2(\omega_T)$, and $\{f_n\}_{n\in\mathbb{N}} \subset F_+$ the sequence of solutions to the corresponding problems*

$$\min_{f\in F_+} \frac{1}{2}\|u(f) - z_n\|^2_{\mathrm{L}^2(\omega_T)} + \frac{\lambda}{2}\|f\|^2_{\mathrm{L}^2(Q_T)}, \qquad n \in \mathbb{N}, \quad \text{subject to (3.1)}. \tag{3.17}$$

Then, $\{f_n\}_{n\in\mathbb{N}}$ converges strongly to the solution $f_\lambda^\varepsilon \in F_+$ of the problem (3.5) in $\mathrm{L}^2(Q_T)$.

Proof. Owing to Theorem 3.3, for each $n \in \mathbb{N}$, there exists a unique minimizer $f_n \in F_+$ of the problem (3.17). For all $f \in F_+$, we have

$$\frac{1}{2}\|u(f_n) - z_n\|^2_{\mathrm{L}^2(\omega_T)} + \frac{\lambda}{2}\|f_n\|^2_{\mathrm{L}^2(Q_T)} \le \frac{1}{2}\|u(f) - z_n\|^2_{\mathrm{L}^2(\omega_T)} + \frac{\lambda}{2}\|f\|^2_{\mathrm{L}^2(Q_T)},$$

which implies the boundedness of $\{f_n\}_{n\in\mathbb{N}}$ in $\mathrm{L}^2(Q_T)$. By following the technique as in Theorem 3.3, we conclude the existence of an element $f_\lambda^\varepsilon \in F_+$ and a (not relabed) weakly convergent subsequence $\{f_n\}_{n\in\mathbb{N}}$ such that $f_n \rightharpoonup f_\lambda^\varepsilon$ in $\mathrm{L}^2(Q_T)$. Moreover, as $n \to \infty$, it holds

$$u(f_n) \rightharpoonup u(f_\lambda^\varepsilon) \qquad \text{in } \mathrm{X},$$

up to taking a further subsequence. Together with the strong convergence of the sequence $\{z_n\}_{n\in\mathbb{N}}$ to z_d^ε in $\mathrm{L}^2(\omega_T)$, one gets $u(f_n) - z_n \rightharpoonup u(f_\lambda^\varepsilon) - z_d^\varepsilon$ in

$L^2(\omega_T)$, and hence

$$\liminf_{n\to\infty} \|u(f_n) - z_n\|^2_{L^2(\omega_T)} \geq \|u(f^\varepsilon_\lambda) - z^\varepsilon_d\|^2_{L^2(\omega_T)}. \tag{3.18}$$

Therefore, for all $f \in F_+$, we deduce that

$$\begin{aligned} J^\varepsilon_\lambda(f) &= \frac{1}{2}\|u(f) - z^\varepsilon_d\|^2_{L^2(\omega_T)} + \frac{\lambda}{2}\|f\|^2_{L^2(Q_T)} \\ &\geq \lim_{n\to\infty}\left(\frac{1}{2}\|u(f) - z_n\|^2_{L^2(\omega_T)} + \frac{\lambda}{2}\|f\|^2_{L^2(Q_T)}\right) \\ &\geq \liminf_{n\to\infty}\left(\frac{1}{2}\|u(f_n) - z_n\|^2_{L^2(\omega_T)} + \frac{\lambda}{2}\|f_n\|^2_{L^2(Q_T)}\right) \\ &\geq \frac{1}{2}\|u(f^\varepsilon_\lambda) - z^\varepsilon_d\|^2_{L^2(\omega_T)} + \frac{\lambda}{2}\|f^\varepsilon_\lambda\|^2_{L^2(Q_T)} = J^\varepsilon_\lambda(f^\varepsilon_\lambda), \end{aligned} \tag{3.19}$$

which means that $f^\varepsilon_\lambda \in F_+$ is the solution of the problem (3.5).

Next, we prove that the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges strongly to f^ε_λ in $L^2(Q_T)$. By contradiction, suppose that the claim is false. Then, we observe that

$$\lim_{n\to\infty}\|f_n\|_{L^2(Q_T)} \neq \|f^\varepsilon_\lambda\|_{L^2(Q_T)},$$

which yields

$$\theta := \limsup_{n\to\infty}\|f_n\|_{L^2(Q_T)} > \liminf_{n\to\infty}\|f_n\|_{L^2(Q_T)} \geq \|f^\varepsilon_\lambda\|_{L^2(Q_T)}. \tag{3.20}$$

Therefore, there exists a (not relabeled) subsequence $\{f_n\}_{n\in\mathbb{N}}$ that satisfying $\lim_{n\to\infty}\|f_n\|_{L^2(Q_T)} = \theta$. By choosing $f = f^\varepsilon_\lambda \in F_+$ in (3.19), we have

$$\begin{aligned} J^\varepsilon_\lambda(j^\varepsilon_\lambda) &= \liminf_{n\to\infty}\left(\frac{1}{2}\|u(f_n) - z_n\|^2_{L^2(\omega_T)} + \frac{\lambda}{2}\|f_n\|^2_{L^2(Q_T)}\right) \\ &= \liminf_{n\to\infty}\frac{1}{2}\|u(f_n) - z_n\|^2_{L^2(\omega_T)} + \frac{\lambda}{2}\theta^2. \end{aligned}$$

Combining with (3.20), we arrive at

$$\frac{1}{2}\|u(f^\varepsilon_\lambda) - z^\varepsilon_d\|^2_{L^2(\omega_T)} =$$

$$= \liminf_{n\to\infty} \frac{1}{2} \|u(f_n) - z_n\|^2_{\mathrm{L}^2(\omega_T)} + \frac{\lambda}{2}\left(\theta^2 - \|f^\varepsilon_\lambda\|^2_{\mathrm{L}^2(Q_T)}\right)$$
$$> \liminf_{n\to\infty} \frac{1}{2} \|u(f_n) - z_n\|^2_{\mathrm{L}^2(\omega_T)},$$

which contradicts with (3.18). The proof is finished. □

On the other hand, regarding the error estimate of regularizing the source $f_+ \in F_+$ in Definition 3.1 by the Tikhonov regularization, let us recall from [29] the following result for the general linear inverse problems with convex constraints:

Lemma 3.6. *Let $f_+ \in F_+$ be the F_+-best approximated source of the problem (3.4) and $f^\varepsilon_\lambda \in F_+$ be the solution of the regularized problem (3.5). Assume that there exists $\xi \in \mathrm{L}^2(\omega_T)$ with the minimal $\mathrm{L}^2(\omega_T)$-norm that satisfies $f_+ = \mathrm{Proj}_{F_+}(A^*\xi)$, where $A^* : \mathrm{L}^2(\omega_T) \to \mathrm{L}^2(Q_T)$ denotes the adjoint operator of the operator A in (3.4). Then, we have the following inequality*

$$\|f_+ - f^\varepsilon_\lambda\|_{\mathrm{L}^2(Q_T)} \le \sqrt{\lambda}\,\|\xi\|_{\mathrm{L}^2(\omega_T)} + \frac{\varepsilon}{\sqrt{\lambda}}.$$

3.3 Finite element discretization

In this section, we discretize the problem (3.5) by combining the interface-fitted space-time finite element method [5] and the variational approach [18].

3.3.1. The discrete regularized problem

We first discretize the regularized state and adjoint. As in section 2.2.1., let us define the discrete state problem: For $\ell \in \mathrm{L}^\infty(Q_T)$ and $f \in \mathrm{L}^2(Q_T)$, find $u_h(f) \in \mathrm{X}_{h,0}$ that satisfies

$$a_h(u_h(f), \varphi_h) = (\ell f, \varphi_h)_{\mathrm{L}^2(Q_T)} \qquad \forall \varphi_h \in \mathrm{Y}_h\,. \tag{3.21}$$

The following discrete stability condition holds

$$\sup_{\varphi_h \in \mathrm{Y}_h \setminus \{0\}} \frac{a_h(u_h(f), \varphi_h)}{|||\varphi_h|||} \geq C|||u_h(f)|||_* \qquad \forall u_h(f) \in \mathrm{X}_{h,0}, \tag{3.22}$$

which ensures that the problem (3.21) is uniquely solvable. A priori estimates for the state error in three different norms have been presented in the previous chapter.

Similarly, we present the interface-fitted space-time method for solving the adjoint problem (3.9). We introduce the space

$$\mathrm{X}_{h,T} = \{\varphi_h \in \mathrm{Y}_h \mid \varphi_h = 0 \text{ on } \Omega \times \{T\}\} \subset \mathrm{X}_T\,.$$

The discrete adjoint problem reads as: Determine $p_h(f) \in \mathrm{X}_{h,T}$ such that

$$a_h'(p_h(f), \phi_h) = (\chi_{\omega_T}(u(f) - z_d^\varepsilon), \phi_h)_{\mathrm{L}^2(Q_T)} \qquad \forall \phi_h \in \mathrm{Y}_h, \tag{3.23}$$

with the bilinear form $a_h' : \mathrm{X}_T \times \mathrm{Y} \to \mathbb{R}$ given by

$$a_h'(p, \phi) = -\langle \partial_t p, \phi \rangle + \int_0^T \int_\Omega -(\mathbf{v} \cdot \nabla p)\,\phi + \kappa_h \nabla p \cdot \nabla \phi \, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t.$$

Employing the technique as in (3.22), we establish the following stability condition

$$\sup_{\phi_h \in \mathrm{Y}_h \setminus \{0\}} \frac{a_h'(p_h(f), \phi_h)}{|||\phi_h|||} \geq C|||p_h(f)|||_* \qquad \forall p_h(f) \in \mathrm{X}_{h,T}, \tag{3.24}$$

and concludes the unique solvability of the problem (3.23). By using the arguments for the state problem, one can derive a priori error estimates for the adjoint. For completeness, let us state the main results. We shall need the space

$$\mathrm{V}_0 := \left\{ \psi \in \mathrm{X} \mid \psi(\cdot,0) = \frac{\gamma_0 \left(p(f) - p_h(f)\right)(\cdot,0)}{\|(p(f) - p_h(f))(\cdot,0)\|^{-1}_{\mathrm{L}^2(\Omega)}} \text{ in } \Omega \right\},$$

where $\gamma_0 > 0$ is a sufficiently large number, $p(f) \in \mathrm{X}_T$ and $p_h(f) \in \mathrm{X}_{h,T}$ be the solutions of the problems (3.9) and (3.23), respectively. Assume that there exists $y' \in \mathrm{V}_0$ and $z' \in \mathrm{X}$ that satisfy

$$\langle \partial_t y', \varphi \rangle + \int_0^T \int_\Omega (\mathbf{v} \cdot \nabla y')\, \varphi + \kappa \nabla y' \cdot \nabla \varphi \, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t = 0 \qquad \forall \varphi \in \mathrm{Y}, \quad (3.25)$$

and

$$\langle \partial_t z', \varphi \rangle + \int_0^T \int_\Omega (\mathbf{v} \cdot \nabla z')\, \varphi + \kappa \nabla z' \cdot \nabla \varphi \, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t =$$

$$= \|p(f) - p_h(f)\|^{-1}_{\mathrm{L}^2(Q_T)} \int_0^T \int_\Omega (p(f) - p_h(f))\, \varphi \, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t \quad \forall \varphi \in \mathrm{Y}\,. \quad (3.26)$$

with $z'(\cdot,0) \in \mathrm{H}^1_0(\Omega)$. We have $y', z' \in \mathrm{H}^1(Q_T) \cap \mathrm{H}^s(Q_1 \cup Q_2)$ with $s > \frac{d+3}{2}$. Furthermore, assume that there exists a constant $C > 0$ independent of $p(f)$ and $p_h(f)$ such that

$$\|y'\|_{\mathrm{H}^s(Q_1 \cup Q_2)} \leq C, \quad (3.27)$$

and

$$\|z'\|_{\mathrm{H}^s(Q_1 \cup Q_2)} \leq C. \quad (3.28)$$

Lemma 3.7. *For $f \in F_+$, let $p(f) \in \mathrm{X}_T$ and $p_h(f) \in \mathrm{X}_{h,T}$ be the solutions of the problems (3.9) and (3.23), respectively.*

a) Assume that Assumption 2.1 is satisfied. Then, we have the following estimate

$$\left\| p\left(f\right) - p_h\left(f\right) \right\|_* \leq Ch \left\| p\left(f\right) \right\|_{\mathrm{H}^s(Q_1 \cup Q_2)}. \tag{3.29}$$

b) Moreover, if the problem (3.25) has a solution $y' \in \mathrm{V}_0$ that satisfies (3.27), then there holds the estimate

$$\left\| \left(p\left(f\right) - p_h\left(f\right)\right)\left(\cdot, 0\right) \right\|_{\mathrm{L}^2(\Omega)} \leq Ch^2 \left\| p\left(f\right) \right\|_{\mathrm{H}^s(Q_1 \cup Q_2)}.$$

c) Furthermore, if there exists a solution $z' \in \mathrm{X}$ of the problem (3.26) that satisfies (3.28), then the following estimate holds

$$\left\| p\left(f\right) - p_h\left(f\right) \right\|_{\mathrm{L}^2(Q_T)} \leq Ch^2 \left\| p\left(f\right) \right\|_{\mathrm{H}^s(Q_1 \cup Q_2)}. \tag{3.30}$$

The final step is to discretize the regularized source. In this work, we invoke the variational approach [18], in which we turn the discretization of the regularized source into the discrete treatment for a term that involves the regularized adjoint. The discrete inverse source problem reads as

$$\min_{f_h \in F_+} J^\varepsilon_{\lambda,h}\left(f_h\right) := \frac{1}{2} \left\| u_h\left(f_h\right) - z^\varepsilon_{d,h} \right\|^2_{\mathrm{L}^2(\omega_T)} + \frac{\lambda}{2} \left\| f_h \right\|^2_{\mathrm{L}^2(Q_T)}, \quad \text{subject to (3.21)}, \tag{3.31}$$

where $z^\varepsilon_{d,h} := U^\varepsilon_d - u^*_{h\,|\,\omega_T} \in \mathrm{L}^2\left(\omega_T\right)$ denotes the discrete data. Here, $u^*_h \in \mathrm{X}_{h,0}$ approximates $u^* \in \mathrm{X}_0$ in (3.2), also by the interface-fitted space-time method. Therefore, it can be defined similarly as $u_h\left(f\right) \in \mathrm{X}_{h,0}$ in (3.21).

Analogue to the problem (3.13), for $f_h \in F_+$ and $\delta f_h \in \mathrm{L}^2\left(Q_T\right)$, the function $u_h\left(\delta f_h\right) = u_h\left(f_h + \delta f_h\right) - u_h\left(f_h\right) \in \mathrm{X}_{h,0}$ will be the solution of the problem

$$a_h\left(u_h\left(\delta f_h\right), \varphi_h\right) = \left(\ell \delta f_h, \varphi_h\right)_{\mathrm{L}^2(Q_T)} \qquad \forall \varphi_h \in \mathrm{Y}_h. \tag{3.32}$$

Furthermore, the technique as in (3.16) gives us

$$\left(u_h\left(\delta f_h\right), u_h\left(f_h\right) - z^{\varepsilon}_{d,h}\right)_{\mathrm{L}^2(\omega_T)} = \left(\ell p_h\left(f_h\right), \delta f_h\right)_{\mathrm{L}^2(Q_T)}, \tag{3.33}$$

where $u_h\left(f_h\right) \in \mathrm{X}_{h,0}$ and $p_h\left(f_h\right) \in \mathrm{X}_{h,T}$ be the solutions of the problems (3.21) and (3.23) with the corresponding right-hand sides $\ell f_h \in \mathrm{L}^2\left(Q_T\right)$ and $\chi_{\omega_T}\left(u_h\left(f_h\right) - z^{\varepsilon}_{d,h}\right) \in \mathrm{L}^2\left(Q_T\right)$. By employing this equality, we can prove the following discrete optimality conditions:

Lemma 3.8. *Let $f^{\varepsilon}_{\lambda,h} \in F_+$ be the solution of the problem (3.31), $u^{\varepsilon}_{\lambda,h} \in \mathrm{X}_{h,0}$ and $p^{\varepsilon}_{\lambda,h} \in \mathrm{X}_{h,T}$ denote the corresponding state and adjoint. Then, the following optimality system is satisfied*

$$a_h\left(u^{\varepsilon}_{\lambda,h}, \varphi_h\right) = \left(\ell f^{\varepsilon}_{\lambda,h}, \varphi_h\right)_{\mathrm{L}^2(Q_T)} \qquad \forall \varphi_h \in \mathrm{Y}_h, \tag{3.34}$$

and

$$a'_h\left(p^{\varepsilon}_{\lambda,h}, \phi_h\right) = \left(\chi_{\omega_T}\left(u^{\varepsilon}_{\lambda,h} - z^{\varepsilon}_{d,h}\right), \phi_h\right)_{\mathrm{L}^2(Q_T)} \qquad \forall \phi_h \in \mathrm{Y}_h, \tag{3.35}$$

and the variational inequality

$$\left(\ell p^{\varepsilon}_{\lambda,h} + \lambda f^{\varepsilon}_{\lambda,h}, f_h - f^{\varepsilon}_{\lambda,h}\right)_{\mathrm{L}^2(Q_T)} \geq 0 \qquad \forall f_h \in F_+. \tag{3.36}$$

3.3.2. Error and convergence estimates

Let $f_+ \in F_+$ be the F_+-best approximated solution of the problem (3.4) and $f^{\varepsilon}_{\lambda,h} \in F_+$ be the solution of the problem (3.31). In this section, we estimate the errors $f_+ - f^{\varepsilon}_{\lambda,h}$ in the $\mathrm{L}^2\left(Q_T\right)$-norm in terms of the parameter λ, the mesh size h and the noise level ε. Moreover, we suggest an a priori choice for λ, depending on h and ε, such that $f^{\varepsilon}_{\lambda,h}$ converges strongly to f_+

in $\mathrm{L}^2(Q_T)$ as $\lambda \to 0$. Let us start with the triangle inequality

$$\left\|f_+ - f^\varepsilon_{\lambda,h}\right\|_{\mathrm{L}^2(Q_T)} \leq \|f_+ - f^\varepsilon_\lambda\|_{\mathrm{L}^2(Q_T)} + \left\|f^\varepsilon_\lambda - f^\varepsilon_{\lambda,h}\right\|_{\mathrm{L}^2(Q_T)}. \tag{3.37}$$

The first term on the right-hand side of (3.37) is treated in Lemma 3.6. Hence, our aim is first to estimate the second error term in (3.37), then provide an a priori choice for λ that ensures the desired convergence, depending on the total error.

The main result of this subsection is stated in Theorem 3.12 and Corollary 3.13. To begin, we denote by $u_h(f^\varepsilon_\lambda) \in \mathrm{X}_{h,0}$ and $p_h(f^\varepsilon_\lambda) \in \mathrm{X}_{h,T}$ the solutions of the following problems

$$a_h\left(u_h(f^\varepsilon_\lambda), \varphi_h\right) = \left(\ell f^\varepsilon_\lambda, \varphi_h\right)_{\mathrm{L}^2(Q_T)} \qquad \forall \varphi_h \in \mathrm{Y}_h, \tag{3.38}$$

and

$$a'_h\left(p_h(f^\varepsilon_\lambda), \phi_h\right) = \left(\chi_{\omega_T}\left(u_h(f^\varepsilon_\lambda) - z^\varepsilon_{d,h}\right), \phi_h\right)_{\mathrm{L}^2(Q_T)} \qquad \forall \phi_h \in \mathrm{Y}_h. \tag{3.39}$$

Lemma 3.9. *Let the triples* $\left(u_h(f^\varepsilon_\lambda), p_h(f^\varepsilon_\lambda), f^\varepsilon_\lambda\right) \in \mathrm{X}_{h,0} \times \mathrm{X}_{h,T} \times F_+$ *and* $\left(u^\varepsilon_{\lambda,h}, p^\varepsilon_{\lambda,h}, f^\varepsilon_{\lambda,h}\right) \in \mathrm{X}_{h,0} \times \mathrm{X}_{h,T} \times F_+$ *be the solutions of the problems* (3.38)-(3.39)*,* (3.12) *and* (3.34)-(3.36)*, respectively. Then, it holds that*

$$\left|\left|\left|u_h(f^\varepsilon_\lambda) - u^\varepsilon_{\lambda,h}\right|\right|\right|_* + \left|\left|\left|p_h(f^\varepsilon_\lambda) - p^\varepsilon_{\lambda,h}\right|\right|\right|_* \leq C\left\|f^\varepsilon_\lambda - f^\varepsilon_{\lambda,h}\right\|_{\mathrm{L}^2(Q_T)},$$

and

$$\left\|u_h(f^\varepsilon_\lambda) - u^\varepsilon_{\lambda,h}\right\|_{\mathrm{L}^2(Q_T)} + \left\|p_h(f^\varepsilon_\lambda) - p^\varepsilon_{\lambda,h}\right\|_{\mathrm{L}^2(Q_T)} \leq C\left\|f^\varepsilon_\lambda - f^\varepsilon_{\lambda,h}\right\|_{\mathrm{L}^2(Q_T)}.$$

Proof. First, let us prove the first inequality. By subtracting (3.34) from (3.38), we obtain

$$a_h\left(u_h(f^\varepsilon_\lambda) - u^\varepsilon_{\lambda,h}, \varphi_h\right) = \left(\ell f^\varepsilon_\lambda - \ell f^\varepsilon_{\lambda,h}, \varphi_h\right)_{\mathrm{L}^2(Q_T)} \qquad \forall \varphi_h \in \mathrm{Y}_h.$$

The stability condition (3.22) gives us

$$C\left|\left|\left|u_h\left(f_\lambda^\varepsilon\right)-u_{\lambda,h}^\varepsilon\right|\right|\right|_* \leq \sup_{\varphi_h\in \mathrm{Y}_h\setminus\{0\}} \frac{a_h\left(u_h\left(f_\lambda^\varepsilon\right)-u_{\lambda,h}^\varepsilon,\varphi_h\right)}{|||\varphi_h|||}.$$

On the other hand, from the inequality (1.1), one has

$$\sup_{\varphi_h\in \mathrm{Y}_h\setminus\{0\}} \frac{a_h\left(u_h\left(f_\lambda^\varepsilon\right)-u_{\lambda,h}^\varepsilon,\varphi_h\right)}{|||\varphi_h|||} = \sup_{\varphi_h\in \mathrm{Y}_h\setminus\{0\}} \frac{\left(\ell f_\lambda^\varepsilon-\ell f_{\lambda,h}^\varepsilon,\varphi_h\right)_{\mathrm{L}^2(Q_T)}}{|||\varphi_h|||}$$

$$\leq C\left\|f_\lambda^\varepsilon-f_{\lambda,h}^\varepsilon\right\|_{\mathrm{L}^2(Q_T)}.$$

Therefore, we have

$$\left|\left|\left|u_h\left(f_\lambda^\varepsilon\right)-u_{\lambda,h}^\varepsilon\right|\right|\right|_* \leq C\left\|f_\lambda^\varepsilon-f_{\lambda,h}^\varepsilon\right\|_{\mathrm{L}^2(Q_T)}. \tag{3.40}$$

We continue by subtracting (3.35) from (3.39) to get

$$a_h'\left(p_h\left(f_\lambda^\varepsilon\right)-p_{\lambda,h}^\varepsilon,\phi_h\right)=\left(\chi_{\omega_T}\left(u_h\left(f_\lambda^\varepsilon\right)-u_{\lambda,h}^\varepsilon\right),\phi_h\right)_{\mathrm{L}^2(Q_T)} \qquad \forall\phi_h\in \mathrm{Y}_h\,.$$

Invoke the inequality (3.24), the technique as in (3.40), and the inequality (3.40) itself, one obtains

$$\left|\left|\left|p_h\left(f_\lambda^\varepsilon\right)-p_{\lambda,h}^\varepsilon\right|\right|\right|_* \leq C\left\|u_h\left(f_\lambda^\varepsilon\right)-u_{\lambda,h}^\varepsilon\right\|_{\mathrm{L}^2(\omega_T)} \leq C\left\|f_\lambda^\varepsilon-f_{\lambda,h}^\varepsilon\right\|_{\mathrm{L}^2(Q_T)}. \tag{3.41}$$

The first inequality follows by combining (3.40) and (3.41). The second one is a consequence of the first one, thanks to (1.1). □

Lemma 3.10. *Let* $(p_\lambda^\varepsilon, f_\lambda^\varepsilon)\in \mathrm{X}_T\times F_+$ *and* $p_h\left(f_\lambda^\varepsilon\right)\in \mathrm{X}_{h,T}$ *be the solutions of the problems* (3.11)-(3.12) *and* (3.39)*, respectively. Let* $f_{\lambda,h}^\varepsilon\in F_+$ *be the solution of the problem* (3.36) *in case of variational discretization. Then, the following estimate holds*

$$\left\|f_\lambda^\varepsilon-f_{\lambda,h}^\varepsilon\right\|_{\mathrm{L}^2(Q_T)} \leq \frac{C}{\lambda}\left\|p_\lambda^\varepsilon-p_h\left(f_\lambda^\varepsilon\right)\right\|_{\mathrm{L}^2(Q_T)}.$$

Proof. We choose $f = f^{\varepsilon}_{\lambda,h} \in F_+$ in (3.12) and $f_h = f^{\varepsilon}_{\lambda} \in F_+$ in (3.36), then add the corresponding inequalities to get

$$\begin{aligned}
&\lambda \left\| f^{\varepsilon}_{\lambda} - f^{\varepsilon}_{\lambda,h} \right\|^2_{\mathrm{L}^2(Q_T)} \leq \\
&\leq \left(\ell p^{\varepsilon}_{\lambda} - \ell p^{\varepsilon}_{\lambda,h}, f^{\varepsilon}_{\lambda,h} - f^{\varepsilon}_{\lambda} \right)_{\mathrm{L}^2(Q_T)} \\
&= \left(\ell p^{\varepsilon}_{\lambda} - \ell p_h \left(f^{\varepsilon}_{\lambda} \right), f^{\varepsilon}_{\lambda,h} - f^{\varepsilon}_{\lambda} \right)_{\mathrm{L}^2(Q_T)} + \left(\ell p_h \left(f^{\varepsilon}_{\lambda} \right) - \ell p^{\varepsilon}_{\lambda,h}, f^{\varepsilon}_{\lambda,h} - f^{\varepsilon}_{\lambda} \right)_{\mathrm{L}^2(Q_T)}.
\end{aligned} \tag{3.42}$$

By using the Cauchy inequality, we estimate the first term on the right-hand side of (3.42) as follows

$$\begin{aligned}
&\left(\ell p^{\varepsilon}_{\lambda} - \ell p_h \left(f^{\varepsilon}_{\lambda} \right), f^{\varepsilon}_{\lambda,h} - f^{\varepsilon}_{\lambda} \right)_{\mathrm{L}^2(Q_T)} \leq \\
&\leq \|\ell\|_{\mathrm{L}^\infty(Q_T)} \|p^{\varepsilon}_{\lambda} - p_h \left(f^{\varepsilon}_{\lambda} \right)\|_{\mathrm{L}^2(Q_T)} \left\| f^{\varepsilon}_{\lambda,h} - f^{\varepsilon}_{\lambda} \right\|_{\mathrm{L}^2(Q_T)} \\
&\leq \frac{1}{2\lambda} \|\ell\|^2_{\mathrm{L}^\infty(Q_T)} \|p^{\varepsilon}_{\lambda} - p_h \left(f^{\varepsilon}_{\lambda} \right)\|^2_{\mathrm{L}^2(Q_T)} + \frac{\lambda}{2} \left\| f^{\varepsilon}_{\lambda,h} - f^{\varepsilon}_{\lambda} \right\|^2_{\mathrm{L}^2(Q_T)}.
\end{aligned} \tag{3.43}$$

To handle the second term on the right-hand side of (3.42), we utilize the equality (3.33) twice. We have

$$\begin{aligned}
&\left(\ell p_h \left(f^{\varepsilon}_{\lambda} \right) - \ell p^{\varepsilon}_{\lambda,h}, f^{\varepsilon}_{\lambda,h} - f^{\varepsilon}_{\lambda} \right)_{\mathrm{L}^2(Q_T)} = \\
&= \left(\ell p_h \left(f^{\varepsilon}_{\lambda} \right), f^{\varepsilon}_{\lambda,h} - f^{\varepsilon}_{\lambda} \right)_{\mathrm{L}^2(Q_T)} - \left(\ell p^{\varepsilon}_{\lambda,h}, f^{\varepsilon}_{\lambda,h} - f^{\varepsilon}_{\lambda} \right)_{\mathrm{L}^2(Q_T)} \\
&= \left(u^{\varepsilon}_{\lambda,h} - u_h \left(f^{\varepsilon}_{\lambda} \right), u_h \left(f^{\varepsilon}_{\lambda} \right) - z^{\varepsilon}_{d,h} \right)_{\mathrm{L}^2(\omega_T)} - \left(u^{\varepsilon}_{\lambda,h} - u_h \left(f^{\varepsilon}_{\lambda} \right), u^{\varepsilon}_{\lambda,h} - z^{\varepsilon}_{d,h} \right)_{\mathrm{L}^2(\omega_T)} \\
&= - \left\| u^{\varepsilon}_{\lambda,h} - u_h \left(f^{\varepsilon}_{\lambda} \right) \right\|^2_{\mathrm{L}^2(\omega_T)} \leq 0.
\end{aligned} \tag{3.44}$$

By combining (3.42), (3.43), and (3.44), we obtain the result. □

We next estimate the right-hand side of the inequality in lemma 3.10. In doing so, let us introduce $\widetilde{p_h} \left(f^{\varepsilon}_{\lambda} \right) \in \mathrm{X}_{h,T}$ as the solution of the problem

$$a'_h \left(\widetilde{p_h} \left(f^{\varepsilon}_{\lambda} \right), \phi_h \right) = \left(\chi_{\omega_T} \left(u^{\varepsilon}_{\lambda} - z^{\varepsilon}_{d} \right), \phi_h \right)_{\mathrm{L}^2(Q_T)} \qquad \forall \phi_h \in \mathrm{Y}_h. \tag{3.45}$$

Lemma 3.11. *Let* $(u_\lambda^\varepsilon, p_\lambda^\varepsilon, f_\lambda^\varepsilon) \in \mathrm{X}_0 \times \mathrm{X}_T \times F_+$ *and* $p_h(f_\lambda^\varepsilon) \in \mathrm{X}_{h,T}$ *be the solutions of the problems* (3.10)-(3.12) *and* (3.39)*, respectively. Let* $u^* \in \mathrm{X}_0$ *be the solution of the problem* (3.2)*. Assume that the assumptions of Lemma* 2.11 *and Lemma* 3.7*c are satisfied. Then, there holds the following*

$$\begin{aligned} &|||p_\lambda^\varepsilon - p_h(f_\lambda^\varepsilon)|||_* \leq \\ &\leq Ch^2\left(\|u_\lambda^\varepsilon\|_{\mathrm{H}^s(Q_1\cup Q_2)} + \|u^*\|_{\mathrm{H}^s(Q_1\cup Q_2)}\right) + Ch\,\|p_\lambda^\varepsilon\|_{\mathrm{H}^s(Q_1\cup Q_2)}, \end{aligned} \tag{3.46}$$

and

$$\begin{aligned} &\|p_\lambda^\varepsilon - p_h(f_\lambda^\varepsilon)\|_{\mathrm{L}^2(Q_T)} \leq \\ &\leq Ch^2\left(\|u_\lambda^\varepsilon\|_{\mathrm{H}^s(Q_1\cup Q_2)} + \|u^*\|_{\mathrm{H}^s(Q_1\cup Q_2)} + \|p_\lambda^\varepsilon\|_{\mathrm{H}^s(Q_1\cup Q_2)}\right). \end{aligned} \tag{3.47}$$

Proof. We first derive the estimate (3.46). By using the triangle inequality, one has

$$|||p_\lambda^\varepsilon - p_h(f_\lambda^\varepsilon)|||_* \leq |||p_\lambda^\varepsilon - \widetilde{p_h}(f_\lambda^\varepsilon)|||_* + |||\widetilde{p_h}(f_\lambda^\varepsilon) - p_h(f_\lambda^\varepsilon)|||_*, \tag{3.48}$$

where $\widetilde{p_h}(f_\lambda^\varepsilon) \in \mathrm{X}_{h,T}$ is the solution of the problem (3.45). A priori error estimate (3.29) yields

$$|||p_\lambda^\varepsilon - \widetilde{p_h}(f_\lambda^\varepsilon)|||_* \leq Ch\,\|p_\lambda^\varepsilon\|_{\mathrm{H}^s(Q_1\cup Q_2)}. \tag{3.49}$$

For dealing with the second term on the right-hand side of (3.48), we subtract (3.39) from (3.45) to get

$$\begin{aligned} &a_h'\left(\widetilde{p_h}(f_\lambda^\varepsilon) - p_h(f_\lambda^\varepsilon), \phi_h\right) = \\ &= \left(\chi_{\omega_T}\left(u_\lambda^\varepsilon - u_h(f_\lambda^\varepsilon) - z_d^\varepsilon + z_{d,h}^\varepsilon\right), \phi_h\right)_{\mathrm{L}^2(Q_T)} \qquad \forall \phi_h \in \mathrm{Y}_h. \end{aligned} \tag{3.50}$$

We invoke the technique as in (3.41) and a priori error estimate (2.24) to arrive at

$$|||\widetilde{p_h}(f_\lambda^\varepsilon) - p_h(f_\lambda^\varepsilon)|||_* \leq C\left(\|u_\lambda^\varepsilon - u_h(f_\lambda^\varepsilon)\|_{\mathrm{L}^2(\omega_T)} + \|u^* - u_h^*\|_{\mathrm{L}^2(\omega_T)}\right)$$

$$\leq Ch^2 \left(\|u_\lambda^\varepsilon\|_{\mathrm{H}^s(Q_1\cup Q_2)} + \|u^*\|_{\mathrm{H}^s(Q_1\cup Q_2)} \right). \quad (3.51)$$

The estimate (3.46) follows by substituting (3.49) and (3.51) into (3.48). We employ the same arguments to prove the estimate (3.47). Indeed, we have

$$\|p_\lambda^\varepsilon - p_h(f_\lambda^\varepsilon)\|_{\mathrm{L}^2(Q_T)} \leq \|p_\lambda^\varepsilon - \widetilde{p}_h(f_\lambda^\varepsilon)\|_{\mathrm{L}^2(Q_T)} + \|\widetilde{p}_h(f_\lambda^\varepsilon) - p_h(f_\lambda^\varepsilon)\|_{\mathrm{L}^2(Q_T)}.$$

By a priori error estimate (3.30), it holds

$$\|p_\lambda^\varepsilon - \widetilde{p}_h(f_\lambda^\varepsilon)\|_{\mathrm{L}^2(Q_T)} \leq Ch^2 \|p_\lambda^\varepsilon\|_{\mathrm{H}^s(Q_1\cup Q_2)}.$$

On the other hand, we apply the technique as in (3.41), and the inequality (3.51) to obtain

$$\begin{aligned} \|\widetilde{p}_h(f_\lambda^\varepsilon) - p_h(f_\lambda^\varepsilon)\|_{\mathrm{L}^2(Q_T)} &\leq \|\widetilde{p}_h(f_\lambda^\varepsilon) - p_h(f_\lambda^\varepsilon)\|_* \\ &\leq C\left(\|u_\lambda^\varepsilon - u_h(f_\lambda^\varepsilon)\|_{\mathrm{L}^2(\omega_T)} + \|u^* - u_h^*\|_{\mathrm{L}^2(\omega_T)} \right) \\ &\leq Ch^2 \left(\|u_\lambda^\varepsilon\|_{\mathrm{H}^s(Q_1\cup Q_2)} + \|u^*\|_{\mathrm{H}^s(Q_1\cup Q_2)} \right). \end{aligned} \quad (3.52)$$

The proof is complete. □

We arrive at the first main result of this section by combining Lemmas 3.9, 3.10, and 3.11 with the triangle inequality.

Theorem 3.12. *Let* $(u_\lambda^\varepsilon, p_\lambda^\varepsilon, f_\lambda^\varepsilon) \in \mathrm{X}_0 \times \mathrm{X}_T \times F_+$ *and* $\left(u_{\lambda,h}^\varepsilon, p_{\lambda,h}^\varepsilon, f_{\lambda,h}^\varepsilon\right) \in \mathrm{X}_{h,0} \times \mathrm{X}_{h,T} \times F_+$ *be the solutions of the problems* (3.10)-(3.12) *and* (3.34)-(3.36) *in case of variational discretization, respectively. Let* $u^* \in \mathrm{X}_0$ *be the solution of the problem* (3.2)*. Assume that the assumptions of Lemma 2.11 and Lemma 3.7c are satisfied. Then, the following estimates hold*

$$\left\|\left|u_\lambda^\varepsilon - u_{\lambda,h}^\varepsilon\right\|\right|_* + \left\|\left|p_\lambda^\varepsilon - p_{\lambda,h}^\varepsilon\right\|\right|_* + \left\|f_\lambda^\varepsilon - f_{\lambda,h}^\varepsilon\right\|_{\mathrm{L}^2(Q_T)} \leq$$

$$\le C\left(1+\frac{1}{\lambda}\right)h^2\left(\|u_\lambda^\varepsilon\|_{\mathrm{H}^s(Q_1\cup Q_2)}+\|u^*\|_{\mathrm{H}^s(Q_1\cup Q_2)}\right)+Ch\,\|u_\lambda^\varepsilon\|_{\mathrm{H}^s(Q_1\cup Q_2)}$$
$$+C\left(1+\frac{1}{\lambda}\right)h\,\|p_\lambda^\varepsilon\|_{\mathrm{H}^s(Q_1\cup Q_2)},$$

and

$$\left\|u_\lambda^\varepsilon-u_{\lambda,h}^\varepsilon\right\|_{\mathrm{L}^2(Q_T)}+\left\|p_\lambda^\varepsilon-p_{\lambda,h}^\varepsilon\right\|_{\mathrm{L}^2(Q_T)}+\left\|f_\lambda^\varepsilon-f_{\lambda,h}^\varepsilon\right\|_{\mathrm{L}^2(Q_T)}\le$$
$$\le C\left(1+\frac{1}{\lambda}\right)h^2\left(\|u_\lambda^\varepsilon\|_{\mathrm{H}^s(Q_1\cup Q_2)}+\|u^*\|_{\mathrm{H}^s(Q_1\cup Q_2)}+\|p_\lambda^\varepsilon\|_{\mathrm{H}^s(Q_1\cup Q_2)}\right). \tag{3.53}$$

Proof. Let us sketch the proof of the first estimate. Thanks to the triangle inequality, a priori error estimate (2.24), Lemmas 3.9 and 3.10, we have

$$\begin{aligned}
&\left|\left|\left|u_\lambda^\varepsilon-u_{\lambda,h}^\varepsilon\right|\right|\right|_*+\left|\left|\left|p_\lambda^\varepsilon-p_{\lambda,h}^\varepsilon\right|\right|\right|_*+\left\|f_\lambda^\varepsilon-f_{\lambda,h}^\varepsilon\right\|_{\mathrm{L}^2(Q_T)}\le\\
&\le |||u_\lambda^\varepsilon-u_h(f_\lambda^\varepsilon)|||_*+\left|\left|\left|u_h(f_\lambda^\varepsilon)-u_{\lambda,h}^\varepsilon\right|\right|\right|_*+|||p_\lambda^\varepsilon-p_h(f_\lambda^\varepsilon)|||_*\\
&\qquad+\left|\left|\left|p_h(f_\lambda^\varepsilon)-p_{\lambda,h}^\varepsilon\right|\right|\right|_*+\left\|f_\lambda^\varepsilon-f_{\lambda,h}^\varepsilon\right\|_{\mathrm{L}^2(Q_T)}\\
&\le Ch\,\|u_\lambda^\varepsilon\|_{\mathrm{H}^s(Q_1\cup Q_2)}+C\left\|f_\lambda^\varepsilon-f_{\lambda,h}^\varepsilon\right\|_{\mathrm{L}^2(Q_T)}+|||p_\lambda^\varepsilon-p_h(f_\lambda^\varepsilon)|||_*\\
&\le Ch\,\|u_\lambda^\varepsilon\|_{\mathrm{H}^s(Q_1\cup Q_2)}+C\left(1+\frac{1}{\lambda}\right)|||p_\lambda^\varepsilon-p_h(f_\lambda^\varepsilon)|||_*.
\end{aligned}$$

The conclusion follows from the estimate (3.46). The inequality (3.53) is proved similarly. □

Finally, from the inequality (3.37), Lemma 3.6, and the inequality (3.53) in the previous theorem, we have the following result:

Corollary 3.13. *Let $f_+\in F_+$ be the F_+-best approximated solution of the problem* (3.4) *and $f_{\lambda,h}^\varepsilon\in F_+$ be the solution of the problem* (3.31)*. Assume that the assumptions of Theorem 2.11, Lemma 3.6 and Lemma 3.7c are satis-*

fied. There holds the following estimate

$$\begin{aligned}\left\|f_{+}-f_{\lambda,h}^{\varepsilon}\right\|_{\mathrm{L}^2(Q_T)} &\leq \sqrt{\lambda}\,\|\xi\|_{\mathrm{L}^2(\omega_T)}+\frac{\varepsilon}{\sqrt{\lambda}}\\ &+C\left(1+\frac{1}{\lambda}\right)h^2\left(\|u_{\lambda}^{\varepsilon}\|_{\mathrm{H}^s(Q_1\cup Q_2)}+\|u^*\|_{\mathrm{H}^s(Q_1\cup Q_2)}+\|p_{\lambda}^{\varepsilon}\|_{\mathrm{H}^s(Q_1\cup Q_2)}\right).\end{aligned}$$

Moreover, if $\lambda=\mathcal{O}\left(h^{\frac{4}{3}}+\varepsilon\right)$ *then* $f_{\lambda,h}^{\varepsilon}\to f_{+}$ *in* $\mathrm{L}^2\left(Q_T\right)$ *as* $\lambda\to 0$ *with the convergence rate* $\mathcal{O}\left(h^{\frac{2}{3}}+\varepsilon^{\frac{1}{2}}\right)$.

Conclusion and perspectives

Conclusion

We presented the interface-fitted space-time method for the advection-diffusion equation with a moving interface. We showed two optimal order a priori error estimates under some appropriate conditions.

After that, we derived the error and convergence estimates of an inverse source problem governed by an advection-diffusion problem with moving subdomains. The regularized state and adjoint were treated by the interface-fitted space-time method. The regularized source was discretized by using the variational approach. We established the optimal order error estimates of the regularized source, state, and adjoint in two norms. Furthermore, we suggest a priori choice for λ such that $f^{\varepsilon}_{\lambda,h} \to f_+$ in $\mathrm{L}^2\left(Q_T\right)$ as $\lambda \to 0$. The convergence rate was derived in that case.

Future work

In the future, we will extend the presented results to the case of three-dimensional space. Another direction is to develop a priori error estimates in which the parameter λ appears on the numerator of the fractions on the right-hand sides, as in [35], to get a higher convergence rate with respect to h in corollary 3.13.

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