MINISTRY OF EDUCATION AND TRAINING

VIETNAM ACADEMY OF SCIENCE AND TECHNOLOGY

GRADUATE UNIVERSITY OF SCIENCE AND TECHNOLOGY

Do Hoang Viet

NORMAL MONOMIAL IDEALS VERSUS NORMAL POLYTOPES

MASTER THESIS IN MATHEMATICS

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Ngành: Đại số và Lý thuyết số
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NGƯỜI HƯỚNG DẪN KHOA HỌC:
GS. TSKH. NGÔ VIỆT TRUNG

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Declaration

I declare that this thesis titled "Normal monomial ideals versus normal polytopes" is entirely my own work and has not been previously included in a thesis or dissertation submitted for a degree or any other qualification in this graduate university or any other institutions. I will take responsibility for the above declaration.

Hanoi, September 2024

Signature of Student

Do Hoang Viet

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Contents

List of symbols

Symbol	Meaning
I	a (monomial) ideal
$[[I]]$	the set of monomials in I
$\overline{I}$	the integral closure of I
$N(I)$	Newton polyhedra of I
$N^*(I)$	the "dual" Newton polytope of I
$\mathcal{A}$	a set of integral points
$\mathbb{N}\mathcal{A}$	the monoid generated by $\mathcal{A}$
$I_{\mathcal{A}}$	the toric ideal of $\mathcal{A}$
$K[\mathcal{A}]$	the toric ring of $\mathcal{A}$
$\mathrm{Proj}K[\mathcal{A}]$	the set of all homogeneous prime ideals of $K[\mathcal{A}]$ but not the maximal homogeneous ideal
$\overline{S}$	the normalization of the monoid S in $\mathbb{Z}^n$
$\overline{K[S]}$	the integral closure of the ring $K[S]$
$R(I)$	the Rees Algebra of I
M	a matrix
$\nu^*(\mathbf{w}, M)$	$\max\{\mathbf{1}\cdot\mathbf{y} \mid \mathbf{y}\in\mathbb{R}^m_{\geq 0}, M\mathbf{y}^T \leq \mathbf{w}^T\}$
$\nu(\mathbf{w}, M)$	$\max\{\mathbf{1}\cdot\mathbf{y} \mid \mathbf{y}\in\mathbb{N}^m, M\mathbf{y}^T \leq \mathbf{w}^T\}$
$\tau^*(\mathbf{w}, M)$	$\min\{\mathbf{1}\cdot\mathbf{y} \mid \mathbf{y}\in\mathbb{R}^m_{\geq 0}, M\mathbf{y}^T \geq \mathbf{w}^T\}$
$\tau(\mathbf{w}, M)$	$\min\{\mathbf{1}\cdot\mathbf{y} \mid \mathbf{y}\in\mathbb{N}^m, M\mathbf{y}^T \geq \mathbf{w}^T\}$
$\mathrm{vol}(\Delta)$	the (relative) volume of Δ

Introduction

Let I be a monomial ideal in $R = K[x_1, ..., x_n]$. In Commutative Algebra, the normality of I plays an important role in the research into various algebraic characteristic of this monomial ideal. Therefore, the method for verifying the normality of a monomial ideal receives a lot of considerations of experts; and this is the motivation of this dissertation.

Thanks to various prior researches, we have two following valuable results that are so meaningful in theoretical arguments.

- The Lejeune-Teissier's Criterion (Proposition 1.2.3): I is integrally closed if and only if I has the property: $\mathbf{x}^{t\mathbf{a}} \in I^t$ implies $\mathbf{x}^{\mathbf{a}} \in I$;
- The Reid-Roberts-Vitulli Criterion (Proposition 1.4.2): I is normal if and only if I^t is integrally closed for $t = 1, ..., n-1$.

However, two above criterion are not effective in the sense that there is no algorithm to test this property in terms of the generators of I. Clearly, we cannot verify all of monomials in R by the Lejeune-Teissier's Criterion. This motivated us to find an useful sufficient condition for the normality of the monomial ideal I. In this dissertation, we show a criterion (Proposition 4.3.1) from Discrete Geometry. Indeed, we only can find a sufficient condition for the normality of I because all necessary and sufficient criteria must be equivalent to the Reid-Roberts-Vitulli Criterion that is impossible to utilize. Meanwhile, the Proposition 4.3.1 is a geometric property that can be verified by some elementary techniques. Now, we go through the brief process to achieve Proposition 4.3.1

In this dissertation, Proposition 2.5.2 shows that the normality of I is equivalent to the normality of the Rees ring $R[It]$. This ring is a toric ring of the form $K[M]$ where M is a set of monomials. Let E denote the set of the exponents of M. The second Lejeune-Teissier's Criterion can illustrate the relation between the normality of $R[It]$ and geometric concepts.

Lejeune-Teissier's Criterion (Proposition 2.4.2): Let S be the additive semigroup generated by the exponents of E. Then $K[M]$ is normal iff S is normal, i.e. $S = C(E) \cap G(E)$, where $C(E)$ is the cone spanned by E and $G(E)$ is the additive group generated by S.

If $K[M]$ is a standard graded toric ring, we may assume that the generator of E lies on a hyperplane. Let P denote the convex polytope spanned by E. We say that P is normal if S is normal (Definition 4.2.1). Let tP be the polytopes spanned by tE. We can illustrate the normality of P geometrically by following theorem.

The Bruns-Gubeladze-Trung Theorem ([1]): P is normal if and only if tP is full (i.e. tP contains all lattice points inside) for $t = 1, ..., n-1$.

Finally, thanks to various results in this dissertation, we can reach the main result of this dissertation (Proposition 4.3.1).

To be more detailed, in the Chapter 3, we will associate with I a lower comprehensive polytope P such that I is normal if and only if P is normal (Proposition 3.8.1). This plays the role as techniques for proving propositions in the last chapter. Moreover, also in this chapter, we mentions concepts of Linear Programming to illustrate the relation between three mathematical areas: Commutative Algebra, Discrete Geometry and Linear Programming.

This thesis is divided into four chapters.

1. Chapter 1 introduces necessary points in Commutative Algebra that will be utilized frequently later. In particular, the concept of the normality of monomial ideals will play the kernel role in this dissertation.

2. Chapter 2 mentions the main algebraic objects, such as toric rings and Rees algebra. They will play as equipment in the proofs later.

3. Chapter 3 shows the relation between algebraic properties and concepts in Linear Programming and Discrete Geometry: integer rounding properties of matrices, the integral decomposition property of polytopes and the normality of monomial ideals.

4. Chapter 4 contains the definition of unimodular covering and its role in the main result of this thesis.

Chapter 1

Normal Monomial Ideals

1.1 Basic definitions

In the dissertation, we do not recall the basic algebraic concepts, such as polynomial rings and ideals. The readers can find their definitions and properties in [2]. We start the dissertation with monomial ideals of a polynomial ring.

Let $K[x_1, \ldots, x_n]$ be the polynomial ring over an algebraically closed field K . Recall that monomials in this ring are the products of an element in K (is called coefficient) and some variables (with multiplicities), such as $cx_1^{a_1} \cdots x_n^{a_n}$. In this form, $c \in K$ is the coefficient of the monomial and the multiplicity of x_i is $a_i \in \mathbb{N}$ for every $1 \leq i \leq n$. In order to be simple, we write the monomial $x_1^{a_1} \cdots x_n^{a_n}$ by $\mathbf{x}^{\mathbf{a}}$, where $\mathbf{a} = (a_1, \ldots, a_n)$. It is routine to verify that $\mathbf{x}^{\mathbf{a}} \cdot \mathbf{x}^{\mathbf{b}} = \mathbf{x}^{\mathbf{a}+\mathbf{b}}$, so the expression is similar to the ordinary way of writing monomials. Two non-zero monomials $c\mathbf{x}^{\mathbf{a}}$ and $c'\mathbf{x}^{\mathbf{b}}$ are *similar* if they share the same variable term, i.e, $\mathbf{a} = \mathbf{b}$, such as $2x_1x_2^2$ and $-3x_1x_2^2$.

An expression of a polynomial as a sum of non-similar monomials is called the *polynomial expression* of this polynomial. To illustrate, the following first and second expressions are the non-polynomial and polynomial ones respectively

$$P(x) = x^2 + 3x + 2x - 4 + 2x^2 = 3x^2 + 5x - 4.$$

The *terms* of a polynomial are defined to be monomials in its polynomial expression, such as $3x^2, 5x$ and -4 in case of the above polynomial $P(x)$. As for an ideal in the ring, we call a set as its *system of generators* if the ideal can be generated by the set. We come to the first definition.

Definition 1.1.1. Let $K[x_1, \ldots, x_n]$ be the polynomial ring over a field K. An ideal I of this ring is called to be a *monomial ideal* if I is generated by monomials.

More exactly, thanks to Hilbert's Basis Theorem or Dickson's Lemma [3] (Corollary 1.17 and Theorem 1.2 respectively), every ideal of a polynomial ring is generated by finitely many elements and we usually can find a finite set of generators from every (infinite) system of generators. Therefore, every monomial ideal I is generated by finitely many monomials. Besides, we symbolize $[[I]]$ as the set of all monomials in I, so $I = ([[I]])$. Now, we show some basic properties of monomial ideals, which will be useful in the following chapters.

Proposition 1.1.2. $K[x_1, \ldots, x_n]$ be the polynomial ring over a field K. As for an ideal I of this ring, the following are equivalent:

(i) I is a monomial ideal;

(ii) If a polynomial f belongs to I, every term of f is also an element of I.

Proof. See [4] (Lemma 3, page 71). □

The following proposition allows us to imagine the set of monomials in monomial ideals. Specially, the (iv) of this proposition helps us to find a system of generators of the intersection of two monomial ideals. This is an important characteristic of monomial ideals since it is so difficult to find the system of generators of the intersection of two general ideals of a polynomial ring.

Proposition 1.1.3. Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring; and I and J are monomial ideals. Then

(i) $I \subseteq J$ if and only if $[[I]] \subseteq [[J]]$;

(ii) $I = J$ if and only if $[[I]] = [[J]]$;

(iii) $I + J$ and IJ are also monomial ideals and $[[I + J]] = [[I]] \cup [[J]]$;

(iv) $I \cap J$ is also a monomial ideal and $[[I \cap J]] = [[I]] \cap [[J]]$;

(v) The colon ideal $(I : J) := \{f \in R \mid fJ \subseteq I\}$ is a monomial ideal.

Proof. (i)(ii) It is easy to verify them thanks to the equalities $I = [[I]]R$ and $J = [[J]]R$.

(iii) $I + J$ is a monomial ideal obviously since it is generated by the union of systems of generators of I and J respectively. Moreover, the inclusion $I \cup J \subset I + J$ implies $[[I]] \cup [[J]] \subseteq [[I + J]]$. Conversely, for every monomial $\mathbf{x}^{\mathbf{a}} \in [[I + J]]$, $\mathbf{x}^{\mathbf{a}} \in I + J$

is a sum of two elements of I and J respectively. Since both I and J are monomial ones, $\mathbf{x}^{\mathbf{a}}$ is a sum of some monomials in I or J. Because $\mathbf{x}^{\mathbf{a}}$ is a monomial itself, it must be one among the monomials in this sum, so $\mathbf{x}^{\mathbf{a}}$ belongs to either I or J (or both). This ensures $[[I+J]] \subseteq [[I]] \cup [[J]]$. As for IJ, it is easy to verify it is also a monomial ideal.

(iv) Thanks to Proposition 1.1.2, it is routine to verify that all terms of every polynomial in $I \cap J$ also belong to both I and J, so $I \cap J$ is a monomial ideal. The second statement is also obvious!

(v) For an arbitrary polynomial $f \in (I : J)$, we have $f[[J]] \subseteq I$, so $\mathbf{x}^{\mathbf{a}} f \in I$ for every monomial $\mathbf{x}^{\mathbf{a}}$ in J. For each term $\mathbf{x}^{\mathbf{b}}$ of f, the monomial $\mathbf{x}^{\mathbf{a}} \cdot \mathbf{x}^{\mathbf{b}}$ is also a term of $\mathbf{x}^{\mathbf{a}} f \in I$. Since I is a monomial ideal, $\mathbf{x}^{\mathbf{a}} \cdot \mathbf{x}^{\mathbf{b}} \in I$ for every $\mathbf{x}^{\mathbf{a}} \in J$. Thus, $\mathbf{x}^{\mathbf{b}}[[J]] \subseteq I$, so $\mathbf{x}^{\mathbf{b}} J = \mathbf{x}^{\mathbf{b}}([[J]]) \subseteq I$, or equivalently $\mathbf{x}^{\mathbf{b}} \in (I : J)$ for every term $\mathbf{x}^{\mathbf{b}}$ of f. Thanks to Proposition 1.1.2, $(I : J)$ is a monomial ideal.

□

There are various other properties of monomial ideals, but we only mention some valuable ones that we will use later in this section. The readers can find a full set of nice properties of monomial ideals in [4]. Now, we come to the section of integral closure of a monomial ideal.

1.2 Integral Closure

Throughout the thesis, we write points (vectors in vector spaces) and their coordinates (real or rational numbers) in a bold and an usual way respectively, such as an illustration $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$. Additionally, I denote the zero vector and vector whose coordinates are all 1's by $\mathbf{0}$ and $\mathbf{1}$, respectively. If I write $\mathbf{x} < \mathbf{y}$ ($\mathbf{x} \leq \mathbf{y}$), it means every coordinate of $\mathbf{x}$ is lower than (or equal to) the respective coordinate of $\mathbf{y}$. It is routine to verify that space $\mathbb{R}^n$ with the order is a partially ordered set. We will write $\mathbf{x} \cdot \mathbf{y}$ as the inner product $\mathbf{x}\mathbf{y}^T$, the sum of products of respective coordinates of $\mathbf{x}$ and $\mathbf{y}$. For every number $a \in \mathbb{R}$, we use the symbol $\lfloor a \rfloor$ ($\lceil a \rceil$) as the greatest (lowest) integers that are not larger (lower) than a.

We continue to pay attention to the polynomial ring $R = K[x_1, \ldots, x_n]$ and a homogeneous ideal $I \subset R$.

Definition 1.2.1. An element $z \in K[x_1, \dots, x_n]$ is *integral* over I if z is a root of the following monic polynomial:

$$z^d + c_1 z^{d-1} + \cdots + c_{d-1} z + c_d = 0, \tag{1.2.1.1}$$

for some degree d and $c_i \in I^i$ for every $1 \le i \le d$.

Definition and Proposition 1.2.2. The set of integral elements over I in $K[x_1 \dots, x_n]$ is called *integral closure* of I in the polynomial ring, namely $\overline{I}$. Moreover, it is an ideal of $K[x_1 \dots, x_n]$; and if I is a monomial ideal, $\overline{I}$ is so. The ideal I is *integrally closed* in $K[x_1 \dots, x_n]$ if $\overline{I} = I$.

Proof. See [5] (Proposition 4.3.3 and 12.1.1). □

When I is a monomial ideal, we symbolize $\log(I) := \{\mathbf{a} \in \mathbb{N}^n \mid \mathbf{x}^{\mathbf{a}} \in I\}$, or equivalently, $[[I]] = \{\mathbf{x}^{\mathbf{a}} \mid \mathbf{a} \in \log(I)\}$. At that case, the integral closure of I is of a clear form as the following proposition.

Proposition 1.2.3 (The Lejeune-Teissier's Criterion)**.** For every positive integer k and a monomial ideal I,

$$\overline{I^k} = (\mathbf{x}^{\mathbf{a}} \mid \mathbf{x}^{d\mathbf{a}} \in I^{dk} \text{ for some } d) = (\mathbf{x}^{\mathbf{a}} \mid \mathbf{a} \in \operatorname{conv}\{\log(I^k)\} \cap \mathbb{Z}^n),$$

where $\operatorname{conv}\{\log(I^k)\} = \left\{\sum_{j=1}^t \alpha_j \mathbf{b}_j \mid t \in \mathbb{Z}_+, \alpha_j > 0, \sum_{j=1}^t \alpha_j = 1, \mathbf{b}_j \in \log(I^k)\right\}$ is the convex hull of $\log(I^k)$.

In particular, I is integral closed if and only if $I = (\mathbf{x}^{\mathbf{a}} \mid \mathbf{x}^{d\mathbf{a}} \in I^d \text{ for some } d)$.

Proof. If a monomial $\mathbf{x}^{\mathbf{a}}$ satisfies $\mathbf{x}^{d\mathbf{a}} \in I^{dk}$, then it is a root of the monic polynomial $z^d - \mathbf{x}^{d\mathbf{a}}$, in which $\mathbf{x}^{d\mathbf{a}} \in (I^k)^d$, so $\mathbf{x}^{\mathbf{a}}$ is integral over I^k. Conversely, for every monomial $\mathbf{x}^{\mathbf{a}} \in \overline{I^k}$, $\mathbf{x}^{\mathbf{a}}$ is a root of some monic polynomial

$$z^d + c_1 z^{d-1} + \cdots + c_{d-1} z + c_d = 0,$$

where $c_i \in I^{ik}$. By comparing the terms of exponent $d\mathbf{a}$ in both sides, we only need to pay attention to the $i\mathbf{a}$-exponent term of c_i (since I^{ik} is a monomial ideal, this term belongs to I^{ik}), so we can assume that each $c_i \in I^{ik}$ is either similar to $\mathbf{x}^{i\mathbf{a}}$ or zero for all $1 \le i \le d$. Since $\mathbf{x}^{\mathbf{a}} \neq 0$, there exists some i so that $0 \neq c_i \in I^{ik}$. Thus, $\mathbf{x}^{i\mathbf{a}} \in I^{ik}$. We have prove the equality $\overline{I^k} = (\mathbf{x}^{\mathbf{a}} \mid \mathbf{x}^{d\mathbf{a}} \in I^{dk} \text{ for some } d)$.

Now, if a monomial $\mathbf{x}^{\mathbf{a}}$ satisfies $\mathbf{x}^{d\mathbf{a}} \in I^{dk}$, $\mathbf{x}^{d\mathbf{a}}$ is similar to a product $\mathbf{x}^{\mathbf{a}_1} \cdots \mathbf{x}^{\mathbf{a}_{dk}}$ in which $\mathbf{x}^{\mathbf{a}_i} \in I$. Thus, $d\mathbf{a} = \sum_{i=1}^{dk} \mathbf{a}_i$, so $\mathbf{a} = \frac{1}{d}\sum_{i=1}^{d} \mathbf{b}_i$ where $\mathbf{b}_i := \sum_{j=(i-1)k+1}^{ik} \mathbf{a}_j \in \log(I^k)$. Conversely, for $\mathbf{a} \in \operatorname{conv}\{\log(I^k)\} \cap \mathbb{Z}^n$, clearly $\mathbf{a} = \sum_{i=1}^{t} \alpha_i \mathbf{a}_i$ where $\mathbf{a}_i \in \log(I)$, $\alpha_i > 0$ and $\sum_{i=1}^{t} \alpha_i = k$. Since all of $\mathbf{a}$ and $\mathbf{a}_i$ are integral, the α_i's can be chose as rational numbers. Therefore, there exists a constant d such that $d\alpha_i \in \mathbb{N}$ for all $1 \le i \le t$. Hence, $d\mathbf{a} \in \log(I^{dk})$, or equivalently, $\mathbf{x}^{d\mathbf{a}} \in I^{dk}$. The proof is completed. $\square$

Remark 1.2.4. We do not have the general result $\overline{I \cap J} = \overline{I} \cap \overline{J}$ where I and J are monomial ideals. In particular, we can consider the case $I = (x_1^5, x_2^3)$ and $J = (x_2^5, x_3^3)$ in the polynomial ring $K[x_1, x_2, x_3]$. We have $x_1^2x_2^2x_3^2 \in \overline{I} \cap \overline{J}$. Indeed, by Proposition 1.2.3, the facts that $(2,2,2) > \left(\frac{5}{3}, 2, 0\right) \in \operatorname{conv}\{(5,0,0),(0,3,0)\}$ and $(2,2,2) > \left(0, \frac{5}{3}, 2\right) \in \operatorname{conv}\{(0,5,0),(0,0,3)\}$ imply $x_1^2x_2^2x_3^2$ belongs to both $\overline{I}$ and $\overline{J}$.

Now we find $\overline{I \cap J}$. Initially, we find a system of generators of $I \cap J$. By (iv) of Proposition 1.1.3, we have

$$\begin{aligned}
[[I \cap J]] &= [[I]] \cap [[J]] \\
&= \left(\{\mathbf{x}^{(a,b,c)} \mid a \ge 5\} \cup \{\mathbf{x}^{(a,b,c)} \mid b \ge 3\}\right) \cap \left(\{\mathbf{x}^{(a,b,c)} \mid b \ge 5\} \cup \{\mathbf{x}^{(a,b,c)} \mid c \ge 3\}\right) \\
&= \{\mathbf{x}^{(a,b,c)} \mid a, b \ge 5\} \cup \{\mathbf{x}^{(a,b,c)} \mid b \ge 5\} \cup \{\mathbf{x}^{(a,b,c)} \mid a \ge 5, c \ge 3\} \cup \{\mathbf{x}^{(a,b,c)} \mid b, c \ge 3\} \\
&= \{\mathbf{x}^{(a,b,c)} \mid b \ge 5\} \cup \{\mathbf{x}^{(a,b,c)} \mid a \ge 5, c \ge 3\} \cup \{\mathbf{x}^{(a,b,c)} \mid b, c \ge 3\}.
\end{aligned}$$

Therefore, $I \cap J = (x_1^5x_3^3, x_2^5, x_2^3x_3^3)$. We assume that $x_1^2x_2^2x_3^2 \in \overline{I \cap J}$. By Proposition 1.2.3, there exists $d, u, v, t \in \mathbb{N}$ such that $d(2,2,2) \ge u(5,0,3) + v(0,5,0) + t(0,3,3)$ and $d = u + v + t$. Thus, we obtain a system of linear inequalities

$$\begin{aligned}
2d &\ge 5u \\
2d &\ge 5v + 3t \\
2d &\ge 3u + 3t \\
d &= u + v + t.
\end{aligned}$$

By replacing d by $u + v + t$ in the three above inequalities, we obtain that

$$\begin{aligned}
2u + 2t &\ge 3u \\
2u &\ge 3v + t. \\
2v &\ge u + t
\end{aligned}$$

Combining two last inequalities, we have

$$2u + 2v \ge u + 3v + 2t = -u + 3v + (2u + 2t) \ge -u + 3v + 3u = 2u + 3v.$$

Thus, v must be zero, so u, v, t are all zero, a contradiction! This means that $x_1^2x_2^2x_3^2 \notin \overline{I \cap J}$. To sum up, $\overline{I \cap J} \subsetneq \overline{I} \cap \overline{J}$ generally.

Now, we come to a quantitative result about the integral closure of a monomial ideal. This indicates that the system of generators of integral closure $\overline{I}$ is not so different from that of I in terms of the degree of monomials.

Proposition 1.2.5. If $I \subseteq K[x_1, \ldots, x_n]$ is an ideal generated by monomials of degree at most k, then $\overline{I}$ is generated by monomials of degree at most $n + k - 1$.

Proof. We assume $I = (\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_m})$, in which the sum of all coordinates of each $\mathbf{a}_i$ is k. Set $\mathcal{S} = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$. It is sufficient to show that if $\mathbf{a} \in \mathbb{N}^n$ satisfies $\mathbf{x}^{\mathbf{a}} \in \overline{I}$ and the sum of coordinates of $\mathbf{a}$ is at least $n + k$, then there exists $\mathbb{N}^n \ni \mathbf{b} < \mathbf{a}$ such that $\mathbf{x}^{\mathbf{b}}$ also belongs to $\overline{I}$. Indeed, since $\mathbf{x}^{\mathbf{a}} \in \overline{I}$, $\mathbf{a} \in \operatorname{conv}\{\log(I)\}$ by Proposition 1.2.3. Hence, there exists the convex combination (since all of points in the combination are integral, we can assume all coefficients are rational numbers):

$$\mathbf{a} = \sum_{j=1}^{t} \alpha_j \mathbf{a}'_j,$$

where $\alpha_j \in \mathbb{Q}_+$, $\sum_{j=1}^{t} \alpha_j = 1$, and $\mathbf{a}'_j \in \log(I)$ for all $1 \leq j \leq t$. Because $\mathbf{a}'_j \in \log(I)$, there exists $\mathbf{a}''_j \in \mathcal{S}$ such that $\mathbf{a}''_j \leq \mathbf{a}'_j$ for every $1 \leq j \leq t$. Therefore,

$$\mathbf{a} = \sum_{j=1}^{t} \alpha_j \mathbf{a}'_j \geq \sum_{j=1}^{t} \alpha_j \mathbf{a}''_j =: \mathbf{b}' \in \operatorname{conv}\{\mathcal{S}\} \cap \mathbb{Q}^n \subset \operatorname{conv}\{\log(I)\} \cap \mathbb{Q}^n.$$

It is important to notice that the sum of coordinates of $\mathbf{b}'$ is at most k since $\mathbf{b}'$ is a convex combination of the $\mathbf{a}''_j$'s. Set $\mathbf{b} = \lceil \mathbf{b}' \rceil \in \mathbb{N}^n$. Since $\mathbb{N}^n \ni \mathbf{a} \geq \mathbf{b}'$, $\mathbf{a} \geq \mathbf{b}$. We will show that the point $\mathbf{b}$ is the one we need to find.

Now, we prove that $\mathbf{x}^{\mathbf{b}} \in \overline{I}$. Indeed, we have $d\mathbf{b}' \in \mathbb{N}^n$ where d satisfies $d\alpha_j \in \mathbb{N}$ for all j. At that time, $\mathbf{x}^{d\mathbf{b}'} \in I^d$, which implies $x^{d\mathbf{b}} \in I^d$ as $\mathbf{b} \geq \mathbf{b}'$. By Proposition 1.2.3, $\mathbf{x}^{\mathbf{b}} \in \overline{I}$. Finally, we show that $\mathbf{a} > \mathbf{b}$. We have that every coordinate of $\mathbf{b} - \mathbf{b}'$ is strictly lower than 1, so the sum of coordinates of $\mathbf{b}$ is strictly lower that $n + k$ that is the sum of coordinates of $\mathbf{a}$. Thus, $\mathbf{a} > \mathbf{b}$. The proof is completed. $\square$

In order to imagine more virtually, we need to know the geometric description of monomial ideals and their integral closures.

1.3 The geometric description of the integral closure

Let $I = (\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_m})$ be a monomial ideal of $K[x_1 \dots, x_n]$. We can choose the set $\mathcal{S} := \{\mathbf{a}_1, \dots, \mathbf{a}_m\}$ so that $\mathbf{a}_i \not\leq \mathbf{a}_j$ for all $i \neq j$. Clearly, $\log(I) = \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \leq \mathbf{a}\}$. We come to the following geometric concept.

Definition 1.3.1. The convex hull $\operatorname{conv}\{\log(I)\}$, denoted by $N(I)$, is called *the Newton polyhedron* of I.

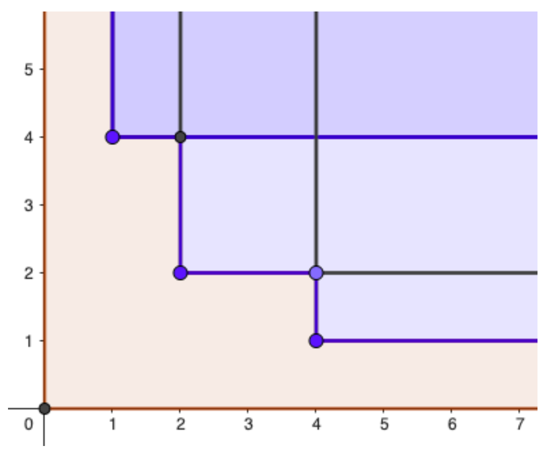
Example 1.3.2. In the polynomial ring $K[x_1, x_2]$, the monomial ideal

$$I = (x_1x_2^4, x_1^2x_2^2, x_1^4x_2)$$

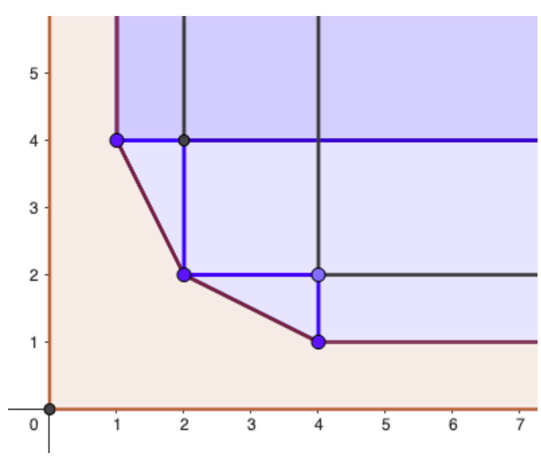
is generated by $\mathbf{x}^{\mathbf{a}_i}$'s, $1 \leq i \leq 3$, where $\mathbf{a}_1 = (1, 4), \mathbf{a}_2 = (2, 2), \mathbf{a}_3 = (4, 1)$. We have

$$[[I]] = \{x_1^a x_2^b \mid \exists i \text{ such that } \mathbf{a}_i \leq (a, b)\},$$

or equivalently, $\log(I) = \{\mathbf{a} \in \mathbb{N}^2 \mid \exists i \text{ such that } \mathbf{a}_i \leq \mathbf{a}\}$. The following picture illustrates $[[I]]$ in the plane.



And the Newton polyhedron of I is the convex hull of $\log(I)$ as follows.



Remark 1.3.3. Every monomial ideal has only one Newton polyhedron, but some different monomial ideals can share a same Newton polyhedron. Particularly, we can see I and $\overline{I}$ have the same Newton polyhedron as the following proposition.

Proposition 1.3.4. In the polynomial ring $K[x_1, \ldots, x_n]$, every monomial ideal I and its integral closure $\overline{I}$ share the same Newton Polyhedron.

Proof. It is sufficient to show that $\text{conv}\{\log(\overline{I})\} \subseteq \text{conv}\{\log(I)\}$. By Proposition 1.2.3, we obtain that $\left[\left[\overline{I}\right]\right] = \{\mathbf{x}^{\mathbf{a}} \mid \mathbf{a} \in \text{conv}\{\log(I)\} \cap \mathbb{N}^n\}$, so $\log\left(\overline{I}\right) = \text{conv}\{\log(I)\} \cap \mathbb{N}^n$, the set of integral points in the Newton polyhedron of I. Thus, $\log(\overline{I}) \subseteq \text{conv}\{\log(I)\}$, which implies $\text{conv}\{\log(\overline{I})\} \subseteq \text{conv}\left\{\text{conv}\{\log(I)\}\right\} = \text{conv}\{\log(I)\}$. The proof is completed. $\square$

Now, we find the geometric version of the proof of Proposition 1.2.5. In that proof, we shown that for a point $\mathbf{a} \in \text{conv}\{\log(I)\}$, $\mathbf{a} \geq \mathbf{b}' \in \text{conv}\{\mathcal{S}\}$ for some $\mathbf{b} \in \mathbb{Q}^n$. Recall that $\mathcal{S}$ is the set of minimal integral points in the Newton Polyhedron $\text{conv}\{\log(I)\}$. Thus, we can obtain that $\mathbf{a}$ is greater than or equal to some point $\mathbf{b} \in \{\lceil \mathbf{b}' \rceil \mid \mathbf{b}' \in \text{conv}\{\mathcal{S}\} \cap \mathbb{Q}^n\}$. As a result, $\overline{I}$ can be generated by monomials $\mathbf{x}^{\lceil \mathbf{b}' \rceil}$'s, where $\mathbf{b}' \in \text{conv}\{\mathcal{S}\}$.

$$\overline{I} = \left(\mathbf{x}^{\lceil \mathbf{b}' \rceil} \mid \mathbf{b}' \in \text{conv}\{\mathcal{S}\}\right).$$

Moreover, we can see that the set of minimal integral points of the Newton Polyhedron $N(I)$ is equal to the set of exponents of monomials in the minimal system of generators of $\overline{I}$. In the other word, if we write $\overline{I} = (\mathbf{x}^{\mathbf{b}_1}, \ldots, \mathbf{x}^{\mathbf{b}_t})$ in which $\mathbf{b}_i \not\leq \mathbf{b}_j$ for all $i \neq j$, then $\{\mathbf{b}_1, \ldots, \mathbf{b}_t\}$ is the set of all minimal integral points of $\{\lceil \mathbf{b}' \rceil \mid \mathbf{b}' \in \text{conv}\{\mathcal{S}\}\}$, and it is also equal to the set of minimal integral points of the Newton Polyhedron of I.

Example 1.3.5. In $K[x_1, x_2]$, the monomial ideal $I = (x_1 x_2^7, x_1^3 x_2^4, x_1^8 x_2)$ has the Newton Polyhedron as follows. By setting $\mathcal{S} = \{(1, 7), (3, 4), (8, 1)\}$, $\text{conv}\{\mathcal{S}\}$ is the green triangle.

The triangle conv$\{\mathcal{S}\}$ has 7 integral points $(1,7),(3,4),(8,1),(2,6),(3,5),(4,4),(5,3)$. But two red points $(3,5)$ and $(4,4)$ are not minimal integral points in the polyhedron since they are greater than $(3,4)$ both. Outside the triangle, the point $(7,2)$ is the upper integral part of $(6.5,2)$ in the triangle. Thus, 6 blue points $(1,7),(3,4),(8,1),(2,6),(5,3),(7,2)$ are all of minimal integral points in the Newton Polyhedron of I. Hence,

$$\overline{I}=(x_1x_2^7,x_1^3x_2^4,x_1^8x_2,x_1^2x_2^6,x_1^5x_2^3,x_1^7x_2^2).$$

1.4 Normal monomial ideals

Let I be a homogeneous ideal of the polynomial ring $K[x_1,\ldots,x_m]$. We come to the following important definition that will be the main object of this dissertation.

Definition 1.4.1. The homogeneous ideal I is called *normal* if all of its powers are integrally closed, i.e, $\overline{I^k}=I^k$ for every $k\in\mathbb{Z}_+$.

In the case where I is a monomial ideal, we have the following result.

Proposition 1.4.2 (The Reid-Roberts-Vitulli Criterion)**.** The monomial ideal I of the polynomial ring $K[x_1,\ldots,x_n]$ is normal if and only if $\overline{I^k}=I^k$ for every $1\le k\le n-1$.

Proof. The main idea of the proof is similar to that of Proposition 1.2.5. By mimicking this method, we can show that $\overline{I^k}=I\overline{I^{k-1}}$ for every $k\ge n$ (see Proposition 1.5.1). Therefore, $\overline{I^k}=I^{k-n+1}\overline{I^{n-1}}$ for every $k\ge n$. At that time, if $\overline{I^{n-1}}=I^{n-1}$, then we can obtain every I^k with $k\ge n$ is integrally closed. More exactly, we can see [6] (Proposition 3.1). $\square$

In order to verify the normality of a monomial ideal, we can use the Propositions 1.2.3 and 1.4.2. However, this method is impossible to utilize because we cannot check all of monomials of $K[x_1, \dots, x_n]$ in the way of Proposition 1.2.3. Therefore, we suggest another method (Proposition 4.3.1) that allows us to run finitely many steps. However, we must agree with the fact that the Proposition 4.3.1 is only a sufficient condition but not a necessary one since a sufficient and necessary property must be equivalent to the Propositions 1.2.3 and 1.4.2 again, so it is still impossible. The way we suggest in this thesis is a result combining various concepts of Linear Programming and Discrete Geometry.

1.5 A small nice extra observation

This part contains a nice exploration of the author of this dissertation. Although it is not so significant, this result is nice because it can be proven by elementary arguments. Initially, we come to the following proposition.

Proposition 1.5.1. Let I be a monomial ideal of the polynomial ring $K[x_1, \dots, x_n]$. Then, $I\overline{I^q} = \overline{I^{q+1}}$ for all $q \geq n-1$.

Proof. Clearly, $I\overline{I^q} \subseteq \overline{I^{q+1}}$ for all q. Thus, we only need to show that $\overline{I^{q+1}} \subseteq I\overline{I^q}$ with $q \geq n-1$. Since they are all monomial ideals, we only pay attention to monomials. For every $\mathbf{x}^{\mathbf{a}} \in \overline{I^{q+1}}$, by Proposition 1.2.3, there exists t such that $\mathbf{x}^{t\mathbf{a}} \in I^{t(q+1)}$. This means that

$$t\mathbf{a} \geq \sum_{i=1}^{n} c_i \mathbf{a}_i,$$

for some positive integers c_i's such that $\sum_{i=1}^{n} c_i = t(q+1)$. Because $q+1 \geq n$, there exists j such that $c_j \geq t$. Therefore,

$$t(\mathbf{a} - \mathbf{a}_j) \geq \sum_{i \neq j} c_i \mathbf{a}_i + (c_j - t)\mathbf{a}_j.$$

Hence, $\mathbf{x}^{t(\mathbf{a}-\mathbf{a}_j)} \in I^{tq}$ which implies $\mathbf{x}^{\mathbf{a}-\mathbf{a}_j} \in \overline{I^q}$, so $\mathbf{x}^{\mathbf{a}} = \mathbf{x}^{\mathbf{a}-\mathbf{a}_j} \cdot \mathbf{x}^{\mathbf{a}_j} \in I\overline{I^q}$. □

Thanks to the Proposition 1.5.1, we can reach the following result. It shows that the normality and integral closedness of a monomial ideal are equivalent in case of two variables.

Proposition 1.5.2. In the polynomial ring $K[x, y]$, a monomial ideal I is normal if and only if it is integrally closed.

Proof. Thanks to the Proposition 1.5.1, $\overline{I^{q+1}} = I\overline{I^q}$ for all $q \geq 1$. If I is integrally closed, i.e, $\overline{I} = I$, then $\overline{I^2} = I\overline{I} = I^2$. then, by induction on q, we can show that $\overline{I^q} = I^q$ for all $q \geq 2$, equivalently, I is normal. The converse statement is obvious due to the definition of the normality. □

Now, we come to the main definition of this section.

Definition 1.5.3. Let I be a monomial ideal of the polynomial ring $K[x_1, \ldots, x_n]$. Then $r(I)$ is defined to be the smallest number q satisfying $I\overline{I^q} = \overline{I^{q+1}}$.

Thanks to the Proposition 1.5.1, we have $r(I) \leq n-1$. Now, we consider a case where $r(I) = n-1$. In this case, I is generated by powers of variables, i.e, $I = (x_1^{a_1}, \ldots, x_n^{a_n})$. We set $\mathbf{a}_i = a_i\mathbf{e}_i$, where $\mathbf{e}_i$ is the i-th standard vector. The following proposition is the main result of this part, in which its solution only consists of simple arguments. Moreover, the presentation of this proposition is also understandable clearly.

Proposition 1.5.4. Let $I = (x_1^{a_1}, \ldots, x_n^{a_n})$ be a monomial ideal. Then, $r(I) = n-1$ if and only if

$$\frac{1}{a_1} + \cdots + \frac{1}{a_n} \leq 1.$$

Proof. We need to find an equivalent condition of the inclusion $I\overline{I^{n-2}} \subsetneq \overline{I^{n-1}}$. This is equivalent to the existence of an element $\mathbf{x^a} \in \overline{I^{n-1}} \setminus I\overline{I^{n-2}}$. By interpreting the fact to the geometric language, we obtain that

$$t\mathbf{a} = \sum_{i=1}^{n} c_i\mathbf{a}_i,$$

where $c_i \in \mathbb{Q}$ and $\sum_{i=1}^{n} c_i \geq t(n-1)$ for some t. Since $\mathbf{x^a} \notin I\overline{I^{n-2}}$, by imitating the above argument, we have all $c_i < t$. Thus, we have every component of $t\mathbf{a}$ is lower than the respective one of $t\sum_{i=1}^{n}\mathbf{a}_i$, or equivalent $\mathbf{a} \leq (a_1 - 1, \cdots, a_n - 1)$. By combining all of inequalities, we obtain that $c_ia_i \leq t(a_i - 1)$ for all i's, so

$$t(n-1) \leq \sum_{i=1}^{n} c_i \leq \sum_{i=1}^{n} \frac{t(a_i - 1)}{a_i},$$

so

$$\frac{1}{a_1} + \cdots + \frac{1}{a_n} \leq 1.$$

Conversely, if $\frac{1}{a_1} + \cdots + \frac{1}{a_n} \leq 1$, then we can see that $\mathbf{x}^{(a_1-1,\cdots,a_n-1)} \in \overline{I^{n-1}}$ since $\mathbf{x}^{t(a_1-1,\cdots,a_n-1)} \in I^{t(n-1)}$, where $t = \prod_{i=1}^n a_i$. Indeed,

$$t(a_1-1,\cdots,a_n-1) = \left(\prod_{i=1}^n a_i\right) \cdot (a_1-1,\cdots,a_n-1) = \sum_{i=1}^n \left(\left(\prod_{j\neq i} a_j\right)(a_i-1)\mathbf{a}_i\right),$$

and

$$\sum_{i=1}^n \left(\left(\prod_{j\neq i} a_j\right)(a_i-1)\right) = \left(\prod_{i=1}^n a_i\right) \cdot \sum_{i=1}^n \left(1 - \frac{1}{a_i}\right) \geq (n-1)\prod_{i=1}^n a_i = t(n-1).$$

However, it is easy to verify that $\mathbf{x}^{(a_1-1,\cdots,a_n-1)} \notin I\overline{I^{n-2}}$ simply because it cannot be an element of I. Thus $I\overline{I^{n-2}} \subsetneq \overline{I^{n-1}}$. The proof is completed. □

Now, we end the first chapter of this dissertation. In the following chapters, we will consider wider picture of geometric and algebraic concepts. Thanks to them, we will understand the role of the normality of monomial ideals more deeply.

Chapter 2

General toric varieties

In this chapter, K is still an algebraically closed field. For every subset $S \subseteq \mathbb{Z}^d$, we denote by $C(S)$ the positive cone generated by S,

$$C(S) = \{\lambda_1 s_1 + \ldots \lambda_t s_t \mid \text{ for all } t \in \mathbb{N} \text{ and } \lambda_i \in \mathbb{R}_{\geq 0}, s_i \in S\}.$$

In this dissertation, we only consider *polyhedral cones*, the cones are generated by finitely many integral points. Equivalently, every polyhedral cone is an intersection of finitely many halfspaces. For two subsets A and B in $\mathbb{Z}^d$, we write $A \pm B = \{\mathbf{a} \pm \mathbf{b} \mid \mathbf{a} \in A, \mathbf{b} \in B\}$ and denote by cA and $c \cdot A$ the sum of c copies of A and $\{c\mathbf{a}, \mathbf{a} \in A\}$ respectively. A cone C is called *strongly convex* if $C \cap (-C) = \{0\}$ (see [7] (Proposition 1.2.12)).

In this chapter, the section 2.1 mentions the general definitions of toric rings. The sections 2.2 and 2.3 are additional ones that illustrate the relation between algebraic concepts and geometric properties. Although the sections are not included in the main flow of this thesis, we can obtain an interesting perspective on the motivation of this thesis. Section 2.4 will be a simpler case of general concepts in Section 2.1 and it is a preparation for the principle of this dissertation in the following chapters.

2.1 General toric rings and varieties

In this thesis, we will define toric rings and toric varieties via affine semigroups (indeed they are monoids). There are some equivalent concepts of toric varieties (see [7] (Theorem 1.1.16)), but from the perspective on affine semigroups, we have the best method of expressing the them in a combinatorial way.

Initially, we fix a subset $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ of $\mathbb{Z}^n$. Each vector $\mathbf{a}_i$ is identified with a monomial $\mathbf{t}^{\mathbf{a}_i}$ in the Laurent polynomial ring $K[\mathbf{t}^{\pm 1}] := K[t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1}]$. Consider the semigroup homomorphism

$$\pi : \mathbb{N}^m \to \mathbb{Z}^n$$
$$\mathbf{u} = (u_1, \ldots, u_m) \mapsto u_1\mathbf{a}_1 + \cdots + u_m\mathbf{a}_m.$$

The image of π is the semigroup

$$\mathbb{N}\mathcal{A} = \{\lambda_1\mathbf{a}_1 + \cdots + \lambda_n\mathbf{a}_m \mid \lambda_i \in \mathbb{N}\}.$$

The map π lifts to a homomorphism of semigroup algebra:

$$\hat{\pi} : K[x_1, \ldots, x_m] \to K[\mathbf{t}^{\pm 1}]$$
$$x_i \mapsto \mathbf{t}^{\mathbf{a}_i}.$$

Definition 2.1.1. The kernel of $\hat{\pi}$, denoted by $I_{\mathcal{A}}$, is called the *toric ideal* of $\mathcal{A}$.

Remark 2.1.2. The set of the exponent of monomials in the image of $\hat{\pi}$ is indeed $\mathbb{N}\mathcal{A}$. Moreover, $I_{\mathcal{A}}$ is a prime ideal since $K[\mathbf{t}^{\mathbf{a}} \mid \mathbf{a} \in \mathbb{N}\mathcal{A}]$ is an integral domain, and hence the affine variety $X_{\mathcal{A}}$ of zeros of $I_{\mathcal{A}}$ in K^m is irreducible. Clearly, $X_{\mathcal{A}}$ is the Zariski closure of the set of points $(\mathbf{p}^{\mathbf{a}_1}, \ldots, \mathbf{p}^{\mathbf{a}_m}) \in K^m$, where $\mathbf{p} \in (K \setminus \{0\})^n$ (the notion $\mathbf{p}^{\mathbf{a}}$ means $p_1^{a_1} \cdots p_n^{a_n} \in K$ when $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{a} = (a_1 \ldots, a_n)$).

Definition 2.1.3. A variety of the form $X_{\mathcal{A}}$ is an *affine toric variety* . The K-algebra $K[\mathbf{t}^{\mathbf{a}} \mid \mathbf{a} \in \mathbb{N}\mathcal{A}]$ is called the *toric ring* generated by $\mathcal{A}$.

Remark 2.1.4. Indeed, a toric ring is the quotient ring respect to a toric ideal $I_{\mathcal{A}}$, and it is also the coordinate ring of the affine toric variety $X_{\mathcal{A}}$. Moreover, clearly the toric ring $K[\mathbf{t}^{\mathbf{a}} \mid \mathbf{a} \in \mathbb{N}\mathcal{A}]$ is exactly $K[\mathbf{t}^{\mathbf{a}_1}, \ldots, \mathbf{t}^{\mathbf{a}_m}]$. Hence toric rings are algebras generated by finite sets of monomials.

We denote by rank$\mathcal{A}$ the rank of the $n \times m$ matrix whose columns are vectors $\mathbf{a}_i$ for $1 \leq i \leq n$.

Proposition 2.1.5. With the above notion, rank$\mathcal{A}$ is equal to the Krull dimension of the coordinate ring $K[x_1, \ldots, x_m]/I_{\mathcal{A}} \cong K[\mathbf{t}^{\mathbf{a}} \mid \mathbf{a} \in \mathbb{N}\mathcal{A}]$.

Proof. See [8] (Lemma 4.2). □

Proposition 2.1.6. $I_{\mathcal{A}}$ is homogeneous if and only if there exists $\mathbf{0} \neq \mathbf{w} \in \mathbb{Q}^n$ such that $\langle \mathbf{w}, \mathbf{a}_i \rangle = 1$ for all i. This means that all $\mathbf{a}_i$'s are in a hyperplane.

Proof. See [8] (Lemma 4.14). □

In the case where $I_{\mathcal{A}}$ is homogeneous, the toric ideal $I_{\mathcal{A}}$ of such a subset $\mathcal{A}$ defines an affine toric variety $X_{\mathcal{A}}$ in K^m and a projective toric variety $Y_{\mathcal{A}}$ in $\mathbb{P}_k^{m-1}$. Moreover, $Y_{\mathcal{A}}$ has an open cover consisting of the affine toric varieties $X_{\mathcal{A}-\mathbf{a}_i}$, where $\mathbf{a}_i$ runs over the vertices of the polytope conv($\mathcal{A}$) (see [8] (Lemma 13.10)). In Proposition 2.2.2, we will characterize the smoothness of $Y_{\mathcal{A}}$ via these $X_{\mathcal{A}-\mathbf{a}_i}$'s. An affine semisubgroup S in $\mathbb{Z}^n$ is called a *graded affine semigroup* if $S = \mathbb{N}\mathcal{A}$ for some finite subset $\mathcal{A}$ in a hyperplane. In this case, the toric ring $K[\mathbf{t}^{\mathbf{a}} \mid \mathbf{a} \in \mathbb{N}\mathcal{A}]$ is called a *graded affine semigroup ring*.

2.2 Smooth toric varieties and regular rings

In this section, we only pay attention to the case where $I_{\mathcal{A}}$ is homogeneous and will characterize the smoothness of a projective toric variety in a combinatorial way. Let S be a graded affine semisubgroup of $\mathbb{Z}^n$ such that $C(S)$ is strongly convex. This means that S is finitely generated, namely $S = \mathbb{N}\mathbf{a}_1 + \cdots + \mathbb{N}\mathbf{a}_m$, and S contains the neutral element $\mathbf{0} \in \mathbb{Z}^n$. Without loss of generality, we choose $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ to be the *Hilbert basis* of (S), i.e., the (unique) minimal generating set of the semigroup S, and $\mathbf{a}_1, \ldots, \mathbf{a}_t$ are vertices of conv($\mathcal{A}$). The algorithm for computing the Hilbert basis was shown in [8] (page 128)). Since S is graded, S has a generating set in a hyperplane, and it is routine to verify that $\mathcal{A}$ is also a subset of this hyperplane. Clearly, $C(S)$ is spanned by t vectors $\mathbf{a}_1, \ldots, \mathbf{a}_t$. Now, we will characterize the smoothness of the projective toric variety $Y_{\mathcal{A}}$ in terms of vertices $\mathbf{a}_1, \ldots, \mathbf{a}_t$.

Because $\mathcal{A}$ is a subset of a hyperplane, there exists a vector $\mathbf{w} \in \mathbb{Q}^n$ such that $\langle \mathbf{w}, \mathbf{a}_i \rangle = 1$ for all $1 \leq i \leq m$. In the affine semigroup ring $K[\mathcal{A}] := K[\mathbf{t}^{\mathbf{a}} \mid \mathbf{a} \in S] \subset K[\mathbf{t}^{\pm 1}]$, we set $\deg(\mathbf{t}^{\mathbf{a}}) = \langle \mathbf{w}, \mathbf{a} \rangle$ for every $\mathbf{t}^{\mathbf{a}} \in K[\mathcal{A}]$. It is routine to show that $K[\mathcal{A}]$ is a positively graded ring with respect to the this setting. In this chapter, we use this grade for $K[\mathcal{A}]$.

We denote the set consisting of all homogeneous prime ideals of $K[\mathcal{A}]$ but not the maximal ideal $(\mathbf{t}^{\mathbf{a}_i}, 1 \leq i \leq m)$ by Proj$K[\mathcal{A}]$. For every $\mathcal{P} \in$ Proj$K[\mathcal{A}]$, the localization $K[\mathcal{A}]_{\mathcal{P}}$ is graded naturally. We denote its 0-th component by $K[\mathcal{A}]_{(\mathcal{P})}$ and call it the *homogeneous localization* of $K[\mathcal{A}]$ with respect to $\mathcal{P}$. To be more detailed, $K[\mathcal{A}]_{(\mathcal{P})}$ consists of forms $\frac{f}{g}$, where $f, g \in K[\mathcal{A}]$ and $g \notin \mathcal{P}$, such that $\deg(f) = \deg(g)$. It is routine to verify that $K[\mathcal{A}]_{(\mathcal{P})}$ is a local ring with the maximal ideal $K[\mathcal{A}]_{(\mathcal{P})} \cap \mathcal{P}K[\mathcal{A}]_{\mathcal{P}}$ for every $\mathcal{P} \in$ Proj$K[\mathcal{A}]$.

Definition 2.2.1. The projective variety $Y_{\mathcal{A}}$ is called *smooth* if the homogeneous local ring $K[\mathcal{A}]_{(\mathcal{P})}$ is a regular local ring for every $\mathcal{P} \in \mathrm{Proj}K[\mathcal{A}]$.

For a homogeneous element $f \in K[\mathcal{A}]$, the localization $K[\mathcal{A}]_f$ at the set $\{1, f, f^2, \dots\}$ is a $\mathbb{Z}$-graded $K[\mathcal{A}]$-module. We also call its 0-th component the *homogeneous localization* and denote by $K[\mathcal{A}]_{(f)}$. We prove the following result.

Proposition 2.2.2. With the above assumption, $Y_{\mathcal{A}}$ is smooth if and only if $K[\mathcal{A}]_{(\mathbf{t}^{\mathbf{a}_i})}$ is a regular ring for all $1 \le i \le t$.

Proof. Since $\mathbf{a}_1, \dots, \mathbf{a}_t$ are vertices of $\mathrm{conv}(\mathcal{A})$, every $\mathbf{a}_i$ is a convex combination of $\mathbf{a}_1, \dots, \mathbf{a}_t$. Because all $\mathbf{a}_i$'s are integral, the convex combination has rational coefficients. Therefore, $c \cdot \mathcal{A} \subseteq \mathbb{N}\mathbf{a}_1 + \cdots + \mathbb{N}\mathbf{a}_t$ for some $c \in \mathbb{N}$. Thus, $(\mathbf{t}^{\mathbf{a}_1}, \dots, \mathbf{t}^{\mathbf{a}_m}) = \sqrt{(\mathbf{t}^{\mathbf{a}_1}, \dots, \mathbf{t}^{\mathbf{a}_t})}$.

For an arbitrary ideal $\mathcal{P} \in \mathrm{Proj}K[\mathcal{A}]$, if $\mathbf{t}^{\mathbf{a}_i} \in \mathcal{P}$ for all $1 \le i \le t$, then $(\mathbf{t}^{\mathbf{a}_1}, \dots, \mathbf{t}^{\mathbf{a}_m}) \subseteq \mathcal{P}$, a contradiction to the assumption that $\mathcal{P} \in \mathrm{Proj}K[\mathcal{A}]$. Therefore, each $\mathcal{P} \in \mathrm{Proj}K[\mathcal{A}]$ does not contain $\mathbf{t}^{\mathbf{a}_i}$ for some $1 \le i \le t$, so $\mathcal{P}K[\mathcal{A}]_{\mathbf{t}^{\mathbf{a}_i}}$ is a prime ideal of $K[\mathcal{A}]_{\mathbf{t}^{\mathbf{a}_i}}$. Thus, $\mathcal{P}K[\mathcal{A}]_{\mathbf{t}^{\mathbf{a}_i}} \cap K[\mathcal{A}]_{(\mathbf{t}^{\mathbf{a}_i})} =: \mathcal{P}'$ is a prime ideal of $K[\mathcal{A}]_{(\mathbf{t}^{\mathbf{a}_i})}$. Moreover, it is routine to verify that every homogeneous prime ideal $\mathcal{P}'$ of $K[\mathcal{A}]_{(\mathbf{t}^{\mathbf{a}_i})}$ is of this form where $\mathcal{P} = \mathcal{P}'K[\mathcal{A}]_{\mathbf{t}^{\mathbf{a}_i}}$. By the definition of a regular ring, the condition that $K[\mathcal{A}]_{(\mathbf{t}^{\mathbf{a}_i})}$ is regular is equivalent to the requirement $(K[\mathcal{A}]_{(\mathbf{t}^{\mathbf{a}_i})})_{\mathcal{P}'} \cong K[\mathcal{A}]_{(\mathcal{P})}$ is a regular local ring for all prime ideal $\mathcal{P}'$ of $K[\mathcal{A}]_{(\mathbf{t}^{\mathbf{a}_i})}$. This means that $Y_{\mathcal{A}}$ is smooth if and only if $K[\mathcal{A}]_{(\mathbf{t}^{\mathbf{a}_i})}$ is a regular ring for all $1 \le i \le t$. The proof is completed. □

It is routine to verify that $K[\mathcal{A}]_{(\mathbf{t}^{\mathbf{a}_i})} = K[\mathbf{t}^{\mathbf{a}_j - \mathbf{a}_i}, 1 \le j \le m, j \ne i] = K[\mathcal{A} - \mathbf{a}_i]$. Since each $\mathbf{a}_i$, $1 \le i \le t$, is a vertex of $\mathrm{conv}(\mathcal{A})$, the cone $C(\mathcal{A} - \mathbf{a}_i)$ generated by $\mathbf{a}_j - \mathbf{a}_i$, $j \ne i$, is strongly convex. By the exercise 6.1.11 in [9], the condition that $K[\mathcal{A} - \mathbf{a}_i]$ is regular is equivalent to $\mathbb{N}(\mathcal{A} - \mathbf{a}_i) \cong \mathbb{N}^{\mathrm{rank}\mathcal{A}-1}$.

For $\mathbf{a} \in \{\mathbf{a}_1, \dots, \mathbf{a}_t\}$, because $\mathbf{a} - \mathbf{a} = \{\mathbf{a}_j - \mathbf{a}, 1 \le j \le m, \mathbf{a}_j \ne \mathbf{a}\}$ is a generating set of $\mathbb{N}(\mathcal{A} - \mathbf{a})$, there exist $\mathbf{a}'_1, \dots, \mathbf{a}'_{t-1} \in \mathcal{A}$ such that

$$\begin{aligned} \mathbb{N}^{\mathrm{rank}\mathcal{A}-1} &\leftrightarrow \mathbb{N}(\mathcal{A} - \mathbf{a}) \\ \mathbf{e}_j &\mapsto \mathbf{a}'_j - \mathbf{a}, \end{aligned} \tag{2.2.2.1}$$

where $\mathbf{e}_j$ is the j-th standard unit vector for $1 \le j \le \mathrm{rank}\mathcal{A} - 1$. This means that for every $\mathbf{a}_j \in \mathcal{A} \setminus \{\mathbf{a}\}$, $\mathbf{a}_j - \mathbf{a}$ is a sum of vectors belonging to $\{\mathbf{a}'_1 - \mathbf{a}, \dots, \mathbf{a}'_{t-1} - \mathbf{a}\}$. Conversely, it is routine to verify that if for all $\mathbf{a} \in \{\mathbf{a}_1, \dots, \mathbf{a}_t\}$, there exist $\mathbf{a}'_1, \dots, \mathbf{a}'_{t-1} \in \mathcal{A}$ such that $\mathbb{N}(\mathcal{A} - \mathbf{a})$ is generated by $\{\mathbf{a}'_1 - \mathbf{a}, \dots, \mathbf{a}'_{t-1} - \mathbf{a}\}$ as a semigroup, then $\mathbb{N}(\mathcal{A} - \mathbf{a}) \cong \mathbb{N}^{t-1}$, and hence, $Y_{\mathcal{A}}$ is smooth by the exercise 6.1.11 in [9]. Indeed, this is the combinatorial characterization of the smoothness of a projective toric variety of a simplicial affine semigroup.

2.3 Normal semigroups and Finite Macaulayfication

In this section, we suppose the set $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\} \subset \mathbb{N}^n$ is simplicial with the set of n vertices $\mathcal{A}' = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ of $\mathrm{conv}(\mathcal{A})$. We will mention the regular (local) ring and the Cohen-Macaulay ring. To be simpler, we do not recall these concepts; the readers can find them in chapter 2 of the book [9]. We call $K[\mathcal{A}]$ to have a *finite Macaulayfication* if there is a ring extension R^* of $K[\mathcal{A}]$ in $K[\mathbf{t}]$ such that R^* is Cohen-Macaulay and $R^*/K[\mathcal{A}]$ has finite length as $K[\mathcal{A}]$-modules. In this section, I will show that $K[\mathcal{A}]$ has a finite Macaulayfication if $Y_{\mathcal{A}}$ is smooth.

For the affine semisubgroup $S = \mathbb{N}\mathcal{A}$ of $\mathbb{N}^n$, it is worth to notice that $\mathbb{Z}S = \mathbb{Z}\mathcal{A}$ is the smallest subgroup of $\mathbb{Z}^n$ that contains S.

Definition 2.3.1. The *normalization* of S, denoted by $\overline{S}$, is the intersection of this group and the cone generated by S, that is $\overline{S} = \mathbb{Z}S \cap C(S) \supseteq S$.

Remark 2.3.2. It is routine to verify that

$$\overline{S} = \{\mathbf{x} \in \mathbb{Z}S \mid p\mathbf{x} \in S \text{ for some } p \in \mathbb{N}\},$$

and hence $\overline{S}$ is also a semigroup.

Proposition 2.3.3. If $\mathcal{A}$ is a subset of a hyperplane, then $\overline{S}$ is generated by finitely many elements, or equivalently, $\overline{S}$ is also an affine semigroup.

Proof. The condition is equivalent to S is a graded affine semigroup . We fix the vector $\mathbf{w} \in \mathbb{Q}^n$ satisfying $\langle \mathbf{w}, \mathbf{a} \rangle = 1$ for all $\mathbf{a} \in \mathcal{A}$. It is routine to verify that $\mathbb{Z}S \subseteq \{\mathbf{x} \in \mathbb{Z}^n \mid \langle \mathbf{x}, \mathbf{w} \rangle \in \mathbb{Z}\}$ and $C(S) \subseteq \{\mathbf{x} \in \mathbb{R}^n_{\geq 0} \mid \langle \mathbf{x}, \mathbf{w} \rangle \geq 0\}$, so $\overline{S} \subseteq \{\mathbf{x} \in \mathbb{N}^n \mid \langle \mathbf{x}, \mathbf{w} \rangle \in \mathbb{N}\}$. It is worth to notice that $\mathrm{conv}(r\mathcal{A}) = \mathrm{conv}(r \cdot \mathcal{A}) = r\mathrm{conv}(\mathcal{A}) = r \cdot \mathrm{conv}(\mathcal{A})$.

By setting $\mathcal{A}_r = \mathbb{Z}\mathcal{A} \cap \operatorname{conv}(r\mathcal{A})$, we obtain that

$$\begin{aligned}\overline{S} &= \overline{\mathbb{N}\mathcal{A}}\\ &= \mathbb{Z}\mathcal{A} \cap C(S)\\ &= \bigsqcup_{r=0}^{\infty}\{\mathbf{x} \in \mathbb{Z}\mathcal{A} \cap C(S) \mid \langle \mathbf{w}, \mathbf{x}\rangle = r\}\\ &= \bigsqcup_{r=0}^{\infty}\{\mathbf{x} \in \mathbb{Z}\mathcal{A} \mid \mathbf{x} = \sum_{i=1}^{n} c_i\mathbf{a}_i, c_i \in \mathbb{R}_{\geq 0}, \sum_{i=1}^{n} c_i = r\}\\ &= \bigsqcup_{r=0}^{\infty}\mathcal{A}_r.\end{aligned}$$

□

Proposition 2.3.4. $\overline{S}$ is generated by the finite set $\bigsqcup_{r=0}^{n}\mathcal{A}_r$ as a semigroup. As a consequence, $\overline{S}$ is an affine semigroup.

Proof. Because $\mathcal{A}' \subseteq \mathcal{A} \subseteq \mathcal{A}_1$, we have

$$\mathcal{A}_r + \mathcal{A}' \subseteq \mathcal{A}_r + \mathcal{A}_1 \subseteq \mathbb{N}\left(\bigsqcup_{i=0}^{r}\mathcal{A}_i\right).$$

Therefore, it is sufficient to show that $\mathcal{A}_r \subseteq \mathcal{A}_{r-1} + \mathcal{A}'$, for every $r > n$. Indeed, for every $\mathbf{x} \in \mathcal{A}_r$, $\mathbf{x} = \sum_{i=1}^{n} c_i\mathbf{a}_i$, where $c_i \in \mathbb{R}_{\geq 0}$ and $\sum_{i=1}^{n} c_i = r > n$. Since $r > n$, there exists $1 \leq j \leq n$ such that $c_j > 1$, so $\mathbf{x} - \mathbf{a}_j = (c_j - 1)\mathbf{a}_j + \sum_{1\leq i\neq j\leq n} c_i\mathbf{a}_i$ is a linear combination of vectors belonging to $\mathbf{a}$ with non-negative coefficients, or equivalently $\mathbf{x} - \mathbf{a}_j \in C(S)$. Therefore, $\frac{\mathbf{x}-\mathbf{a}_j}{r-1} \in \operatorname{conv}(\mathcal{A})$, or equivalently $\mathbf{x} - \mathbf{a}_j \in \operatorname{conv}((r-1)\mathcal{A})$. As a result, $\mathbf{x}-\mathbf{a}_j \in \mathbb{Z}\mathcal{A}\cap\operatorname{conv}((r-1)\mathcal{A}) = \mathcal{A}_{r-1}$, so $\mathbf{x} \in \mathcal{A}_{r-1}+\mathcal{A}'$. The proof is completed. □

Definition 2.3.5. A semigroup $S = \mathbb{N}\mathcal{A}$ is called to be *normal* if $\overline{S} = S$.

It is routine to verify $\overline{S}$ is a normal affine semigroup when S is graded. By [10] (Corollary 4.7)), $K[\overline{S}]$ is Cohen-Macaulay. Now, we suppose that $Y_{\mathcal{A}}$ is smooth, we will show that $K[\overline{S}]/k[\mathcal{A}]$ has finite length as a $K[\mathcal{A}]$-module in order to show $K[\overline{S}]$ is a finite Macaulayfication of $K[\mathcal{A}]$.

Lemma 2.3.6. *With the above assumption about the graded affine semigroup $\mathbb{N}\mathcal{A}$, if $Y_{\mathcal{A}}$ is smooth, then $\mathcal{A}_r \subseteq \mathbb{N}(\mathcal{A}-\boldsymbol{a})+r\boldsymbol{a}$ for every $r \in \mathbb{N}$ and $\boldsymbol{a} \in \mathcal{A}'$ being a vertex of conv($\mathcal{A}$).*

Proof. For an arbitrary vector $\mathbf{x} \in \mathcal{A}_r = \mathbb{Z}\mathcal{A} \cap r\operatorname{conv}(\mathcal{A})$, $\mathbf{x} = \sum_{i=1}^{m} z_i\mathbf{a}_i = r\sum_{i=1}^{n} c_i\mathbf{a}_i$, where $z_i \in \mathbb{Z}$, $c_i \geq 0$ and $\sum_{i=1}^{n} c_i = 1$. The equality $\langle \mathbf{w}, \mathbf{x}\rangle = \langle \mathbf{w}, r\sum_{i=1}^{n} c_i\mathbf{a}_i\rangle = r$ implies

$\sum_{i=1}^{m} z_i = r$. Therefore,

$$\begin{aligned} \mathbf{x} - r\mathbf{a} &= \sum_{i=1}^{m} z_i(\mathbf{a}_i - \mathbf{a}) \\ &= r\sum_{i=1}^{n} c_i(\mathbf{a}_i - \mathbf{a}). \end{aligned} \tag{2.3.6.1}$$

Because $Y_{\mathcal{A}}$ is smooth, there exist $\mathbf{a}'_1, \ldots, \mathbf{a}'_{n-1} \in \mathcal{A}$ such that $\mathbf{a}_i - \mathbf{a}$ is a sum of vectors belonging to $\{\mathbf{a}'_1 - \mathbf{a}, \ldots, \mathbf{a}'_{n-1} - \mathbf{a}\}$ for $1 \le i \le n$ thanks to (2.2.2.1). Therefore, the first equality of (2.3.6.1) allows $\mathbf{x} - r\mathbf{a}$ is a linear combination of vectors belonging to $\{\mathbf{a}'_1 - \mathbf{a}, \ldots, \mathbf{a}'_{n-1} - \mathbf{a}\}$ with integral coefficients. Moreover, the second equality of (2.3.6.1) allows $\mathbf{x} - r\mathbf{a}$ is a linear combination of vectors belonging to $\{\mathbf{a}'_1 - \mathbf{a}, \ldots, \mathbf{a}'_{n-1} - \mathbf{a}\}$ with non-negative coefficients. Because $\{\mathbf{a}'_1 - \mathbf{a}, \ldots, \mathbf{a}'_{n-1} - \mathbf{a}\}$ is linearly independent, the two linear combinations are the same. This means that $\mathbf{x} - r\mathbf{a}$ is a linear combination of vectors belonging to $\{\mathbf{a}'_1 - \mathbf{a}, \ldots, \mathbf{a}'_{n-1} - \mathbf{a}\}$ with non-negative integral coefficients, or equivalently $\mathbf{x} - r\mathbf{a} \in \mathbb{N}(\mathcal{A} - \mathbf{a})$. Thus, $\mathbf{x} \in \mathbb{N}(\mathcal{A} - \mathbf{a}) + r\mathbf{a}$. The proof is completed. □

Lemma 2.3.7. *With the same assumption, there exists natural number $\delta(r)$ depending on r such that $\mathcal{A}_r + \delta(r)\mathcal{A}' \subseteq (r + \delta(r))\mathcal{A}$.*

Proof. Initially, we fix $\mathbf{a} \in \mathcal{A}'$. For each $\mathbf{x} \in \mathcal{A}_r$, since $\mathbf{x} \in \mathbb{N}(\mathcal{A} - \mathbf{a}) + r\mathbf{a}$ by Lemma 2.3.6, there exists $t_i \in \mathbb{N}$ such that $\mathbf{x} + t_i\mathbf{a} \in \mathbb{N}\mathcal{A}$. Because $\mathcal{A}_r$ is a finite set, we can choose a large enough number $t_{\mathbf{a}} \in \mathbb{N}$ such that $\mathcal{A}_r + t_{\mathbf{a}}\mathbf{a} \subset \mathbb{N}\mathcal{A}$. Then, by allowing $\mathbf{a}$ to run on the finite set $\mathcal{A}'$, we obtain the number $\delta(r) := \sum_{\mathbf{a} \in \mathcal{A}'} t_{\mathbf{a}} = \sum_{i=1}^{n} t_{\mathbf{a}_i}$ satisfies the inclusion $\mathcal{A}_r + \delta(r)\mathcal{A}' \subseteq \mathbb{N}\mathcal{A}$. Finally, we have $\langle \mathbf{w}, \mathbf{y} \rangle = r + \delta(r)$ for all $\mathbf{y} \in \mathcal{A}_r + \delta(r)\mathcal{A}'$, so $\mathcal{A}_r + \delta(r)\mathcal{A}' \subseteq (r + \delta(r))\mathcal{A}$. □

Proposition 2.3.8. $K[\overline{S}]/K[\mathcal{A}]$ has finite length. Therefore, $K[\overline{S}]$ is a finite Macaulayfication of $K[\mathcal{A}]$.

Proof. Since $(r + \delta(r))\mathcal{A} \subseteq \mathcal{A}_{r+\delta(r)} \subseteq \mathcal{A}_r + \delta(r)\mathcal{A}' \subseteq (r + \delta(r))\mathcal{A}$, for $r > n$, by the proof of Proposition 2.3.4 and Lemma 2.3.7, $\mathcal{A}_{r+\delta(r)} = (r + \delta(r))\mathcal{A}$ for every $r > n$. Thus, we can fix a number $s > n$ such that $\mathcal{A}_s = s\mathcal{A}$. It is routine to verify that

$$s'\mathcal{A} \subseteq \mathcal{A}_{s'} \subseteq \mathcal{A}_s + (s' - s)\mathcal{A}' \subseteq s\mathcal{A} + (s' - s)\mathcal{A}' \subseteq s'\mathcal{A},$$

for every $s' \ge s$, so $\mathcal{A}_{s'} = s'\mathcal{A}$ for every $s' \ge s$. This implies $M := K[\mathbf{t}^{\mathbf{a}}, \mathbf{a} \in \mathcal{A}_{s'}, s' \ge s] = K[\mathbf{t}^{\mathbf{a}}, \mathbf{a} \in s'\mathcal{A}, s' \ge s]$ is a submodule of both $K[\overline{S}] = K[\mathbf{t}^{\mathbf{a}}, \mathbf{a} \in \mathcal{A}_s, s \ge 1]$ and $K[\mathcal{A}]$. It is routine to verify that

$$\frac{K[\overline{S}]}{M} \cong \sum_{\mathbf{a}\in\mathcal{A}_r, 1\le r\le s-1} K\mathbf{t}^{\mathbf{a}}$$

and

$$\frac{K[\mathcal{A}]}{M} \cong \sum_{\mathbf{a}\in r\mathcal{A}, 1\le r\le s-1} K\mathbf{t}^{\mathbf{a}}$$

are K-vector spaces of finite dimension. Therefore, both of them have finite lengths. As a result,

$$\frac{K[\overline{S}]}{K[\mathcal{A}]} \cong \frac{K[\overline{S}]/M}{K[\mathcal{A}]/M}$$

has finite length. The proof is completed. □

2.4 Normal toric rings

Since this section until the end of this thesis, we only concentrate on the set of points $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\} \subset \mathbb{N}^n$ instead of $\mathbb{Z}^n$ as our work in the section 2.1. The set-up of the equivalence between the semigroup $S := \mathbb{N}\mathcal{A}$ and the monomials in $K[x_1, \ldots, x_n]$ (instead of the Laurent polynomial ring $K[\mathbf{t}^{\pm 1}]$) is similar to that in the prior sections. This implies $K[S] = K[\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_m}]$ and it is also called the *toric ring*.

Let $K[x_1, \ldots, x_n]$ be the polynomial ring with n variables x_i, and $K[S] \subset K[x_1, \ldots, x_n]$ is a toric ring. Now, we define the *integral closure* of the toric ring $K[S]$ (generally it is the definition of the integral closure of an arbitray communicative domain with the unity).

Definition and Proposition 2.4.1. Let $K[S]$ be a toric ring in the polynomial ring $K[x_1, \ldots, x_n]$ and $Q(K[S])$ be the quotient field of $K[S]$. An element $z \in Q(K[S])$ is called to be *integral over* $K[S]$ if z is a root of the monic polynomial

$$z^d + c_1 z^{d-1} + \cdots + c_{d-1} z + c_d = 0,$$

where $c_i \in K[S]$ for all $1 \le i \le d$. The set of all integral elements over $K[S]$, denoted by $\overline{K[S]}$, is a ring is also a subring of $Q(K[S])$ and is called to be the *integral closure* of $K[S]$. Moreover, $K[S]$ is called to be *normal* if $\overline{K[S]} = K[S]$.

Proof. See [11] (Proposition 5.1). □

Thanks to the following proposition, we can see the relation between the normality of a toric ring and that of corresponding semigroup.

Proposition 2.4.2. $\overline{K[S]} = K[\overline{S}]$. Then, the toric ring $K[S]$ is normal if and only if the semigroup S is normal.

Proof. See [5] (Theorem 9.1.1). □

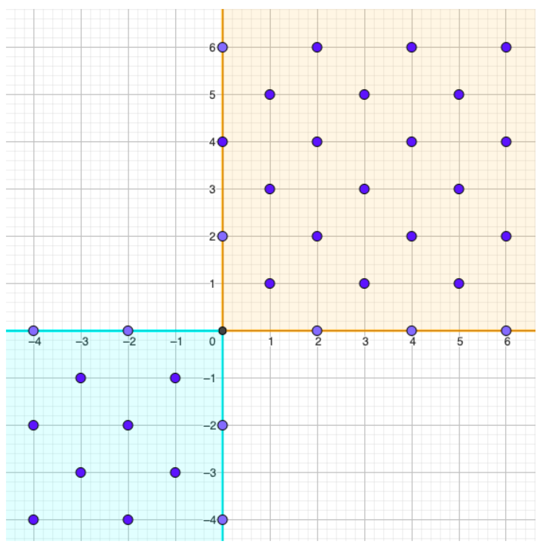
Example 2.4.3. Let S be the additive monoid generated by the vectors $(2,0),(1,1),(0,2)$ in $\mathbb{N}^2$. Then $K[S] = K[x_1^2, x_1x_2, x_2^2]$. We have

$$\mathbb{Z}S = \{(a,b) \in \mathbb{Z}^2 \mid a+b \equiv 0 \text{ mod } 2\},$$

and $C(S) = \mathbb{R}^2_{\geq 0}$. Hence

$$\overline{S} = \mathbb{Z}S \cap C(S) = \{(a,b) \in \mathbb{N}^2 \mid a+b \equiv 0 \text{ mod } 2\} = S.$$

Therefore, $K[S]$ is normal.



2.5 Rees algebra

We introduce a brief description of Rees algebras which will play an important role in the following chapters.

Definition 2.5.1. Let $I = (\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_m})$ be an ideal of the polynomial ring $R = K[x_1, \ldots, x_n]$. Then the *Rees algebra* of I is the graded ring

$$R(I) := \bigoplus_{k=0}^{\infty} I^k t^k = K[x_1, \ldots, x_n, \mathbf{x}^{\mathbf{a}_1}t, \ldots, \mathbf{x}^{\mathbf{a}_m}t] \subset K[x_1, \ldots, x_n, t],$$

where t is an indeterminate.

If we set $S' = \{(\mathbf{e}_1, 0), \ldots, (\mathbf{e}_n, 0), (\mathbf{a}_1, 1), \ldots, (\mathbf{a}_m, 1)\}$ that is the set of exponents of generators of $R(I)$ over K, then we obtain the following proposition.

Proposition 2.5.2. With the above assumption, we have

$$\overline{R(I)} = K[\overline{S'}] = \bigoplus_{k=0}^{\infty} \overline{I^k} t^k.$$

Therefore, $R(I)$ is normal if and only if ideal I is normal.

Proof. See [5] (Part 12.3). □

Now, we complete the second chapter. During two first chapters, we come through all of necessary algebraic content in this dissertation. In the following chapter, we will concentrate on concepts of Linear Programming and Discrete Geometry in order to construct a relationship between the fields.

Chapter 3

Integer rounding properties

3.1 Motivation

Initially, we mention some definitions of integer programming and convex geometry, which will be used for the rest of the dissertation.

Let M be an non-zero $n \times m$ matrix with non-negative rational entries. For each vector $\mathbf{w} \in \mathbb{N}^n$, we define the following values:

$$\nu^*(\mathbf{w}, M) = \max\{\mathbf{1} \cdot \mathbf{y} \mid \mathbf{y} \in \mathbb{R}^m_{\geq 0}, M\mathbf{y}^T \leq \mathbf{w}^T\},$$

$$\nu(\mathbf{w}, M) = \max\{\mathbf{1} \cdot \mathbf{y} \mid \mathbf{y} \in \mathbb{N}^m, M\mathbf{y}^T \leq \mathbf{w}^T\},$$

$$\tau^*(\mathbf{w}, M) = \min\{\mathbf{1} \cdot \mathbf{y} \mid \mathbf{y} \in \mathbb{R}^m_{\geq 0}, M\mathbf{y}^T \geq \mathbf{w}^T\},$$

$$\tau(\mathbf{w}, M) = \min\{\mathbf{1} \cdot \mathbf{y} \mid \mathbf{y} \in \mathbb{N}^m, M\mathbf{y}^T \geq \mathbf{w}^T\}.$$

It is routine to verify that $\nu^*(\mathbf{w}, M) \geq \nu(\mathbf{w}, M)$ and $\tau^*(\mathbf{w}, M) \leq \tau(\mathbf{w}, M)$. From the perspective of integer programming, we pay attention to when $\nu(\mathbf{w}, M)$ and $\nu^*(\mathbf{w}, M)$ (similarly $\tau(\mathbf{w}, M)$ and $\tau^*(\mathbf{w}, M)$) are most approximate. With this motivation, we come into the following definition.

Definition 3.1.1. An $n \times m$-matrix M has the *integer round-down property* (*integer round-up property*) if for every $\mathbf{w} \in \mathbb{N}^n$, $\lfloor \nu^*(\mathbf{w}, M) \rfloor = \nu(\mathbf{w}, M)$ (respectively $\lceil \tau^*(\mathbf{w}, M) \rceil = \tau(\mathbf{w}, M)$).

Now, we come into geometric concepts that have a close relationship with the integer rounding properties that we can see in Proposition 3.1.3.

Definition 3.1.2. Let P be a polyhedron in $\mathbb{R}^n_{\geq 0}$.

(i) We call P *upper comprehensive* (*lower comprehensive*) if for every $\mathbf{y} \geq \mathbf{x} \in P$ ($\mathbf{0} \leq \mathbf{y} \leq \mathbf{x} \in P$), $\mathbf{y}$ must belong to P.

(ii) P is said to have the *integral decomposition property* if every integral vector of $kP = \{k\mathbf{a} \mid \mathbf{a} \in P\}$ is a sum of k integral vectors of P for all integers $k \geq 1$.

In 1981, S. Baum and Jr. L. E. Trotter shown the relationship between above concepts that motivated Professor Ngo Viet Trung and I to investigate the additional relationships between them and even algebra.

Proposition 3.1.3. Let $P \subsetneq \mathbb{R}^n_+$ be a nonempty polyhedron with nonempty interior satisfying its extreme points are all integral.

(i) If P is upper comprehensive, then for the matrix M whose columns are precisely minimal integral points of P, P has the integral decomposition property if and only if M has the integer round-down property.

(ii) If P is bounded and lower comprehensive, then for the matrix M whose columns are exactly maximal integral points of P, P has the integral decomposition property if and only if M has the integer round-up property.

Proof. See [12] (Theorem 1). □

By considering Proposition 3.1.3, we can realize that both integer rounding properties are strictly associated to the integral decomposition property of upper and lower comprehensive polyhedra. This poses a question straightforwardly that whether the integer rounding properties are equivalent or not. In terms of geometry, it is routine to realize that there are certainly not polyhedra being both upper and lower comprehensive. Therefore, if we want to build a bridge between two rounding properties, then we must construct pairs of "dual" polyhedra, an upper and a lower comprehensive ones. We can see the construction in next section and realize that two integer rounding properties are simply different versions of an algebraic condition, the normality of monomial ideals, in the last section.

3.2 Newton polyhedra

Let $I = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\} \subset \mathbb{N}^n$ be a set of finitely many integral points such that $\mathbf{a}_i \not\leq \mathbf{a}_j$ for all $i \neq j$ (later we can consider I as the monomial ideal $(\mathbf{x}^{\mathbf{a}_n}, \ldots, \mathbf{x}^{\mathbf{a}_m})$ straightforwardly).

We are going to use the symbols for the rest of the thesis. Note that we will not consider the case $I = \{\mathbf{0}\}$ due to its inconvenience. By writing $\mathbf{a}_i = (a_{i1}, \dots, a_{in}) \in \mathbb{N}^n$ for $1 \le i \le m$, we put $(a_{max})_j := \max\{a_{ij} \mid 1 \le i \le m\}$, the maximal j-coordinate among that of the points, and $\mathbf{a}_{max} := ((a_{max})_1, \dots, (a_{max})_n)$. We can see that $\mathbf{a}_{max} \ge \mathbf{a}_i$ for every $1 \le i \le m$, so we can define $\mathbf{a}_i^* := \mathbf{a}_{max} - \mathbf{a}_i \in \mathbb{N}^n$.

With the set (or monomial ideal) I above, in order to be consistent with the Definition 1.3.1, the *Newton polyhedron* associated to I is defined to be

$$N(I) := \text{conv}\{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \le \mathbf{a}\}.$$

With the aim in the first section, we define a "dual" polytope that is bounded as follows:

$$N^*(I) := \text{conv}\{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i^* \ge \mathbf{a}\}.$$

We define M and M^* as the $n \times m$ matrices whose columns are $\mathbf{a}_1^T, \dots, \mathbf{a}_m^T$ and $\mathbf{a}_1^{*T}, \dots, \mathbf{a}_m^{*T}$, respectively. Indeed, $N(I)$ and $N^*(I)$ will play roles as polyhedra in Proposition 3.1.3. In order to ensure they satisfy the conditions in Proposition 3.1.3, we, initially, need to prove their upper and lower comprehensiveness. The process belongs to this section and the next where more complex technical properties will be obtained. Now, we show some useful basic properties of $N(I)$ and $N^*(I)$.

Lemma 3.2.1. *If $\boldsymbol{u} \in \{\boldsymbol{a} \in \mathbb{N}^n \mid \exists \boldsymbol{a}_i^* \ge \boldsymbol{a}\}$, then $\boldsymbol{a}_{max} - \boldsymbol{u} \in \{\boldsymbol{a} \in \mathbb{N}^n \mid \exists \boldsymbol{a}_i \le \boldsymbol{a}\}$. Consequently, if $\boldsymbol{u} \in N^*(I)$, then $\boldsymbol{a}_{max} - \boldsymbol{u} \in N(I)$.*

Proof. The first statement is obvious. For any $\mathbf{u} \in N^*(I)$, we can express $\mathbf{u} = \sum_{i=1}^t \lambda_i \mathbf{u}_i$, where $\lambda_i \ge 0$, $\sum_{i=1}^t \lambda_i = 1$ and $\mathbf{u}_i \in \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i^* \ge \mathbf{a}\}$. Therefore, $\mathbf{a}_{max} - \mathbf{u} = \sum_{i=1}^t \lambda_i(\mathbf{a}_{max} - \mathbf{u}_i) \in N(I)$ because $\mathbf{a}_{max} - \mathbf{u}_i \in \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \le \mathbf{a}\}$ by the previous statement. $\square$

From the above lemma, we can pose a natural question that if $\mathbf{u} \in N(I)$ and $\mathbf{u} \le \mathbf{a}_{max}$, then we can whether reach a conclusion that $\mathbf{a}_{max} - \mathbf{u} \in N^*(I)$ or not? We can mimic the proof of Lemma 3.2.1 for the case as follows: if $\mathbf{u} \in N(I)$, then $\mathbf{u}$ can be express as a convex combination $\mathbf{u} = \sum_{i=1}^t \lambda_i \mathbf{u}_i$, where $\mathbf{u}_i \in \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \le \mathbf{a}\}$. However, we cannot obtain an equality $\mathbf{a}_{max} - \mathbf{u} = \sum_{i=1}^t \lambda_i(\mathbf{a}_{max} - \mathbf{u}_i)$. This is because we do not know whether $\mathbf{u}_i \le \mathbf{a}_{max}$ or not! Indeed, we can also obtain the result later (Lemma 3.3.4) when we construct some technical properties that ensure that we can choose the $\mathbf{u}_i$'s satisfying the inequality (Lemma 3.3.3). Now we come into another property of Newton polyhedra.

Lemma 3.2.2. *The Newton polyhedron $N(I)$ is upper comprehensive, so is $kN(I)$ for every $k \geq 1$.*

Proof. For any two points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_+$ satisfying $\mathbf{y} \geq \mathbf{x} \in N(I)$, we need to prove $\mathbf{y} \in N(I)$. Initially, we show the statement holds for integral points $\mathbf{x}$ and $\mathbf{y}$. Indeed, according to the definition of $N(I)$, we can express $\mathbf{x} = \sum_{i=1}^t \lambda_i \mathbf{u}_i$, where $\lambda_i \geq 0$, $\sum_{i=1}^t \lambda_i = 1$ and $\mathbf{u}_i \in \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \leq \mathbf{a}\}$ for $1 \leq i \leq t$. This results in $\mathbf{y} - \mathbf{x} + \mathbf{u}_i \in \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \leq \mathbf{a}\}$ since $\mathbf{y} \geq \mathbf{x}$. Thus, the equality $\mathbf{y} = \sum_{i=1}^t \lambda_i(\mathbf{y} - \mathbf{x} + \mathbf{u}_i)$ leads $\mathbf{y}$ to be an element of $N(I)$.

Afterwards, we prove the statement also holds for a real point $\mathbf{y}$ and an integral point $\mathbf{x}$. By setting $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n_+$, we obtain that $(\lfloor y_1 \rfloor, \ldots, \lfloor y_n \rfloor) \geq \mathbf{x}$ because $\mathbf{x}$ is integral. Therefore, by the previous statement, it must satisfy that $(\lfloor y_1 \rfloor, \ldots, \lfloor y_n \rfloor)$ belongs to $N(I)$, so does every vertex of the cube $[\lfloor y_1 \rfloor, \lfloor y_1 \rfloor + 1] \times \cdots \times [\lfloor y_n \rfloor, \lfloor y_n \rfloor + 1]$. Because $\mathbf{y}$ is an element of the cube, $\mathbf{y}$ is a convex combination of these vertices, which leads $\mathbf{y}$ to belong to $\mathrm{conv}\{N(I)\}$, or equivalently $\mathbf{y} \in N(I)$.

Next, we show the conclusion of this lemma holds for real points $\mathbf{x}$ and $\mathbf{y}$. Because $\mathbf{x} \in N(I)$, we can write $\mathbf{x} = \sum_{i=1}^t \lambda_i \mathbf{u}_i$, where $\mathbf{u}_i \in \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \leq \mathbf{a}\}$ and $\lambda_i \geq 0$ and $\sum_{i=1}^t \lambda_i = 1$. We have $\mathbf{y} - \mathbf{x} + \mathbf{u}_i \geq \mathbf{u}_i \in \mathbb{N}^n \cap N(I)$, so $\mathbf{y} - \mathbf{x} + \mathbf{u}_i \in N(I)$ by the last statement. Therefore, the equality $\mathbf{y} = \sum_{i=1}^t \lambda_i(\mathbf{y} - \mathbf{x} + \mathbf{u}_i)$ results in $\mathbf{y} \in \mathrm{conv}\{N(I)\} = N(I)$. Finally, the similar property for $kN(I)$ is obtained straightforwardly. □

Similarly, we will also have $N^*(I)$ is lower comprehensive (Lemma 3.3.5). But at that time, our tools are not effective enough to prove that. Now, we indicate a property of $N^*(I)$ that will help us to show its lower comprehensiveness in the next section.

Lemma 3.2.3. *$N^*(I)$ is closed in terms of Euclidean norm in $\mathbb{R}^n$.*

Proof. It is routine to verify that the number of elements of $\{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i^* \geq \mathbf{a}\}$ is finite, and we put m as the number. Fix an (arbitrary) order $\mathbf{b}_i$'s, $1 \leq i \leq m$, of elements of the set and let A be the matrix $\begin{bmatrix} \mathbf{b}_1^T & \cdots & \mathbf{b}_m^T \end{bmatrix}$. Now, for an arbitrary point $\mathbf{y} \in \mathbb{R}^n$ satisfying that there exists a sequence of vectors $\{\mathbf{y}_i\}_{i=1}^\infty$ of $N^*(I)$ converging to $\mathbf{y}$, we claim that $\mathbf{y} \in N^*(I)$. Indeed, since each $\mathbf{y}_i$ is a convex combination of vectors belonging to $\{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i^* \geq \mathbf{a}\}$, we have a vector $\mathbf{z}_i := (y_{i1}, y_{i2}, \ldots, y_{im}) \in [0,1]^m$ such that $\mathbf{1} \cdot \mathbf{z}_i = \sum_{j=1}^m y_{ij} = 1$ and $A\mathbf{z}_i^T = \sum_{j=1}^m y_{ij} \mathbf{b}_j^T = \mathbf{y}_i^T$, for each i. Due to the compactness of $[0,1]^m$, we obtain a subsequence $\{\mathbf{z}_{i'}\}$ of $\{\mathbf{z}_i\}$ that converges to a vector $\mathbf{z} = (y_1, \ldots, y_m)$ belonging to $[0,1]^m$. Since the linear map from $\mathbb{R}^m$ to $\mathbb{R}^n$ defined by A is continuous, the fact that $\mathbf{y}_{i'}^T = A\mathbf{z}_{i'}^T$ tends to $\mathbf{y}^T$ allows $\mathbf{y}^T$ to be $A\mathbf{z}^T$. The obvious property $\mathbf{1}\cdot\mathbf{z} = 1$ leads $\mathbf{y}^T$ to be a convex combination of columns of A, which is equivalent to $\mathbf{y} \in N^*(I)$. □

3.3 Technical properties

The properties in the previous section were elegant results of polyhedra, but results in this section are only technical ones that allow us to prove latter propositions rather than helping understand the geometric structure of polyhedra.

Lemma 3.3.1. *If a rational point $\boldsymbol{u}$ is a convex combination of rational points $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_t \in \mathbb{Q}^n_{\geq 0}$, which means that $\boldsymbol{u} = \sum_{i=1}^t \lambda_i \boldsymbol{u}_i$, where $\lambda_i \in \mathbb{R}_{\geq 0}$ and $\sum_{i=1}^t \lambda_i = 1$, then there exists a sequence of* ***non-negative rational*** *vectors $(\lambda_1^{(j)}, \ldots, \lambda_t^{(j)}) \in \mathbb{Q}^t_{\geq 0}$, $j \geq 1$, such that $\sum_{i=1}^t \lambda_i^{(j)} = 1$ and $\boldsymbol{u} = \sum_{i=1}^t \lambda_i^{(j)} \boldsymbol{u}_i$ for every j, and $(\lambda_1^{(j)}, \ldots, \lambda_t^{(j)}) \to (\lambda_1, \ldots, \lambda_t)$ as $j \to \infty$.*

Proof. Initially, let S be the subset of $\{1, \ldots, t\}$ such that $s \in S$ if and only if $\lambda_s = 0$. In cases $S \neq \emptyset$, we will write $S = \{s_1, \ldots, s_{|S|}\}$.

After that, when $S \neq \emptyset$, we put S' to be the $|S| \times t$ matrix such that the (i, s_i)-entry of S' is 1 for every $1 \leq i \leq |S|$ and the remaining entries are all zero. Otherwise, we regard S' as the "0×0" matrix. Now we put

$$A := \left[\begin{array}{ccccc} 1 & 1 & \ldots & 1 & 1 \\ \mathbf{u}_1^T & \mathbf{u}_2^T & \ldots & \mathbf{u}_{t-1}^T & \mathbf{u}_t^T \\ \hline & & S' & & \end{array}\right] \text{ and } B := \begin{bmatrix} 1 \\ \mathbf{u}^T \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

which are a $(n+1+|S|) \times t$ matrix and a $(n+1+|S|) \times 1$ matrix, respectively.

We can observe that the linear system $A\mathbf{x}^T = B$ has the solution $\mathbf{x}_0 = (\lambda_1, \ldots, \lambda_t) \in \mathbb{R}^t_{\geq 0}$. If $\operatorname{rank}(A) = t$, then the linear system has the unique solution $(\lambda_1, \ldots, \lambda_t)$. Moreover, $\mathbf{x}_0$ must be rational because we can solve the linear system by Gaussian elimination and all of entries of both A and B are rational. Therefore, the conclusion holds in the case $\operatorname{rank}(A) = t$, provided that we choose $(\lambda_1^{(j)}, \ldots, \lambda_t^{(j)}) = (\lambda_1, \ldots, \lambda_t)$ for all j. Otherwise, if $\operatorname{rank}(A) = r < t$, then there are exactly $t - r$ free variables among t variables of the solution $\mathbf{x} = (x_1, \ldots, x_t)$. Without loss of generality, we assume that $x_1, \ldots, x_r$ are precisely free variables of the solutions. By Linear Algebra, the dependent variable x_h is a linear combination $f_h(1, x_1, \ldots, x_r)$ of 1 and the free variables with rational coefficients for every $r+1 \leq h \leq t$ due to the rationality of entries of A and B.

A remark that should be noticed is $S \cap \{1, \ldots, r\} = \emptyset$. This is because for each $s \in S$, the variable x_s of the solution $\mathbf{x} = (x_1, \ldots, x_t)$ is usually constant zero, so it is not a free variable. This allows us to choose x_i to be positive for $1 \leq i \leq r$. Now we choose positive rational numbers $\lambda_i^{(j)}$, $j \geq 1$ and $1 \leq i \leq r$, such that $\lambda_i^{(j)} \to \lambda_i$ as $j \to \infty$. If we choose free variable x_i to be $\lambda_i^{(j)}$ for $1 \leq i \leq r$, then the dependent variable $\lambda_h^{(j)} := x_h = f_h(1, x_1, \ldots, x_r)$ tends to $f_h(1, \lambda_1, \ldots, \lambda_r) = \lambda_h$ as $j \to \infty$ for every $r + 1 \leq h \leq t$. Moreover, for $h \in S \subseteq \{r + 1, \ldots, t\}$, $\lambda_h^{(j)} = f_h(1, \lambda_1, \ldots, \lambda_r) = 0$ for all j. To sum up, we obtain that $(\lambda_1^{(j)}, \ldots, \lambda_t^{(j)}) \to (\lambda_1, \ldots, \lambda_t)$ as $j \to \infty$ and $\lambda_s^{(j)} = 0$ if $\lambda_s = 0$ for all j. Therefore, the condition that $(\lambda_1, \ldots, \lambda_t) \in \mathbb{R}_+^t$ allows us to obtain $(\lambda_1^{(j)}, \ldots, \lambda_t^{(j)}) \geq \mathbf{0}$, for sufficiently large j. By excluding small indices, we can assume this property holds for every $j \geq 1$. Since each $(\lambda_1^{(j)}, \ldots, \lambda_t^{(j)})$ is a solution of the linear system, it satisfies all of the requirements in the conclusion. The proof is completed. $\square$

The above lemma allows us to change the background of numbers from $\mathbb{R}$ to $\mathbb{Q}$. In latter propositions, we can realize the important role of rational vectors in the thesis. The following lemma helps us to "replace" "free" vectors with vectors "lower than" $\mathbf{a}_{max}$ in Lemma 3.3.3, the main technique of the section.

Lemma 3.3.2. *Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a point of $\mathbb{N}^n$ and k be a positive integer. If there exist elements $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_k$ of $\mathbb{N}^n$ such that $\mathbf{x} \leq \sum_{i=1}^k \boldsymbol{y}_i$, then there exist elements $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_k$ of $\mathbb{N}^n$ satisfying $\mathbf{x} = \sum_{i=1}^k \boldsymbol{z}_i$ and $\boldsymbol{z}_i \leq \boldsymbol{y}_i$.*

Proof. We set y_{ij} to be the j-th coordinate of $\mathbf{y}_i$. We only prove this lemma in cases that $\mathbf{x} < \sum_{i=1}^k \mathbf{y}_i$. Let Ω be the subset of $\{1, \ldots, n\}$ such that $x_j < \sum_{i=1}^k y_{ij}$ if and only if $j \in \Omega$. By the assumption, Ω is a non-empty set. These conditions allow us to define a map

$$\omega : \Omega \to \{0, \ldots, n - 1\}$$

as follows: For each $t \in \Omega$,

- if $x_t < y_{1t}$, then $\omega(t) = 0$;
- otherwise, $\omega(t)$ is chosen to satisfy $\sum_{i=1}^{\omega(t)} y_{it} \leq x_t < \sum_{i=1}^{\omega(t)+1} y_{it}$.

Afterwards, we set $\mathbf{z}_i$'s. For $j \notin \Omega$, we set $z_{ij} := b_{ij}$. For each $t \in \Omega$,

- put $z_{it} := y_{it}$ if $i \leq \omega(t)$,
- $z_{\omega(t)+1,t} := x_t - \sum_{i=1}^{\omega(t)} y_{it}$,

- $z_{it} := 0$ if $i > \omega(t) + 1$ (we regard $\sum_{i=1}^{0} y_{it}$ as 0).

After setting the z_{ij}'s, we set points $\mathbf{z}_i = (z_{i1}, \dots, z_{in})$ for each $1 \le i \le k$. It is routine to verify that the points $\mathbf{z}_i$'s satisfy the conclusion. □

Now, we come into the main technique of this section that allows us to prove the remain questions in the previous section.

Lemma 3.3.3. *We have the following results that allow us to "replace" vector $\boldsymbol{u}$ with a vector $\boldsymbol{v} \le \boldsymbol{a}_{max}$.*

(a) If $\boldsymbol{u} \in \{\boldsymbol{a} \in \mathbb{N}^n \mid \exists \boldsymbol{a}_i \le \boldsymbol{a}\}$, then there exists $\boldsymbol{v} \le \boldsymbol{u}$ satisfying $\boldsymbol{v} \in \{\boldsymbol{a} \in \mathbb{N}^n \mid \exists \boldsymbol{a}_i \le \boldsymbol{a}\}$ and $\boldsymbol{v} \le \boldsymbol{a}_{max}$.

(b) If $\boldsymbol{u} \in \mathbb{N}^n \cap N(I)$, then there exists $\boldsymbol{v} \le \boldsymbol{u}$ satisfying $\boldsymbol{v} \in \mathbb{N}^n \cap N(I)$ and $\boldsymbol{v} \le \boldsymbol{a}_{max}$.

(b') If $\boldsymbol{u} \in \mathbb{Q}^n \cap N(I)$, then there exists $\boldsymbol{v} \le \boldsymbol{u}$ satisfying $\boldsymbol{v} \in \mathbb{Q}^n \cap N(I)$ and $\boldsymbol{v} \le \boldsymbol{a}_{max}$. Moreover, if k is a positive integer satisfying $k\boldsymbol{u} \in \mathbb{N}^n$, then $k\boldsymbol{v} \in \mathbb{N}^n$.

*(c) If $\boldsymbol{v} \in \mathbb{Q}^n \cap N(I)$ and $\boldsymbol{v} \le \boldsymbol{a}_{max}$, then $\boldsymbol{v}$ is a convex combination of vectors belonging to $\{\boldsymbol{a} \in \mathbb{N}^n \mid \exists \boldsymbol{a}_i \le \boldsymbol{a} \le \boldsymbol{a}_{max}\}$ with positive **rational** coefficients.*

Proof. For parts (a), (b) and (b'), we set $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{N}^n$. It is sufficient to prove the lemma in cases that $\mathbf{u} \not\le \mathbf{a}_{max}$. Let J be the subset of $\{1, \dots, n\}$ such that $u_i > (a_{max})_i$ if and only if $i \in J$. By the assumption, it must satisfy $J \ne \emptyset$. For each $j \in J$, we put $v_j := (a_{max})_j$; otherwise, we put $v_i := u_i$ if $i \notin J$. It is routine to verify that $\mathbf{v} := (v_1, \dots, v_n) \le \mathbf{u}$ and $\mathbf{v} \le \mathbf{a}_{max}$.

(a) The rest of the proof is to prove that $\mathbf{v} \in \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \le \mathbf{a}\}$. Indeed, because $\mathbf{u} \in \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \le \mathbf{a}\}$, we can assume $\mathbf{u} \ge \mathbf{a}_1 = (a_{11}, \cdots, a_{in})$ without loss of generality. This results in $u_i \ge \mathbf{a}_{1i}$ for every $i \notin J$. Moreover, because $(a_{max})_j = \max\{a_{ij} \mid 1 \le i \le m\}$, we have $(a_{max})_j \ge a_{1j}$ for $j \in J$. Combining the inequalities, we obtain that $\mathbf{v} \ge \mathbf{a}_1$.

(b) The rest of the proof is to prove that $\mathbf{v} \in \mathbb{N}^n \cap N(I)$. Because $\mathbf{u} \in N(I)$, we can write $\mathbf{u} = \sum_{i=1}^{t} \lambda_i \mathbf{u}'_i$, where $\lambda_i > 0$ and $\sum_{i=1}^{t} \lambda_i = 1$ and $\mathbf{u}'_i \in \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \le \mathbf{a}\}$. We write $\mathbf{u}'_i = (u'_{i1}, \dots, u'_{in})$ for $1 \le i \le t$. For each $1 \le i \le t$, we set $v'_{ij} := (a_{max})_j$ if $j \in J$; otherwise, we put $v'_{ij} := u'_{ij}$ if $j \notin J$. Similarly to (a), we obtain $\mathbf{v}'_i := (v'_{i1}, \dots, v'_{in}) \in \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \le \mathbf{a}\}$. Therefore, it is routine to verify that $\mathbf{v} = \sum_{i=1}^{t} \lambda_i \mathbf{v}'_i \in N(I)$.

(b') The proof of the first statement is similar to that of (b). The way of putting $\mathbf{v}$ we used allows us to obtain the second statement straightforwardly.

(c) Because $\mathbf{v} \in N(I)$, we have a convex combination $\mathbf{v} = \sum_{i=1}^{t} \lambda_i \mathbf{d}_i$, where $\lambda_i \in \mathbb{R}_+$ and $\sum_{i=1}^{t} \lambda_i = 1$ and $\mathbf{d}_i \in \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \leq \mathbf{a}\}$. By Lemma 3.3.1, we can choose $\lambda_i \in \mathbb{Q}_+$ for every $1 \leq i \leq t$. Therefore, there exists $k \in \mathbb{N}$ satisfying $k\lambda_i \in \mathbb{N}$ for all i, which implies $k\mathbf{v}$ is a sum of k integral vectors $\mathbf{d}_i$'s. We rewrite $k\mathbf{v} = \sum_{i=1}^{k} \mathbf{d}_i \in \mathbb{N}^n$, where $\mathbf{d}_i \in \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \leq \mathbf{a}\}$. By (a) of this lemma, there exists $\mathbf{d}'_i \leq \mathbf{d}_i$ for each i such that $\mathbf{d}'_i \in \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \leq \mathbf{a} \leq \mathbf{a}_{max}\}$. As a result,

$$\mathbf{0} \leq k\mathbf{a}_{max} - k\mathbf{v} = \sum_{i=1}^{k} (\mathbf{a}_{max} - \mathbf{d}_i) \leq \sum_{i=1}^{k} (\mathbf{a}_{max} - \mathbf{d}'_i).$$

By Lemma 3.3.2, there exist $\mathbf{c}'_1, \ldots, \mathbf{c}'_k$ satisfying $k\mathbf{a}_{max} - k\mathbf{v} = \sum_{i=1}^{k} \mathbf{c}'_i$ and $\mathbf{0} \leq \mathbf{c}'_i \leq \mathbf{a}_{max} - \mathbf{d}'_i$. Thus, $k\mathbf{v} = \sum_{i=1}^{k} (\mathbf{a}_{max} - \mathbf{c}'_i)$. Because $\mathbf{a}_{max} - \mathbf{c}'_i \geq \mathbf{d}'_i \in \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \leq \mathbf{a}\}$, we have $\mathbf{c}_i := \mathbf{a}_{max} - \mathbf{c}'_i \in \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \leq \mathbf{a} \leq \mathbf{a}_{max}\}$. Finally, we obtain that $\mathbf{v} = \sum_{i=1}^{k} \frac{1}{k} \mathbf{c}_i$. We have just completed the proof.

□

Part (c) of Lemma 3.3.3 allows us to complete the remain questions in the previous section.

Lemma 3.3.4. *If $\boldsymbol{v} \in N(I) \cap \mathbb{Q}^n$ and $\boldsymbol{v} \leq \boldsymbol{a}_{max}$, then $\boldsymbol{a}_{max} - \boldsymbol{v} \in N^*(I)$.*

Proof. By (c) of Lemma 3.3.3, $\mathbf{v} = \sum_{i=1}^{t} \lambda_i \mathbf{d}_i$, where $\sum_{i=1}^{t} \lambda_i = 1$ and $\mathbf{d}_i \in \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \leq \mathbf{a} \leq \mathbf{a}_{max}\}$. Thus, $\mathbf{a}_{max} - \mathbf{v} = \sum_{i=1}^{t} \lambda_i (\mathbf{a}_{max} - \mathbf{d}_i) \in N^*(I)$ since $\mathbf{a}_{max} - \mathbf{d}_i \in \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i^* \geq \mathbf{a}\}$. □

The above lemma and Lemma 3.2.1 help us to see that the rational points of $N(I)$ and $N^*(I)$ are symmetric through the point $\frac{1}{2}\mathbf{a}_{max}$. At that time, we can pose a question how about real points of the polyhedra. Indeed, we also have the similar result for real points, but it is not necessary for our progress. We can utilize the convergent sequences to prove the results about real points. At the end of the section, we show the lower comprehensiveness of $N^*(I)$ by the analysis-related argument.

Lemma 3.3.5. *$N^*(I)$ is lower comprehensive.*

Proof. For any $\mathbf{y} < \mathbf{x} \in N^*(I)$, we have $\mathbf{a}_{max} - \mathbf{y} \geq \mathbf{a}_{max} - \mathbf{x} \in N(I)$ by Lemma 3.2.1, and so $\mathbf{a}_{max} - \mathbf{y} \in N(I)$ by Lemma 3.2.2. If $\mathbf{y} \in \mathbb{Q}^n$ (or equivalently $\mathbf{a}_{max} - \mathbf{y} \in \mathbb{Q}^n$), then $\mathbf{y} \in N^*(I)$ by Lemma 3.3.4. Otherwise, when $\mathbf{y} \in \mathbb{R}^n$ generally, there is a sequence of rational vectors between $\mathbf{x}$ and $\mathbf{y}$ and the sequence converges to $\mathbf{y}$. Thanks to the previous argument, all of the vectors of the sequence belong to $N^*(I)$, so does $\mathbf{y}$ due to the closeness of $N^*(I)$ (Lemma 3.2.3). □

3.4 The integral decomposition property

By results in the previous sections, we can see that $N(I)$ and $N^*(I)$ are quite related. This motivates us to find a common property of both them. And fortunately, the integral decomposition property, the most important characteristic mentioned in the first section, is an one that both $N(I)$ and $N^*(I)$ have or do not have in common.

Proposition 3.4.1. $N(I)$ has the integral decomposition property iff so does $N^*(I)$.

Proof. Assume $N(I)$ has the integral decomposition property. We claim that $N^*(I)$ also has the property. Let k be a positive integer and $\mathbf{a}$ an integral vector of $kN^*(I)$. We have $\mathbf{a} = k\mathbf{b}$ for some $\mathbf{b} \in N^*(I)$. By Lemma 3.2.1, we have $\mathbf{a}_{max} - \mathbf{b} \in N(I)$, or equivalently $k\mathbf{a}_{max} - \mathbf{a} \in kN(I) \cap \mathbb{N}^n$. Because $N(I)$ has the integral decomposition property, there exist integral vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k \in N(I)$ such that $k\mathbf{a}_{max} - \mathbf{a} = \sum_{i=1}^k \mathbf{v}_i$. By (b) of Lemma 3.3.3, we can choose integral elements $\mathbf{v}'_1, \ldots, \mathbf{v}'_k \in N(I)$ such that $\mathbf{v}'_i \leq \mathbf{a}_{max}$ and $\mathbf{v}'_i \leq \mathbf{v}_i$ for every $1 \leq i \leq k$. This implies $k\mathbf{a}_{max} - \mathbf{a} \geq \sum_{i=1}^k \mathbf{v}'_i$, so $\mathbf{a} \leq \sum_{i=1}^k (\mathbf{a}_{max} - \mathbf{v}'_i)$. By Lemma 3.3.2, there exist integral vectors $\mathbf{d}_1, \ldots, \mathbf{d}_k$ such that $\mathbf{a} = \sum_{i=1}^k \mathbf{d}_i$ and $\mathbf{0} \leq \mathbf{d}_i \leq \mathbf{a}_{max} - \mathbf{v}'_i$. This leads to $\mathbf{a}_{max} - \mathbf{d}_i \geq \mathbf{v}'_i \in N(I)$. Combining with Lemma 3.2.2, we obtain $\mathbf{a}_{max} - \mathbf{d}_i \in N(I)$ for every $1 \leq i \leq k$. By lemma 3.3.4, $\mathbf{d}_i \in N^*(I)$. Therefore, $\mathbf{a}$ is a sum of k integral vectors of $N^*(I)$. This means that $N^*(I)$ also has the integral decomposition property.

Conversely, assume $N^*(I)$ has the integral decomposition property. For $k \in \mathbb{N}$ and $\mathbf{a} \in kN(I) \cap \mathbb{N}^n$, we need to show that $\mathbf{a}$ is a sum of k integral vectors of $N(I)$. Indeed, we can express $\mathbf{a} = k\mathbf{b}$, where $\mathbf{b} \in \mathbb{Q}^n \cap N(I)$. By (b') of Lemma 3.3.3, there exists $\mathbf{b}' \leq \mathbf{b}$ satisfying $\mathbf{b}' \in \mathbb{Q}^n \cap N(I)$ and $\mathbf{b}' \leq \mathbf{a}_{max}$. The statement (c) of Lemma 3.3.3 allows us to have $\mathbf{b}' = \sum_{i=1}^t \lambda_i \mathbf{c}_i$, where $\lambda_i \in \mathbb{Q}_+$, $\sum_{i=1}^t \lambda_i = 1$, $\mathbf{c}_i \in \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \leq \mathbf{a} \leq \mathbf{a}_{max}\}$. Therefore, we have $\mathbf{a}_{max} - \mathbf{c}_i \in \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}^*_i \geq \mathbf{a}\}$, so $\mathbf{a}_{max} - \mathbf{b}' = \sum_{i=1}^t \lambda_i (\mathbf{a}_{max} - \mathbf{c}_i) \in N^*(I)$. It is worth noting that $k\mathbf{b}' \in \mathbb{N}^n$ because $k\mathbf{b} = \mathbf{a} \in \mathbb{N}^n$ (see (b') of Lemma 3.3.3).

This results in $k\mathbf{a}_{max} - k\mathbf{b}' \in \mathbb{N}^n \cap kN^*(I)$. Consequently, we obtain $k\mathbf{a}_{max} - k\mathbf{b}' = \sum_{i=1}^{k} \mathbf{v}_i$, where $\mathbf{v}_i \in \mathbb{N}^n \cap N^*(I)$ because of the integral decomposition property of $N^*(I)$. By Lemma 3.2.1, we have $\mathbf{a}_{max} - \mathbf{v}_i \in N(I)$. As a particular consequence, it must satisfy that $\mathbf{a} - k\mathbf{b}' + \mathbf{a}_{max} - \mathbf{v}_1 \in N(I)$ since $\mathbf{a} \geq k\mathbf{b}'$ (see Lemma 3.2.2). To sum up, the equality $\mathbf{a} = (\mathbf{a} - k\mathbf{b}' + \mathbf{a}_{max} - \mathbf{v}_1) + \sum_{i=2}^{k}(\mathbf{a}_{max} - \mathbf{v}_i)$ shows $\mathbf{a}$ is a sum of k integral vectors of $N(I)$. The proof is completed. □

The above proposition is the end of the first module. From the next section, we will apply results in previous sections to algebraic subjects.

3.5 Monomial ideals

Now, let M and M^* denote the matrices whose columns are $\mathbf{a}_i$'s and $\mathbf{a}_i^*$'s, respectively, as the section two. We can see the clear relationship between the monomial ideals and the values ν and ν^* mentioned at the beginning of the chapter.

Proposition 3.5.1. For every $k \in \mathbb{N}$,

(a) $\mathbf{x}^{\mathbf{a}} \in I^k$ if and only if $\nu(\mathbf{a}, M) \geq k$;

(b) $\mathbf{x}^{\mathbf{a}} \in \overline{I^k}$ if and only if $\nu^*(\mathbf{a}, M) \geq k$.

Proof. (a) $\mathbf{x}^{\mathbf{a}} \in I^k$ if and only if $\mathbf{a} \geq \sum_{i=1}^{m} c_i\mathbf{a}_i$ for some $c_i \in \mathbb{N}$ satisfying that $\sum_{i=1}^{m} c_i = k$. Equivalently, the vector $\mathbf{y} := (c_1, \ldots, c_m) \in \mathbb{N}^m$ satisfies $M\mathbf{y}^T \leq \mathbf{a}^T$ and $\mathbf{1} \cdot \mathbf{y} = k$, which means $\nu(\mathbf{a}, M) \geq k$.

(b) By Proposition 1.2.3, $\mathbf{x}^{\mathbf{a}} \in \overline{I^k}$ if and only if $\mathbf{x}^{d\mathbf{a}} \in I^{dk}$ for some $d \in \mathbb{N}$, which is equivalent to $\nu(d\mathbf{a}, M) \geq dk$ by (a). Thus, the condition that $\mathbf{x}^{\mathbf{a}} \in \overline{I^k}$ implies $\nu^*(d\mathbf{a}, M) \geq \nu(d\mathbf{a}, M) \geq dk$, so $\nu^*(\mathbf{a}, M) \geq k$. Conversely, we consider the cases that $\nu^*(\mathbf{a}, M) \geq k$. We need to show that $\nu(d\mathbf{a}, M) \geq dk$ for some $d \in \mathbb{N}$, or equivalently there exists a rational vector $\mathbf{z} \in \mathbb{Q}^m_{\geq 0}$ such that $M\mathbf{z}^T \leq \mathbf{a}^T$ and $\mathbf{1} \cdot \mathbf{z} \geq k$. Indeed, let $\mathbf{y} \in \mathbb{R}^m_+$ be the solution vector for $\nu^*(\mathbf{a}, M)$, i.e., $M\mathbf{y}^T \leq \mathbf{a}^T$ and $\mathbf{1} \cdot \mathbf{y} = \nu^*(\mathbf{a}, M)$. There are some following cases.

When $\mathbf{1} \cdot \mathbf{y} = \nu^*(\mathbf{a}, M) > k$, we choose a sequence of non-negative rational vectors $\{\mathbf{y}_i\}_{i=1}^{\infty}$ satisfying $\mathbf{y}_i \leq \mathbf{y}$ and $\mathbf{y}_i \to \mathbf{y}$ when $i \to \infty$. It is routine to verify that $M\mathbf{y}_i^T \leq \mathbf{a}^T$ for every i and $\mathbf{1} \cdot \mathbf{y}_j \geq k$ for all sufficiently large j. The conclusion holds in this case provided that we choose $\mathbf{z} = \mathbf{y}_j$.

Now, we consider the cases where $\mathbf{1}\cdot\mathbf{y} = \nu^*(\mathbf{a}, M) = k$. The condition that $M\mathbf{y}^T \leq \mathbf{a}^T$ implies that $(a_{1i}, \ldots, a_{mi}) \cdot \mathbf{y} \leq a_i$, where a_{ji} and a_i are the i-th coordinate of $\mathbf{a}_j$ and $\mathbf{a}$, respectively, for every $1 \leq j \leq m$ and $1 \leq i \leq n$. Let S be the set of coordinates i's satisfying $(a_{1i}, \ldots, a_{mi}) \cdot \mathbf{y} = a_i$ iff $i \in S$. When $S = \emptyset$, we choose a sequence of rational vector $\mathbf{y}_i$ such that $\mathbf{y} \leq \mathbf{y}_i$ and $\mathbf{y}_i \to \mathbf{y}$ as $i \to \infty$. Thus, for every $1 \leq i \leq n$, we have $(a_{1i}, \ldots, a_{mi}) \cdot \mathbf{y}_i \to (a_{1i}, \ldots, a_{mi}) \cdot \mathbf{y} < a_i$. Therefore, for sufficiently large j, it must satisfy $(a_{1i}, \ldots, a_{mi}) \cdot \mathbf{y}_j \leq a_i$ for every $1 \leq i \leq n$, or equivalently $M\mathbf{y}_j^T \leq \mathbf{a}^T$. Since $\mathbf{y}_i \geq \mathbf{y}$ for all i, we have $\mathbf{1}\cdot\mathbf{y}_j \geq k$. The conclusion holds in cases $S = \emptyset$ if we choose $\mathbf{z} = \mathbf{y}_j$.

Otherwise, when $S \neq \emptyset$, without loss of generality, we assume that $S = \{1, \ldots, t\}$. Put $\mathbf{u}_i := (a_{i1}, \ldots, a_{it}) \in \mathbb{N}^t$ for every $1 \leq i \leq m$, and $\mathbf{b} = (a_1, \ldots, a_t)$. In the other words, $\mathbf{u}_i$'s and $\mathbf{b}$ are the images of $\mathbf{a}_i$'s and $\mathbf{a}$ of the projection from $\mathbb{R}^n$ onto the vector space $\mathbb{R}^t$ of the first t coordinates, respectively. It is routine to verify that $\frac{\mathbf{b}}{k} = \sum_{i=1}^m \frac{y_i}{k}\mathbf{u}_i$, where y_i is the i-th coordinate of $\mathbf{y}$. By Lemma 3.2.2, there exists a sequence of non-negative rational vectors $(y_1^{(p)}, \ldots, y_m^{(p)})$ such that $\frac{\mathbf{b}}{k} = \sum_{i=1}^m y_i^{(p)}\mathbf{u}_i$ (or equivalently $\frac{a_i}{k} = (a_{1i}, \ldots, a_{mi})\cdot(y_1^{(p)}, \ldots, y_m^{(p)})$ for every $1 \leq i \leq t$), $\sum_{i=1}^m y_i^{(p)} = 1$ and $(y_1^{(p)}, \ldots, y_m^{(p)}) \to \frac{\mathbf{y}}{k}$ as $p \to \infty$. For each $i \notin S$, the fact that $(a_{1i}, \ldots, a_{mi})\cdot\mathbf{y} < a_i$ implies that $(a_{1i}, \ldots, a_{mi})\cdot(y_1^{(p)}, \ldots, y_m^{(p)}) \leq \frac{a_i}{k}$ for sufficiently large p. As a result, for sufficiently large p, we have $M(y_1^{(p)}, \ldots, y_m^{(p)})^T \leq \frac{\mathbf{a}^T}{k}$, so $M(ky_1^{(p)}, \ldots, ky_m^{(p)})^T \leq \mathbf{a}^T$. The conclusion holds provided that we choose $\mathbf{z} = (ky_1^{(p)}, \ldots, ky_m^{(p)})$. The proof is completed.

□

3.6 The integral closeness

In order to have enough conditions to show the final proposition in the chapter, we need to show $N(I)$ and $N^*(I)$ have fully the requirements in Proposition 3.1.3. This is the maximization and minimization of $\mathbf{a}_i^*$'s and $\mathbf{a}_i$'s in $N^*(I)$ and $N(I)$ respectively. Now, we give an equivalent version of Proposition 1.2.3.

Lemma 3.6.1. $\overline{I^k} = (\mathbf{x}^{\boldsymbol{a}} \mid \boldsymbol{a} \in kN(I) \cap \mathbb{N}^n)$.

Proof. By [13] (Proposition 7.3.12, page 237), $\overline{I^k} = (\mathbf{x}^{\mathbf{a}} \mid \mathbf{a} \in \text{ conv } \{\mathbf{u} \mid \mathbf{x}^{\mathbf{u}} \in I^k\} \cap \mathbb{N}^n)$.

Now, we have the following inclusions:

$$\begin{aligned}\overline{I^k} &= (\mathbf{x}^{\mathbf{a}} \mid \mathbf{a} \in \ \text{conv}\{\mathbf{u} \mid \mathbf{x}^{\mathbf{u}} \in I^k\} \cap \mathbb{N}^n) \\ &\subseteq (\mathbf{x}^{\mathbf{a}} \mid \mathbf{a} \in \ \text{conv}\{\mathbf{u} \mid \mathbf{u} \in kN(I)\} \cap \mathbb{N}^n) \\ &= (\mathbf{x}^{\mathbf{a}} \mid \mathbf{a} \in \ \text{conv}\{kN(I)\} \cap \mathbb{N}^n) \\ &= (\mathbf{x}^{\mathbf{a}} \mid \mathbf{a} \in kN(I) \cap \mathbb{N}^n) \text{ (because } N(I) \text{ is convex).}\end{aligned}$$

Conversely, for $\mathbf{a} \in kN(I) \cap \mathbb{N}^n$, we have $\mathbf{a} = k\mathbf{u}$ for some $\mathbf{u} \in N(I) \cap \mathbb{Q}^n$. By Lemma 3.3.1, $\mathbf{u}$ is a convex combination $\sum_{i=1}^t \lambda_i \mathbf{y}_i$, where $\lambda_i \in \mathbb{Q} \cap [0,1]$ and $\mathbf{y}_i \in \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \leq \mathbf{a}\}$. By multiplying both terms by an integer λ so that $\lambda\lambda_i$'s are all integers, we obtain that $\lambda\mathbf{u}$ is a sum of λ vectors of $\{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \leq \mathbf{a}\}$, and so, $\lambda\mathbf{a} = \lambda k \mathbf{u}$ is a sum of $k\lambda$ elements of $\{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \leq \mathbf{a}\}$. This means that $\mathbf{x}^{\lambda\mathbf{a}}$ is divided by $k\lambda$ monomials of I, or equivalently $(\mathbf{x}^{\mathbf{a}})^\lambda \in (I^k)^\lambda$. Thus, $\mathbf{x}^{\mathbf{a}} \in \overline{I^k}$ by Proposition 1.2.3. The proof is completed. □

The main result of this section is the following proposition that gives equivalent conditions of the integral closeness of a monomial ideal.

Proposition 3.6.2. The three following conditions are equivalent:

(a) $N^*(I) \cap \mathbb{N}^n = \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i^* \geq \mathbf{a}\}$;

(b) $N(I) \cap \mathbb{N}^n = \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \leq \mathbf{a}\}$;

(c) I is integrally closed;

(d) $\{\mathbf{a}_i^*, 1 \leq i \leq m\}$ consists of all maximal integral points of $N^*(I)$. In other words, columns of M^* are precisely maximal integral points of $N^*(I)$;

(e) $\{\mathbf{a}_i, 1 \leq i \leq m\}$ consists of all minimal integral points of $N(I)$. In other words, columns of M are precisely minimal integral points of $N(I)$.

Proof. $(a) \Rightarrow (b)$: Assume that $N^*(I) \cap \mathbb{N}^n = \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i^* \geq \mathbf{a}\}$, we claim that $N(I) \cap \mathbb{N}^n = \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \leq \mathbf{a}\}$. Indeed, for each $\mathbf{u} \in N(I) \cap \mathbb{N}^n$, there exists $\mathbf{v} \in N(I) \cap \mathbb{N}^n$ and $\mathbf{v} \leq \mathbf{a}_{max}$ and $\mathbf{v} \leq \mathbf{u}$ by (b) of Lemma 3.3.3. Thus, by Lemma 3.3.4,

$$\mathbf{a}_{max} - \mathbf{v} \in N^*(I) \cap \mathbb{N}^n = \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i^* \geq \mathbf{a}\}.$$

This implies that $\mathbf{v} \geq \mathbf{a}_i$ for some i, and so $\mathbf{u} \in \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \leq \mathbf{a}\}$. Thus, $N(I) \cap \mathbb{N}^n \subseteq \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \leq \mathbf{a}\}$. Additionally, the converse inclusion is verified routinely.

$(b) \Rightarrow (a)$: If $N(I) \cap \mathbb{N}^n = \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \leq \mathbf{a}\}$, then we have

$$\begin{aligned} N^*(I) \cap \mathbb{N}^n &\subseteq \{\mathbf{a}_{max} - \mathbf{a} \mid \mathbf{a} \in N(I) \cap \mathbb{N}^n, \mathbf{a} \leq \mathbf{a}_{max}\} (\text{by Lemma 3.2.1}) \\ &= \{\mathbf{a}_{max} - \mathbf{a} \mid \exists \mathbf{a}_i \leq \mathbf{a} \leq \mathbf{a}_{max}\} \\ &= \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i^* \geq \mathbf{a}\}. \end{aligned}$$

The converse inclusion is verified routinely, so we obtain the equality.

$(b) \Leftrightarrow (c) \Leftrightarrow (e)$: This is a consequence of Lemma 3.6.1.

$(a) \Leftrightarrow (d)$: This is an obvious statement!

□

3.7 Technical results

Afterwards, by using the Proposition 3.6.2, we come into the following result that shows clearly the equivalence between the normality of the monomial ideal I and the integer round-down property of M. Its way of proving is purely technical and its role is also a tool of reaching Proposition 3.8.1. Moreover, the Proposition 3.8.1 is its comprehensive version of the Proposition 3.6.2.

Lemma 3.7.1. *I is normal if and only if M has the integer round-down property.*

Proof. Assume that I is normal. For an arbitrary vector $\mathbf{a} \in \mathbb{N}^n$, we need to show that $\nu(\mathbf{a}, M) = \lfloor \nu^*(\mathbf{a}, M) \rfloor$. By setting $k := \lfloor \nu^*(\mathbf{a}, M) \rfloor$, we have $\mathbf{x}^{\mathbf{a}} \in \overline{I^k}$ by (ii) of Proposition 3.5.1. Due to the normality of I, $\mathbf{x}^{\mathbf{a}}$ belongs to I^k, and so $\nu(\mathbf{a}, M) \geq k$ by (i) of Proposition 3.5.1. Consequently, it must satisfy $\nu(\mathbf{a}, M) = k$.

Conversely, if $\nu(\mathbf{a}, M) = \lfloor \nu^*(\mathbf{a}, M) \rfloor$ for all $\mathbf{a} \in \mathbb{N}^n$. For every $k \in \mathbb{N}$, we claim that $I^k = \overline{I^k}$. Indeed, we get a monomial $\mathbf{x}^{\mathbf{a}} \in \overline{I^k} \setminus I^k$ if $\overline{I^k} \neq I^k$. By Proposition 3.5.1, it must satisfy that $\nu^*(\mathbf{a}, M) \geq k$ and $\nu(\mathbf{a}, M) \leq k - 1$, which contradicts the assumption of the case. The proof is completed. □

After obtaining the relationship between the normality and the integer round-down property, we can pose a question how about the normality and the integer round-up property? Fortunately, we also have the similar result, Proposition 3.8.1. But before reaching it, we need to mention some useful properties that not only help us to understand the structure of polyhedra, but allow us also to prove the final proposition.

Lemma 3.7.2. *If M^* has the integer round-up property, then I is integrally closed.*

Proof. By Proposition 3.6.2, it is sufficient to show that $N^*(I) \cap \mathbb{N}^n = \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i^* \geq \mathbf{a}\}$. Assume that there exists an integral vector $\mathbf{u} \in N^*(I) \cap \mathbb{N}^n \setminus \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i^* \geq \mathbf{a}\}$. Thus, $\tau^*(\mathbf{u}, M^*) \leq 1$ because $\mathbf{u}^T$ is a convex combination of columns of M^*. Meanwhile, $\tau(\mathbf{u}, M^*) > 1$ because $\mathbf{u} \nleq \mathbf{a}_i^*$ for all i. This contradicts the round-up property of M^*. □

After seven sections, now we have enough equipment that allow us to reach the final result.

3.8 The normality

Now, we have the following main result, it is a consequence of the previous. Furthermore, it is another presentation of Proposition 3.1.3 of S. Baum and Jr. L. E. Trotter ([12]) in an algebraic language.

Proposition 3.8.1. The following conditions related to the normality of I are equivalent:

(a) I is integrally closed and $N^*(I)$ has the integral decomposition property;

(b) I is integrally closed and $N(I)$ has the integral decomposition property;

(c) M^* has the integer round-up property;

(d) M has the integer round-down property.

(e) I is normal.

Proof. $(a) \Leftrightarrow (b)$: This is a consequence of Proposition 3.4.1.

$(a) \Rightarrow (c)$: The integral closeness of I allows the columns of M^* to be precisely maximal integral points of $N^*(I)$ by Proposition 3.6.2. By (b) of Proposition 3.1.3 and Lemma 3.3.5, M^* has the integer round-up property.

$(c) \Rightarrow (a)$: If M^* has the integer round-up property, then I is integrally closed by Proposition 3.6.2, and so the columns of M^* are precisely maximal integral points of $N^*(I)$ by Proposition 3.6.2. By (b) of Proposition 3.1.3 and Lemma 3.3.5, $N^*(I)$ has the integral decomposition property.

$(b) \Rightarrow (d)$: We can cope with the part in a similar way in the above part. Meanwhile, we show another method of proving the part by Lemma 3.7.1 to clarify the equivalence between the normality of I and the integral decomposition property of $N(I)$. By Lemma

3.7.1, it is sufficient to show that I is normal, i.e., $I^k = \overline{I^k}$ for every $k \in \mathbb{N}$. However, we can obtain it from the integral decomposition property of $N(I)$ as follows:

$$\begin{aligned}
\overline{I^k} &= (\mathbf{x}^{\mathbf{a}} \mid \mathbf{a} \in kN(I) \cap \mathbb{N}^n)(\text{ Lemma 3.6.1}) \\
&\subseteq \left(\mathbf{x}^{\mathbf{a}} \mid \mathbf{a} = \sum_{i=1}^{k} \mathbf{u}_i, \mathbf{u}_i \in N(I) \cap \mathbb{N}^n\right) (\text{ The integral decomposition property of } N(I)) \\
&= (\mathbf{x}^{\mathbf{u}_1} \cdots \mathbf{x}^{\mathbf{u}_k} \mid \mathbf{u}_i \in N(I) \cap \mathbb{N}^n) \\
&= (\mathbf{x}^{\mathbf{u}_1} \cdots \mathbf{x}^{\mathbf{u}_k} \mid \mathbf{u}_i \in \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \leq \mathbf{a}\}) \\
&= I^k
\end{aligned}$$

The converse inclusion is verified routinely, so we have proved that I is integrally closed.

$(d) \Rightarrow (b)$: By Lemma 3.7.1, I is normal. In particular, I is integrally closed. This implies M consists of all minimal integral points of $N(I)$ by Proposition 3.6.2. By (a) of Proposition 3.1.3 and Lemma 3.2.2, $N(I)$ has the integral decomposition property.

$(d) \Leftrightarrow (e)$: This is the Lemma 3.7.1. The proof is completed. □

Finally, we end the Chapter 3. Through the long process, we can get a comprehensive perspective on the relation between three wide areas: Commutative Algebra, Discrete Geometry and Linear Programming. Moreover, this big picture also provides us with various open problems, and this dissertation is the first step in this research direction. Now, we come to the final chapter where we answer the question posed in the Introduction at the beginning of this thesis.

Chapter 4

Unimodular covering

In this chapter, we mention some geometric definitions that play the important role in the main result of this dissertation, Proposition 4.3.1. This proposition shows a sufficient condition for the normality of a monomial ideal.

4.1 Unimodular polytopes

With the aim of simplifying, we do not recall the definitions in Affine Geometry.

Definition 4.1.1. Let $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_n$ be affinely independent points in $\mathbb{R}^n$, and the simplex $\Delta = \mathrm{conv}\{\mathbf{a}_0, \ldots, \mathbf{a}_n\}$ is the convex hull of these points. Then the *(relative) volume* of Δ, denoted by $\mathrm{vol}(\Delta)$, is defined to be

$$\mathrm{vol}(\Delta) = \frac{\left|\det\begin{pmatrix}\mathbf{a}_0 & 1\\ \vdots & \vdots \\ \mathbf{a}_n & 1\end{pmatrix}\right|}{n!} = \frac{\left|\det\begin{pmatrix}\mathbf{a}_1 - \mathbf{a}_0\\ \vdots \\ \mathbf{a}_n - \mathbf{a}_0\end{pmatrix}\right|}{n!}.$$

And the value $n!\mathrm{vol}(\Delta)$ is called to be the *normalized volume* of Δ.

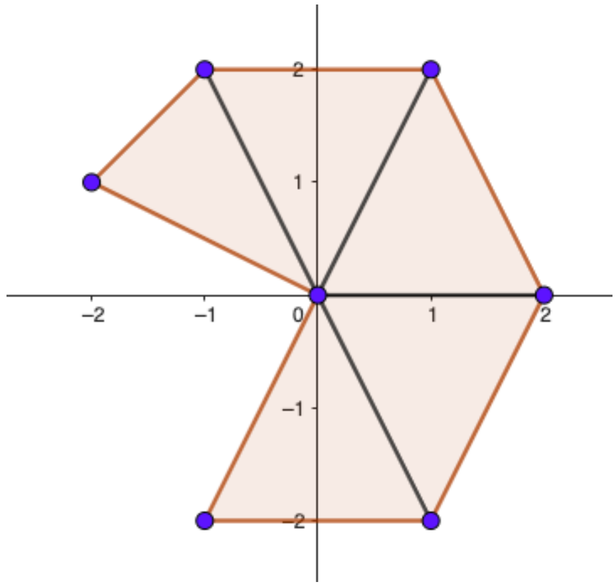
Indeed, the above relative volume of a simplex in $\mathbb{R}^n$ is consistent with the definition of volume in $\mathbb{R}^2$ (area) and $\mathbb{R}^3$ (volume).

Definition 4.1.2. A set Δ in $\mathbb{R}^n$ is called a *lattice d-simplex* if Δ is the convex hull of a set of $d+1$ affinely independent integral points.

Therefore, if Δ is a lattice n-simplex, then

$$\mathrm{vol}(\Delta) \in \mathbb{Z}_+\left(\frac{1}{n!}\right),$$

so $\text{vol}(\Delta) \geq \frac{1}{n!}$. We can observe this property in $\mathbb{R}^2$. Indeed, every polygon whose vertices are all integral has the semi-integral area.



The polygon in this picture has 7 vertices
$(-2,1),(-1,2),(1,2),(2,0),(1,-2),(-1,-2),(0,0)$ and its area is $\frac{19}{2}$.

Now, we come to the definition of a class of simplexes that play the role as the "cells" of polytopes.

Definition 4.1.3. A lattice n-simplex Δ in $\mathbb{R}^n$ is called to be *unimodular* if $\text{vol}(\Delta) = \frac{1}{n!}$, or equivalently its normalized volume is 1.

The relation between Discrete Geometry and Algebra is shown in the following results and definition. They play the role as the bridge connecting algebraic language and geometric perspective in the progress towards Proposition 4.3.1.

Proposition 4.1.4. Let Δ be a lattice n-simplex with vertices $\mathbf{a}_0, \ldots, \mathbf{a}_n$ in $\mathbb{R}^n$. Then the following are equivalent:

(i) Δ is unimodular;

(ii) $\sum_{i=1}^{n} \mathbb{Z}(\mathbf{a}_i - \mathbf{a}_0) = \mathbb{Z}^n$;

(iii) The additive quotient group

$$\frac{\mathbb{Z}^{n+1}}{\sum_{i=0}^{n} \mathbb{Z}(\mathbf{a}_i, 1)}$$

is torsion-free, this means that only element $\overline{0}$ in the group has the finite order.

Proof. They are corollaries of [5] (Proposition 1.2.21). Indeed, the proposition can be showed by the result: if $\sum_{i=1}^{n} \mathbb{R}(\mathbf{a}_i - \mathbf{a}_0) \cap \mathbb{Z}^n = \sum_{i=1}^{n} \mathbb{Z}\gamma_i$ and $\mathbf{a}_i - \mathbf{a}_0 = \sum_{j=1}^{n} c_{ij}\gamma_j$, then $\mathrm{vol}(\Delta) = \frac{|\det(c_{ij})|}{n!}$. By this result, if Δ is unimodular, then $\sum_{i=1}^{n} \mathbb{R}(\mathbf{a}_i - \mathbf{a}_0) \cap \mathbb{Z}^n = \mathbb{R}^n \cap \mathbb{Z}^n = \mathbb{Z}^n$ and $\det(c_{ij}) = \pm 1$, so we can choose $\gamma_i = \mathbf{e}_i$ and (c_{ij}) is invertible. This implies (ii). By mimicking the similar proof, we can prove the whole proposition. □

The following corollary helps us to find whether an integral simplex is not unimodular. To be more detailed, if it contains some integral point apart from its vertices, then it must be not unimodular.

Corollary 4.1.5. *If Δ is an unimodular lattice n-simplex in $\mathbb{R}^n$ with $n+1$ vertices $\mathbf{a}_0, \ldots, \mathbf{a}_n$, then $\Delta \cap \mathbb{Z}^n = \{\mathbf{a}_0, \ldots, \mathbf{a}_n\}$.*

Proof. For an arbitrary point $\mathbf{a} \in \Delta \cap \mathbb{Z}^n$, we have $\mathbf{a} - \mathbf{a}_0 \in \mathbb{Z}^n$, so

$$\mathbf{a} - \mathbf{a}_0 = \sum_{i=1}^{n} c_i(\mathbf{a}_i - \mathbf{a}_0),$$

for some $c_i \in \mathbb{Z}$, by Proposition 4.1.4. On the other hand, since $\mathbf{a} \in \Delta = \mathrm{conv}\{\mathbf{a}_0, \ldots, \mathbf{a}_n\}$, there exist real numbers $d_i \geq 0, 0 \leq i \leq n$, such that

$$\mathbf{a} = \sum_{i=0}^{n} d_i \mathbf{a}_i,$$

and $\sum_{i=0}^{n} d_i = 1$. Therefore, $\mathbf{a} - \mathbf{a}_0 = \sum_{i=1}^{n} d_i(\mathbf{a}_i - \mathbf{a}_0)$. Since $\mathbf{a}_0, \ldots, \mathbf{a}_n$ are affinely independent, the vectors $\mathbf{a}_1 - \mathbf{a}_0, \ldots, \mathbf{a}_n - \mathbf{a}_0$ are linearly independent. Hence, $d_i = c_i \in \mathbb{N}$ for every $1 \leq i \leq n$. As a consequence, the equality $\sum_{i=0}^{n} d_i = 1$ implies either $d_0 = 1$ or exactly one $d_i = 1$ among $1 \leq i \leq n$. Equivalently, we obtain either $\mathbf{a} = \mathbf{a}_0$ or $\mathbf{a} = \mathbf{a}_i$. The proof is completed. □

Remark 4.1.6. The converse of the Corollary 4.1.5 only holds in $\mathbb{R}^2$ and is called Pick's Theorem. In $\mathbb{R}^n$ where $n \geq 3$, there are some integral simplexes that do not contain any integral points apart from their vertices have volume greater then $\frac{1}{n!}$. Indeed, we can consider a tetrahedron with vertices: $(0,0,0), (1,0,0), (0,1,0), (1,1,k)$. Its volume is $\frac{k}{6}$ and it does not contain any integral points apart from the four vertices. As $k \to \infty$, we can obtain tetrahedrons that do not contain any integral points apart from their vertices have volume as big as possible.

We call a polytope with integral vertices as a *lattice polytope* .

Definition 4.1.7. A lattice polytope P of dimention n (equivalently P contains at least $n+1$ affinely independent points) is called to have a *unimodular covering* if there are lattice n-simplex $\Delta_1, \dots, \Delta_t$ satisfying both two following conditions:

(i) $P = \cup_{i=1}^{t} \Delta_i$;

(ii) All simplices Δ_i's are unimodular.

4.2 Normal polytopes

In the prior chapters, we mentioned the normality of monomial ideals and the normality of toric rings; and now, we come to the concept of the normality of polytopes. Afterwards, we will consider the relation between all of the properties in this section.

Definition 4.2.1. Let P be a lattice polytope in $\mathbb{R}^n$ and S be the additive monoid generated by the vectors $(\mathbf{a}, 1)$ where $\mathbf{a} \in P \cap \mathbb{Z}^n$. We call P *normal* if $\overline{S} = S$.

We have two sufficient conditions, Propositions 4.2.2 and 4.2.3, of the normality of a polytope. In particular, Proposition 4.2.2 is an algebraic property, and the Proposition 4.2.3 illustrates a geometric condition. Thus, we can realize that the Definition 4.2.1 is the center of our interpretation between algebraic and geometric languages.

Proposition 4.2.2. $K[S]$ is normal if and only if P is normal.

Proof. This is an equivalent statement of the Proposition 2.4.2. □

Proposition 4.2.3. P is normal if and only if P has the integral decomposition property.

Proof. Initially, we assume that P is normal, which means that $\overline{S} = S$. For every $k \in \mathbb{N}$ and $\mathbf{a} \in kP \cap \mathbb{Z}^n$, we have $(\mathbf{a}, k) \in C(S)$ and $(\mathbf{a}, k) \in \mathbb{Z}^{n+1} = \mathbb{Z}S$. Therefore, $(\mathbf{a}, k) \in \overline{S} = S$. Thus, $(\mathbf{a}, k)$ is a sum of vectors of form $(\mathbf{a}', 1)$ where $\mathbf{a}' \in P \cap \mathbb{Z}^n$. Hence, $(\mathbf{a}, k)$ is the sum of exactly k those vectors, or equivalently, $\mathbf{a}$ is a sum of k integral vectors in P.

Conversely, we assume that P has the integral decomposition property. For every $(\mathbf{a}, k) \in \overline{S} = C(S) \cap \mathbb{Z}^{n+1}$, there exist vectors $\mathbf{a}'_i \in P, 1 \leq i \leq t$, such that

$$(\mathbf{a}, k) = \sum_{i=1}^{t} \lambda_i (\mathbf{a}'_i, 1),$$

where $\lambda_i > 0$. Therefore, $\sum_{i=1}^{t} \lambda_i = k$ and $\mathbf{a} = \sum_{i=1}^{t} \lambda_i \mathbf{a}'_i$. Hence, $\mathbf{a} \in kP$. Since P has the integral decomposition property, $\mathbf{a}$ is a sum of k integral vectors in P. Thus, $(\mathbf{a}, k)$ is the sum of k vectors of form $(\mathbf{a}', 1)$ where $\mathbf{a}' \in P \cap \mathbb{Z}^n$. This implies $(\mathbf{a}, k) \in S$. The proof is completed. □

Now, we mention a helpful proposition with the input being a geometric condition and output being an algebraic property.

Proposition 4.2.4. If P has a unimodular covering, then $K[S]$ is normal.

Proof. See [5] (Proposition 9.3.5). □

4.3 Main result and the following research

The following proposition is the main and final result of this dissertation. This is a sufficient condition of the normality of a monomial ideal. Indeed, it is an equivalent expression of the Bruns-Gubeladze-Trung Theorem ([1]).

Proposition 4.3.1. If $N^*(I)$ has a unimodular covering and I is integrally closed, then I is normal.

Proof. By Proposition 4.2.4, $K[S]$ is normal, so $N^*(I)$ is normal by Proposition 4.2.2. Therefore, $N^*(I)$ has the integral decomposition property by Proposition 4.2.3. Finally, thanks to Proposition 3.8.1, I is normal. □

Example 4.3.2. When $I = (x^4y, x^3y^2, x^2y^3, xy^4) \subset k[x, y]$, we can verify that I is integrally closed by Proposition 3.6.2: $N^*(I) \cap \mathbb{N}^n = \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i^* \geq \mathbf{a}\}$. Moreover, $N^*(I)$ is a polytope in $\mathbb{R}^2$, it is triangularizable, so it has a unimodular covering obviously. Therefore, I is normal by 4.3.1.

We can see that it is easier to verify whether $N^*(I)$ has a unimodular covering or not because it is bounded. Although the Proposition 4.3.1 is only a sufficient condition, it is so helpful for us to utilize. Moreover, the converse of Proposition 4.3.1 often holds in a lot of low-dimensional cases, which is certain that the unimodular covering of the dual Newton polytope corresponding to an integrally closed monomial ideal is "approximate" to its normality. This shows the Proposition 4.3.1 is not only a sufficient condition, but also is "nearly" a necessary one, which is the meaningful role of this result.

In order to find a unimodular covering of a polytope (if it exists), we can use the elementary techniques from Discrete Geometry and this is really our work later. We hope to find an useful algorithm that can provide a unimodular covering of lower comprehensive integral polytopes. This is a potenial direction because there have been various algorithms by both mathematicians and experts in Computer Science that can illustrate the unimodular covering of integral polytopes in small-dimensional spaces (see [14]).

Finally, we complete the dissertation and hope it can provide an interesting perspective on the intersection of Commutative Algebra, Discrete Geometry and Linear Programming. Nowadays, the intersection of three areas has been widened significantly thanks to a lot of effort of mathematicians and became an increasing trend towards modern mathematics which attracts much attention of experts. This thesis is a piece of this development and the author would like to thank all work of previous people who contributed to this direction generally and this thesis partially. In the future, the results in this dissertation will be enhanced and will be a contribution to community.

References

[1] J. Gubeladze W. Bruns and N. V. Trung. “Normal polytopes, triangulations, and Koszul algebras”. In: *Journal für die reine und angewandte Mathematik* 485 (1997), pp. 123–160.

[2] J. J. Rotman. *Advanced Modern Algebra*. Prentice Hall, Upper Saddle River, NJ, 2002. ISBN: 0-13-087868-5.

[3] Jürgen Herzog, Takayuki Hibi, and Hidefumi Ohsugi. *Binomial Ideals*. Springer, 2018. ISBN: 3319953494, 9783319953496.

[4] John Little David Cox and Donal O’Shea. *Ideals, Varieties, and Algorithms*. Springer, 2007. ISBN: 978-0-387-35650-1.

[5] Rafael Villarreal. *Monomial Algebras*. Chapman and Hall/CRC, 2015. ISBN: 9781351229203. URL: https://www.taylorfrancis.com/books/mono/10.1201/b18224/monomial-algebras-rafael-villarreal.

[6] Leslie G. Roberts Les Reid and Marie A. Vitulli. “Some Results On Normal Homogeneous Ideals”. In: *Communications in Algebra* 31.9 (2002), p. 19. DOI: 10.1081/AGB-120022805.

[7] David A. Cox, John B. Little, and Henry K. Schenck. *Toric Varieties*. American Mathematical Society, 2011. ISBN: 978-0821848197. URL: https://www.amazon.com/Toric-Varieties-Graduate-Studies-Mathematics/dp/0821848194.

[8] Bernd Sturmfels. *Grobner Bases and Convex Polytopes*. American Mathematical Society, 1995. ISBN: 978-0821804872. URL: https://www.amazon.com/Grobner-Convex-Polytopes-University-Lecture/dp/0821804871.

[9] Winfried Bruns and H. Jürgen Herzog. *Cohen-Macaulay Rings*. Cambridge University Press, 1998. ISBN: 9780511608681. URL: https://www.cambridge.org/core/books/cohenmacaulay-rings/938BC2204D8A7C99E2CEBA1695A692A4.

[10] Ngô Viêt Trung and Lê Tuân Hoa. “Affine Semigroups and Cohen-Macaulay Rings Generated by Monomials”. In: *Transactions of the American Mathematical Society* 298.1 (1986), pp. 145–167. DOI: https://doi.org/10.2307/2000613.

[11] I. G. MacDonald Michael F. Atiyah. *Introduction To Commutative Algebra*. Cambridge University Press, 1994. ISBN: 978-0201407518. URL: https://www.amazon.com/Introduction-Commutative-Algebra-Addison-Wesley-Mathematics/dp/0201407515.

[12] S. Baum and Jr. L. E. Trotter. "Integer Rounding for Polymatroid and Branching Optimization Problems". In: *SIAM Journal on Algebraic Discrete Methods* 2.4 (1981), pp. 416–425. DOI: 10.1137/0602044.

[13] Rafael Villarreal. *Monomial Algebras*. M. Dekker, 2001. ISBN: 0824705246, 9780585425160. URL: https://ebookst.com/download/monomial-algebras-4735/.

[14] J. Gubeladze W. Bruns and N. V. Trung. "Problems and algorithms for affine semigroups". In: *Semigroup Forum* 64.2 (2001), p. 27. DOI: 10.1007/s002330010099.

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