MINISTRY OF EDUCATION
AND TRAINING

VIETNAM ACADEMY OF
SCIENCE AND TECHNOLOGY

GRADUATE UNIVERSITY OF SCIENCE AND TECHNOLOGY

LIEU LONG HO

HILBERT-SERRE THEOREM
FOR INFINITE DIMENSIONAL POLYNOMIAL RINGS

MASTER THESIS IN MATHEMATICS

Hanoi - 2025

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Declaration

I declare that the master thesis titled "Hilbert-Serre Theorem for Infinite Dimensional Polynomial Rings" is my original work. All the sources of information and ideas used in this thesis have been acknowledged and properly cited. I certify that this work has not previously been submitted in whole, or in part, for any degree or diploma at this or any other tertiary institution. Furthermore, I declare that this thesis represents my own work and that all external sources of information have been appropriately acknowledged. I take full responsibility for its content.

Hanoi, March 2025

Signature of Student

Lieu Long Ho

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Introduction

Consider a polynomial ring $R = K[x_1, ..., x_n]$ defined in n variables over a field K. Let M be a finitely generated graded R-module. Such a module decomposes into its homogeneous components M_i indexed by integers i. The Hilbert series associated with M is then defined as the generating function for the sequence $\{\dim_K(M_i)\}_{i\in\mathbb{Z}}$, where the dimensions of these components as K-vector spaces. The classical Hilbert-Serre theorem states that the Hilbert series of any finitely generated graded module is a rational function, meaning it can be expressed as the quotient of two polynomials. The Hilbert series serves as a fundamental invariant of graded rings and modules, aiding in the determination of essential invariants such as the dimension and degree of projective varieties.

Motivated by applications in algebraic statistics [1, 2] and representation theory [3], recent research has focused on studying ideals in a polynomial ring in infinitely many indeterminates, which are stable under the action of a monoid. Specifically, fix an integer $c \geq 1$ and let $K[X] = K[x_{ij}]$ be a polynomial ring in the variables x_{ij}, where $1 \leq i \leq c$ and $j \geq 1$. Let $\mathrm{Inc}(\mathbb{N})$ be the monoid of strictly increasing maps $p\colon \mathbb{N} \to \mathbb{N}$, with the composition operator. Let $\mathrm{Sym}(\mathbb{N})$ be the set of maps from $\mathbb{N} \to \mathbb{N}$, fixing all but a finite number of positive integers, with the composition operator. In other words, $\mathrm{Sym}(\mathbb{N})$ is the direct limit of symmetric groups on n elements $\mathrm{Sym}(n)$, with the natural embedding $\mathrm{Sym}(n) \to \mathrm{Sym}(n+1)$. There are many research focus on the ideals I in the infinite dimensional ring $K[X]$, which are stable under the action of $\mathrm{Inc}(\mathbb{N})$ or $\mathrm{Sym}(\mathbb{N})$. We know that $K[X]$ is not a noetherian ring, since there are non-finitely generated ideals. However, a result settled by Cohen, Aschenbrenner-Hillar, Hillar-Sullivant states that $K[X]$ is a $\mathrm{Sym}(\mathbb{N})$-noetherian ring, that is every $\mathrm{Sym}(\mathbb{N})$-invariant ideal is generated by the orbits of finitely many elements. The same result is true when we replace $\mathrm{Sym}(\mathbb{N})$ by $\mathrm{Inc}(\mathbb{N})$. This is the infinite dimensional version of the Hilbert's basis theorem.

A natural question arising is whether the classical Hilbert-Serre theorem can be extended to the infinite dimensional case, in particular, for $\mathrm{Inc}(\mathbb{N})$-invariant ideals?

This question is not trivial because even defining the Hilbert series for the ring $K[X]/I$ is not straightforward. A method to study these ideals is to consider the ascending chain of truncated ideals $I_n = I \cap K[X_n]$, where X_n is the set of variables $\{x_{ij} \mid 1 \leq i \leq c, 1 \leq j \leq n\}$. From this perspective, the equivariant (bigraded) Hilbert series of $K[X]/I$ can be defined as the generating function of the sequence of Hilbert series of $K[X_n]/I_n$ for varying n.

This method was introduced by Nagel and Römer in [4], who proved that for any homogeneous ideal $I \subset K[X]$, the equivariant Hilbert series of $K[X]/I$ is also a rational function. This thesis will focus on studying the Nagel-Römer theorem.

This thesis has three chapters.

1. Chapter 1 explains the theories of graded rings and graded modules, which are essential for establishing the classical Hilbert-Serre theorem. Additionally, we review the theory of Gröbner bases and monomial ideals, along with their properties.

2. Chapter 2 establishes the framework to prove the finiteness up to symmetry of equivariant Gröbner bases. Then we apply this framework to prove the Hilbert's basis theorem for infinite dimensional polynomial rings, which is due to Cohen [5], Aschenbrenner-Hillar [1] and Hillar-Sullivant [2]. The main results of this chapter are Kruskal's tree theorem (Theorem 2.14), Higman's lemma (Corollary 2.15) and Hilbert's basis theorem for infinite dimensional polynomial rings (Corollary 2.33).

3. Chapter 3 presents the principal theorems concerning the equivariant Hilbert series, along with its implications and several detailed computational examples of the equivariant Hilbert series. The main theorem of this thesis is Theorem 3.6:

   Theorem 3.6. *Assume $\mathcal{I} = (I_n)_{n\in\mathbb{N}}$ is an $\mathrm{Inc}(\mathbb{N})^i$-invariant chain of homogeneous ideals, where $i \geq 0$ is an integer. Then*

   $$H_{\mathcal{I}}(s,t) = \frac{g(s,t)}{(1-t)^a \cdot \prod_{j=1}^{b}[(1-t)^{c_j} - s \cdot f_j(t)]},$$

   where $a, b, c_j \geq 0$ are integers, $g(s,t) \in \mathbb{Z}[s,t]$, and each $f_j(t) \in \mathbb{Z}[t]$ such that $f_j(1) > 0$.

   The reference for our main result is [4, Theorem 7.2].

Chapter 1
Preliminaries

Throughout this thesis, by "a ring" we always mean a commutative ring with identity element. The primary focus of this work involves graded rings and modules. Our main references are Atiyah-Macdonald [6], Bruns-Herzog [7], and Cox-Little-O'shea [8].

1.1 Graded Rings and Modules

Definition 1.1. A ring R is called $\mathbb{Z}$-*graded* (or simply, graded) if there is a family of additive subgroups $\{R_n\}_{n\in\mathbb{Z}}$ of R, such that

(a) $R = \bigoplus_{n\in\mathbb{Z}} R_n$, and

(b) $R_n R_m \subseteq R_{n+m}$ for all $m, n \in \mathbb{Z}$.

An element $x \in R \setminus \{0\}$ is said to be a *homogeneous element* of degree n if $x \in R_n$. Additionally, R is called $\mathbb{N}$-graded if $R_n = 0$ for all $n < 0$.

Every ring R admits a trivial grading, which is obtained by defining $R_0 = R$ and $R_n = 0$ for all $n \neq 0$. Other non-trivial graded rings are given in the following examples.

Example 1.2. (a) The polynomial ring $R = K[x]$ is $\mathbb{N}$-graded with the n-th graded part $R_n = \{\alpha x^n \colon n \geq 0,\, \alpha \in K\}$.

(b) Similarly, the Laurent polynomial ring $R = K[x, x^{-1}]$ is $\mathbb{Z}$-graded with the n-th graded part $R_n = \{\alpha x^n \colon n \in \mathbb{Z},\, \alpha \in K\}$.

(c) Let $S = R[x_1, \ldots, x_d]$ be a polynomial ring over a ring R. S is an $\mathbb{N}$-graded ring with the n-th graded part

$$S_n = \Big\{ \sum_{m\in\mathbb{N}^d} r_m x_1^{m_1} \ldots x_d^{m_d} \Big| r_m \in R, m_1 + \cdots + m_d = n \Big\},$$

where $m = (m_1, \ldots, m_d) \in \mathbb{N}^d$. By letting $\deg(x_i) = 1$ for all i, this gradation is called the *standard grading* of S.

Proposition 1.3. *Let $R = \bigoplus_{n\in\mathbb{Z}} R_n$ be a $\mathbb{Z}$-graded ring, then we have*

(a) *R_0 is a subring of R, containing 1;*

(b) *R_n is a R_0-module for every $n \in \mathbb{Z}$.*

Proof. By definition, $R_0R_0 \subseteq R_0$ so R_0 is closed under multiplication. It suffices to show that $1 \in R_0$. Suppose that $1 = \sum_{i=-n}^{n} x_i$ for some large n and homogeneous elements x_i, for all m we have

$$x_m = 1.x_m = \sum_{i=-n}^{n} x_i x_m.$$

Comparing the degrees gives $x_m = x_m x_0$ for all m. Now we have

$$x_0 = 1.x_0 = \sum_{i=-n}^{n} x_i x_0 = \sum_{i=-n}^{n} x_i = 1.$$

Hence $1 = x_0 \in R_0$.

The second part is trivial since $R_0R_n \subseteq R_n$ for all n. □

Definition 1.4. Let $R = \bigoplus_{n\in\mathbb{Z}} R_n$ be a graded ring.

(a) A subring $S \subseteq R$ is called a *graded subring* if $S = \bigoplus_{n\in\mathbb{Z}}(S \cap R_n)$.

(b) An ideal $I \subseteq R$ is called a *graded ideal* (or *homogeneous ideal*) if $I = \bigoplus_{n\in\mathbb{Z}}(I \cap R_n)$.

Proposition 1.5. *For an ideal I in a graded ring $R = \bigoplus_{n\in\mathbb{Z}} R_n$, the following two conditions are equivalent:*

(a) *I is a homogeneous ideal.*

(b) *$I = \langle S\rangle$ for some set S containing only homogeneous elements of R.*

Proof. Suppose that I is generated by a set of homogeneous elements S. For each $x \in I$, we have

$$x = \sum_j r_j s_j,$$

where $r_j \in R, s_j \in S$. Decompose r_j as a sum of homogeneous elements and let x_n be the sum of homogeneous elements of degree n in the resulting expression of x. We get $x = \sum_n x_n$. Since S is a generating set of I, $x_n \in I$ for all n. Thus $x_n \in I \cap R_n$ for all n, this implies $I \subseteq \bigoplus_{n\in\mathbb{Z}}(I \cap R_n)$. Hence $I = \bigoplus_{n\in\mathbb{Z}}(I \cap R_n)$. The converse implication is trivial. □

The above proof also implies that if I is a homogeneous ideal of a graded ring R and an element $x \in I$ is presented as a sum of homogeneous elements, then these homogeneous elements are belonged to I. The following corollary stems from this observation.

Corollary 1.6. *Let R be a graded ring and I, J be homogeneous ideals of R. Then the ideals $IJ, I + J, I \cap J$ are homogeneous.*

Lemma 1.7. *Let $R = \bigoplus_{n \in \mathbb{Z}} R_n$ be a graded ring and $I \subseteq R$ be a homogeneous ideal of R. Then I is a prime ideal if and only if $xy \in I$ implies $x \in I$ or $y \in I$ for all homogeneous elements x, y.*

Proof. Let $xy \in I$ and suppose that $y \notin I$. We write x and y as sums:

$$x = x_m + \cdots + x_{m+d},$$
$$y = y_n + \cdots + y_{n+r},$$

where $x_i \in R_i, y_j \in R_j, x_m \neq 0, y_n \neq 0$. If there is some $y_j \in I$, we may replace y by $y - y_j$. Hence we may assume that $y_n, \dots, y_{n+r} \notin I$. Now

$$xy = x_m y_n + (x_m y_{n+1} + x_{m+1} y_n) + \cdots + x_{m+d} y_{n+r} \in I.$$

Since I is homogeneous, all homogeneous elements $x_m y_n, (x_m y_{n+1} + x_{m+1} y_n), \dots$ are contained in I. Now $x_m y_n \in I$ and $y_n \notin I$ imply $x_m \in I$. Next

$$x_{m+1} y_n = (x_m y_{n+1} + x_{m+1} y_n) - x_m y_{n+1} \in I \text{ and } y_n \notin I,$$

hence $x_{m+1} \in I$. Continuing this process repeatedly gives $x_m, \dots, x_{m+d} \in I$, thus $x \in I$. The converse implication is trivial. □

Definition 1.8. Let R, S be $\mathbb{Z}$-graded rings and consider a ring homomorphism $f\colon R \to S$. The map f is said to be a *homogeneous homomorphism* if $f(R_n) \subseteq S_n$ for all $n \in \mathbb{Z}$. A homogeneous homomorphism is also called a *graded homomorphism*.

Definition 1.9. Let R be a $\mathbb{Z}$-graded ring and M an R-module. M is called a *graded R-module* if there is a family of subgroups $\{M_n\}_{n \in \mathbb{Z}}$ of M such that

(a) $M = \bigoplus_{n \in \mathbb{Z}} M_n$, and

(b) $R_m M_n \subseteq M_{m+n}$ for all $m, n \in \mathbb{Z}$.

Each subgroup M_n is called a *homogeneous component* of degree n; a non-zero element $x \in M_n$ is called a *homogeneous element* of degree n.

Definition 1.10. Let $M = \bigoplus_{n\in\mathbb{Z}} M_n$ be a graded R-module and N a submodule of M. For each $n \in \mathbb{Z}$, let $N_n = N \cap M_n$. If $N = \bigoplus_{n\in\mathbb{Z}} N_n$ then N is called a *graded submodule* of M.

Definition 1.11. Let M, N be $\mathbb{Z}$-graded modules and consider a module homomorphism $f\colon M \to N$. The map f is called a *homogeneous morphism of modules* if $f(M_n) \subseteq N_n$ for all $n \in \mathbb{Z}$. A homogeneous morphism of modules is also called a *graded morphism*.

Proposition 1.12. *Let M be a graded R-module and N an arbitrary submodule of M. Then N is a graded submodule if and only if N is generated by homogeneous elements of M.*

Proof. The argument is identical to the proof of Proposition 1.5. □

Corollary 1.13. *Let M be a graded module and N, P be graded submodules of M. Then the modules $N + P, N \cap P$ are graded.*

Definition 1.14. Let $M = \bigoplus_{n\in\mathbb{Z}} M_n$ be a graded module. For an integer s, we define the new graded module $M(s)$ by shifting the degree of each homogeneous component by s, that is

$$M(s) = \bigoplus_{n\in\mathbb{Z}} M_{n+s}.$$

This means that the homogeneous component of degree n of $M(s)$ is $M(s)_n = M_{n+s}$.

Clearly, $M(s)$ and M are equal as sets.

1.2 Monomial Ideals and Dickson's Lemma

We now focus on monomial ideals within polynomial rings in n variables over an arbitrary field K. This section explores some of their fundamental properties, beginning with a key definition.

Definition 1.15. A *monomial* in n variables $x_1, \dots, x_n$ is a product of the form

$$x_1^{a_1} x_2^{a_2} \dots x_n^{a_n},$$

where $a_1, \dots, a_n$ are non-negative integers.

Let $\mathbb{N}_0$ be the set of non-negative integers. The notation for monomials can be simplified by denoting $x^\alpha = x_1^{a_1} \dots x_n^{a_n}$ where $\alpha = (a_1, \dots, a_n) \in \mathbb{N}_0^n$. Note that $x^{(0,\dots,0)} = 1$.

Denoting by $K[x_1, \dots, x_n]$ the polynomial ring in n variables over K. For a subset of polynomials $S \subset K[x_1, \dots, x_n]$, the notation $\langle S \rangle$ denotes the ideal generated by S.

Definition 1.16. An ideal $I \subset K[x_1, \dots, x_n]$ is called a *monomial ideal* if I is generated by a set of monomials.

Any monomial ideal I admits a representation $I = \langle x^\alpha \colon \alpha \in A \rangle$, where A is a subset of $\mathbb{N}_0^n$. The next proposition gives a criterion when a monomial is contained in a monomial ideal.

Proposition 1.17. *Suppose I is a monomial ideal generated by the set of monomials $\{x^\alpha \mid \alpha \in A\}$. Then, for any monomial x^β, membership $x^\beta \in I$ holds if and only if x^β is a multiple of a generator x^α for some $\alpha \in A$.*

Proof. Let $x^\beta \in I$, we write $x^\beta = \sum_{i=1}^s h_i x^{\alpha_i}$, $h_i \in K[x_1, \dots, x_n]$ and $\alpha_i \in A$. Expand each of h_i as a linear combination of monomials. Each term of the right hand side is divisible by x^α for some $\alpha \in A$. Hence x^β on the left hand side must share the same property due to equality. □

The generating set of a monomial ideal I in the above definition is not necessarily finite. However, Dickson's Lemma establishes that I always admits a finite set of monomial generators.

Lemma 1.18 (Dickson's Lemma)**.** *Every monomial ideal in $K[x_1, \dots, x_n]$ is finitely generated.*

The proof for this lemma will be given in Chapter 2, after its second version (Corollary 2.8).

1.3 Gröbner Bases

The theory of Gröbner bases is the study of the division algorithm among polynomials, in which monomial ordering is the key. In the one variable case, this ordering is simply comparing the degree of monomials. The complexity arises while working with more than one variable. For instance, between x^3y and y^5, which monomial should be the "larger" to do the division algorithm for the polynomial $f = x^3y + y^5$? This leads us to the notion of monomial ordering.

Consider the polynomial ring $K[x_1, \ldots, x_n]$.

Definition 1.19. [8, Definition 2.1] A relation on the set of monomials $x^\alpha, \alpha \in \mathbb{N}_0^n$, or, equivalently, a relation $\preceq$ on $\mathbb{N}_0^n$ is called a *monomial ordering* if

(a) $\preceq$ is a total ordering;

(b) for all $\alpha, \beta, \gamma \in \mathbb{N}_0^n$, if $\alpha \preceq \beta$, then $\alpha + \gamma \preceq \beta + \gamma$;

(c) every nonempty subset of $\mathbb{N}_0^n$ has a smallest element with respect to $\preceq$.

There are many monomial orders. We study some representative examples, the first one will be *lexcographic order* (or *lex order*, in short).

Definition 1.20. Given $\alpha = (a_1, \ldots, a_n)$, $\beta = (b_1, \ldots, b_n) \in \mathbb{N}_0^n$. We say that α is less than β with respect to *lex order* (denoted by $\alpha \prec_{lex} \beta$) if the leftmost non-zero entry of the vector $\alpha - \beta \in \mathbb{Z}^n$ is negative. Further, we write $x^\alpha \prec_{lex} x^\beta$ if $\alpha \prec_{lex} \beta$.

Example 1.21. (a) $(0, 5) \prec_{lex} (1, 3)$, and $x_2^5 \prec_{lex} x_1x_2^3$.

(b) $(2, 5, 4) \prec_{lex} (2, 5, 8)$, and $x_1^2x_2^5x_3^4 \prec_{lex} x_1^2x_2^5x_3^8$.

(c) In the polynomial ring $K[x_1, \ldots, x_n]$, we have $x_n \prec_{lex} x_{n-1} \prec_{lex} \cdots \prec_{lex} x_1$, since

$$(0, \ldots, 0, 1) \prec_{lex} (0, \ldots, 0, 1, 0) \prec_{lex} \cdots \prec_{lex} (1, 0, \ldots, 0).$$

In the lex order, we do not regard the degree of monomials. For example, if we let $x \succ_{lex} y \succ_{lex} z$, then $y^5z^2 \prec_{lex} x$. In some cases, we may need to take the degree of monomials into account. This leads us to define *graded lexicographic order* (or *grlex order*, in short).

Definition 1.22. Given $\alpha = (a_1, \ldots, a_n)$, $\beta = (b_1, \ldots, b_n) \in \mathbb{N}_0^n$. We say that α is less than β with respect to grlex order (denoted by $\alpha \prec_{grlex} \beta$) if

$$|\alpha| = \sum_{i=1}^n a_i < |\beta| = \sum_{i=1}^n b_i, \text{ or } |\alpha| = |\beta| \text{ and } \alpha \prec_{lex} \beta.$$

Example 1.23. (a) $(1, 2, 4) \prec_{grlex} (1, 4, 5)$.

(b) $(1, 3, 5) \prec_{grlex} (1, 4, 4)$.

(c) The ordering of variables in the ring $K[x_1, \ldots, x_n]$ in the grlex order is the same as in the lex order.

Definition 1.24. Let $f = \sum_\alpha a_\alpha x^\alpha$, $\alpha \in \mathbb{N}_0^n$ be a non-zero polynomial in $K[x_1, \ldots, x_n]$ and let $\prec$ be a monomial order.

(a) The *multidegree* of f is $\mathrm{mdeg}(f) = \max\{\alpha \colon a_\alpha \neq 0\}$.

(b) The *leading coefficient* of f is $\mathrm{LC}(f) = a_{\mathrm{mdeg}(f)} \in K$.

(c) The *leading monomial* of f is $\mathrm{LM}(f) = x^{\mathrm{mdeg}(f)}$.

(d) The *leading term* of f is $\mathrm{LT}(f) = a_{\mathrm{mdeg}(f)} \cdot x^{\mathrm{mdeg}(f)}$.

Here the maximum is taken under $\prec$.

For example, take $f = 4x^6 + y^4z^3 + 7xz^8 - 5xy - 1$ and consider grlex ordering where $x = x_1, y = x_2, z = x_3$. Then

$$\begin{aligned}
\mathrm{mdeg}(f) &= (1, 0, 8), \\
\mathrm{LM}(f) &= xz^8, \\
\mathrm{LC}(f) &= 7, \\
\mathrm{LT}(f) &= 7xz^8.
\end{aligned}$$

Let $S \subset K[x_1, \ldots, x_n]$ be a set of polynomials. We denote $\mathrm{LT}(S) = \{\mathrm{LT}(f) \colon f \in S\}$, this is called the *set of leading terms* of S. Furthermore, for an ideal $I \subset K[x_1, \ldots, x_n]$, the ideal $\langle \mathrm{LT}(I) \rangle$ is called the *ideal of leading terms* (or *initial ideal*) of I with respect to the given monomial ordering.

Definition 1.25. Fix a monomial order. A finite subset $G = \{g_1, \ldots, g_t\}$ of an ideal $I \subseteq K[x_1, \ldots, x_n]$ is called a *Gröbner basis* of I if $\langle \mathrm{LT}(I) \rangle = \langle \mathrm{LT}(g_1), \ldots, \mathrm{LT}(g_t) \rangle$.

Theorem 1.26 (Hilbert's Basis Theorem)**.** *Every ideal in $K[x_1, \ldots, x_n]$ is finitely generated.*

We may use Dickson's Lemma to prove Hilbert's Basis Theorem. Here is a proof that Dickson's Lemma (Lemma 1.18) implies that every ideal in $K[x_1, \ldots, x_n]$ has a finite Gröbner basis, then satisfied the Hilbert's Basis Theorem.

Proof. Let $I \subseteq R = K[x_1, \ldots, x_n]$ be an ideal. We first consider the ideal $\langle \mathrm{LT}(I) \rangle$. By applying Dickson's Lemma, there exist a finite set of monomials $\{m_1, \ldots, m_t\}$ that generate $\langle \mathrm{LT}(I) \rangle$. Since each m_k is a leading term of some polynomial in I, we can choose a corresponding polynomial $g_k \in I$ such that $\mathrm{LT}(g_k) = m_k$ for each $k = 1, \ldots, t$. The objective is to prove that this finite set of polynomials, $\{g_1, \ldots, g_t\}$, constitutes a generating set for the original ideal I. Observe that the inclusion $\langle g_1, \ldots, g_t \rangle \subseteq I$ is immediate.

For the converse, suppose there exists a polynomial $f \in I \setminus \langle g_1, \ldots, g_t \rangle$. We can choose such an f whose the leading term $\mathrm{LT}(f)$ is minimal among all elements in $I \setminus \langle g_1, \ldots, g_t \rangle$. We have

$$\mathrm{LT}(f) \in \langle \mathrm{LT}(I) \rangle = \langle \mathrm{LT}(g_1), \ldots, \mathrm{LT}(g_t) \rangle.$$

There exist $1 \leq i \leq t$ such that $\mathrm{LT}(f) = \alpha m.\mathrm{LT}(g_i)$, where m is a monomial and $\alpha \in K$. Thus $f - \alpha m g_i \in I \setminus \langle g_1, \ldots, g_t \rangle$. But we also have $\mathrm{LT}(f - \alpha m g_i) < \mathrm{LT}(f)$, which contradicts the minimality of $\mathrm{LT}(f)$. Hence $I = \langle g_1, \ldots, g_t \rangle$. □

This elegant proof immediately yields a standard fact about Gröbner bases.

Corollary 1.27. *Any Gröbner basis of an ideal $I \subseteq K[x_1, \ldots, x_n]$ is a generating set of I, i.e., if $G = \{g_1, \ldots, g_t\}$ is a Gröbner basis of I, then*

$$I = \langle g_1, \ldots, g_t \rangle.$$

The preceding discussion naturally leads to the question of how to compute a Gröbner basis for a given ideal. Bruno Buchberger first addressed this in his 1965 Ph.D. dissertation, introducing the theoretical notion of Gröbner bases alongside his algorithm for their calculation. This thesis will not delve into the mechanics of Buchberger's algorithm; interested readers can consult Cox-Little-O'Shea [8, Chapter 6] for a full treatment.

1.4 The Classical Hilbert-Serre Theorem

1.4.1 Length and Krull Dimension of Modules

Definition 1.28. Let R be a ring and M an R-module. The *length* of R-module M is

$$l_R(M) = \sup\{n \colon \exists M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n\},$$

where all M_i are R-submodules of M. It may happen that $l_R(M) = \infty$. Further, a chain of submodules of M

$$M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n$$

is said to be a *composition series* of M if $n = l_R(M) < \infty$.

Clearly, if $n = l_R(M) < \infty$, a composition series of length n of M must has $M_0 = 0$ and $M_n = M$. We may simply write $l_R(M) = l(M)$ when the ring R is clear.

Proposition 1.29 (Additivity of length)**.** *Let M, N, P be R-modules that fit into a short exact*

$$0 \longrightarrow N \xrightarrow{f} M \xrightarrow{g} P \longrightarrow 0$$

Assume that N and P have finite lengths, then so does M and $l(N) + l(P) = l(M)$.

Proof. Let $n = l(N)$ and $m = l(P)$, consider the composition series of N and P

$$0 = N_0 \subset N_1 \subset \cdots \subset N_n = N,$$
$$0 = P_0 \subset P_1 \subset \cdots \subset P_m = P.$$

One checks that the series

$$\begin{aligned} 0 = f(N_0) \subset f(N_1) \subset \cdots \subset f(N_n) = \operatorname{im}(f) \\ = \ker(g) = g^{-1}(P_0) \subset g^{-1}(P_1) \subset \cdots \subset g^{-1}(P_m) = M \end{aligned}$$

is a composition series of M with length $n + m$. For this, note that N_i/N_{i-1} and P_j/P_{j-1} are simple modules, says $N_i/N_{i-1} \cong R/m_i$, $P_j/P_{j-1} \cong R/m_j$ for some maximal ideals m_i, m_j of R. This implies $f(N_i)/f(N_{i-1})$, $g^{-1}(P_j)/g^{-1}(P_{j-1})$ are simple R-modules. Thus we get the statement on the composition series. $\square$

Corollary 1.30. *Let N be a submodule of an R-module M of finite length, then*

$$l(M) = l(N) + l(M/N).$$

Proof. Applying the above proposition to the short exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0,$$

we obtain the equality. $\square$

Proposition 1.31. *Let* $0 \xrightarrow{f_0} M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} M_n \xrightarrow{f_{n+1}} 0$ *be an exact sequence of* R*-modules. Assume that* $l(M_i) < \infty$ *for all* i*, then*

$$\sum_{i=0}^{n}(-1)^i l(M_i) = 0.$$

Proof. Let $N_i = \operatorname{im}(f_i)$. For each i, we have an induced short exact sequence

$$0 \longrightarrow N_i \longrightarrow M_i \longrightarrow N_{i+1} \longrightarrow 0.$$

By the additivity of length, we have $l(M_i) = l(N_i) + l(N_{i+1})$. Now the alternating sum becomes

$$\begin{aligned}\sum_{i=0}^{n}(-1)^i l(M_i) &= (l(N_0) + l(N_1)) - (l(N_1) + l(N_2)) + \cdots + (-1)^n(l(N_n) + l(N_{n+1})) \\ &= l(N_0) + (-1)^n l(N_{n+1}) = 0,\end{aligned}$$

as claimed. $\square$

For an R-module M, recall that $\operatorname{supp}(M) = \{\text{prime ideals } P \text{ of } R \colon M_P \neq 0\}$.

Definition 1.32. Let M be a finitely generated R-module. The *Krull dimension* of M is

$$\dim(M) = \dim(\operatorname{supp}(M)) = \sup\{n \colon \exists P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n, \text{ where } P_i \in \operatorname{supp}(M)\}.$$

Remark 1.33. Let M be a finitely generated R-module and $\operatorname{ann}(M) = \{x \in R \colon xM = 0\}$. Note that

$$\operatorname{supp}(M) = V(\operatorname{ann}(M)) := \{\text{prime ideals } P \text{ of } R \colon P \supset \operatorname{ann}(M)\}$$

hence we have

$$\dim(M) = \dim(\operatorname{supp}(M)) = \dim(V(\operatorname{ann}(M))) = \dim(R/\operatorname{ann}(M)).$$

This formula is useful for computing the dimension of M.

Proposition 1.34. *Let* M *be an* R*-module.*

(a) *M has finite length if and only if M satisfying both artinian (descending chain condition) and noetherian (ascending chain condition).*

(b) *If the ring R is artinian, then any finitely generated R-module M must have finite length over R.*

Proof. (a) Assume M has finite length, say $l(M) = n$. Consider an ascending chain $M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$. The lengths of these submodules form a non-decreasing sequence of non-negative integers $l(M_1) \leq l(M_2) \leq l(M_3) \leq \cdots$, bounded above by $l(M) = n$. Thus, the sequence of lengths stabilizes, i.e., there exists d such that for all $i \geq d$, $l(M_i) = l(M_{i+1})$. Since $M_i \subseteq M_{i+1}$ and $l(M_i) = l(M_{i+1})$, we must have $M_i = M_{i+1}$ for all $i \geq d$. Hence, the ascending chain stabilizes, and M is noetherian.

Given the descending chain $M_1 \supseteq M_2 \supseteq \cdots$, the corresponding sequence of lengths $l(M_1) \geq l(M_2) \geq \cdots$ is necessarily non-increasing. Since module lengths are non-negative integers, this sequence is bounded below by 0 and therefore must eventually become constant. Thus, the sequence of lengths stabilizes, i.e., there exists d such that for all $i \geq d$, $l(M_i) = l(M_{i+1})$. Since $M_{i+1} \subseteq M_i$ and $l(M_i) = l(M_{i+1})$, we must have $M_i = M_{i+1}$ for all $i \geq d$. Hence, the descending chain stabilizes, and M is artinian.

For the converse, suppose M is both artinian and noetherian. The case $M = 0$ is trivial, giving $l(M) = 0$. Assume $M \neq 0$. As M is noetherian, the collection of proper submodules is non-empty and contains a maximal element (with respect to inclusion), say M_1. By maximality, the quotient M/M_1 is a simple module. If M_1 is non-zero, it inherits the noetherian property from M. Therefore, M_1 also contains a maximal proper submodule, M_2. Continuing this construction inductively, we generate a sequence of submodules $M = M_0 \supset M_1 \supset M_2 \supset \ldots$ where each M_{i+1} is a maximal proper submodule of M_i. This forms a strictly descending chain of submodules. Since M is artinian, it satisfies the descending chain condition, meaning this sequence must terminate. Termination of a strictly descending chain requires that $M_n = 0$ for some integer n. This process yields a finite chain $\{0\} = M_n \subset M_{n-1} \subset \cdots \subset M_1 \subset M_0 = M$, where each quotient M_i/M_{i+1} is simple. This is a composition series for M of length n. Consequently, M has finite length.

(b) Induction on the number of generators of M. Firstly, suppose that M is generated by $x \in M$. Consider the map

$$\begin{aligned} f\colon R &\to M \\ a &\mapsto ax. \end{aligned}$$

By the isomorphism theorem, $R/\ker(f) = R/\mathrm{ann}(x) \cong M$. Now since R is an artinian ring, R is also noetherian, hence R has finite length over itself. Then so is $R/\operatorname{ann}(x)$.

Next, we suppose that $x_1, \ldots, x_n$ are the generators of M, for $n > 1$. Putting $N = Rx_1 + \cdots + Rx_{n-1}$, consider the exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0.$$

Observe that $M/N \cong Rx_n$, thus by the induction hypothesis, N and M/N have finite lengths. Ultimately, M has finite length by the additivity of length.

We complete the proposition. □

1.4.2 Numerical Functions on $\mathbb{Z}$

Now we delve into a numerical method, which is useful for deducing the classical Hilbert-Serre. We will use the symbol $n \gg 0$, that means $n > n_0$ for some $n_0 > 0$ (n_0 is typically large). The symbol $n \ll 0$ is defined similarly.

Definition 1.35. Let $F\colon \mathbb{Z} \to \mathbb{Z}$ be a numerical function. F is called of *polynomial type* of degree d if there is a polynomial $P(x) \in \mathbb{Q}[x]$ with $\deg(P) = d$ such that $F(n) = P(n)$ for $n \gg 0$.

As a convention, the degree of the zero polynomial is -1.

Definition 1.36. The map Δ on the set of numerical functions, defined by

$$(\Delta F)(n) = F(n+1) - F(n),$$

is called the *difference operator*. We also define $\Delta^d F$ recursively by $\Delta^d F = \Delta^{d-1}(\Delta F)$. By convention, $\Delta^0 F = F$.

We study some crucial properties of numerical function.

Lemma 1.37. *Let $F\colon \mathbb{Z} \to \mathbb{Z}$ be a numerical function and d be a non-negative integer. The following are equivalent:*

(a) $(\Delta^d F)(n)$ *is a non-zero constant, for $n \gg 0$;*

(b) *F is of polynomial type of degree d .*

Proof. The implication (a) $\Rightarrow$ (b) uses induction on d. For $d = 0$,

$$(\Delta^0 F)(n) = F(n) = c \neq 0,$$

then clearly $P(x) = c$ is the polynomial we are looking for. Now for $d > 0$,

$$(\Delta^d F)(n) = \Delta^{d-1}(F(n+1) - F(n)) = c \neq 0,$$

for $n \gg 0$. By induction hypothesis, there is $n_0 > 0$ and $P(x) \in \mathbb{Q}[x]$, $\deg(P) = d - 1$ satisfying $F(n+1) - F(n) = P(n)$ for all $n \geq n_0$. We need some computation as follows

$$\begin{aligned}
F(n+1) &= F(n) + P(n) \\
&= F(n-1) + P(n-1) + P(n) \\
&\dots \\
&= F(n_0) + \sum_{k=n_0}^{n} P(k) \\
&= F(n_0) + \sum_{k=0}^{n-n_0} P(n-k) \\
&= F(n_0) + (n - n_0 + 1)n^{d-1} + (\text{lower degree terms with respect to } n).
\end{aligned}$$

This confirms that the preceding sum defines a polynomial in n of degree d.

Conversely, recall that the difference operator Δ acts on polynomials by reducing their degree by exactly one (provided the polynomial is non-constant). Consequently, if F is a numerical function agreeing with a polynomial of degree d for $n \gg 0$, applying the operator Δ d times must result in a constant function; let $(\Delta^d F)(n) = c$. To establish that $c \neq 0$, it is sufficient to analyze the base case where F is linear. Therefore, we can assume $F(n) = an + b$ holds for all $n \gg 0$, with $a \neq 0$. Proceeding under this assumption:

$$(\Delta F)(n) = F(n+1) - F(n) = a \neq 0.$$

This complete the proof. $\square$

For $a, b \in \mathbb{Z}$, the *combinatorial polynomial* is defined by

$$\binom{x+a}{b} = \begin{cases} 0 & \text{for } b < 0, \\ 1 & \text{for } b = 0, \\ \dfrac{1}{b!} \cdot (x+a)(x+a-1)\dots(x+a-(b-1)) & \text{for } b > 0. \end{cases}$$

Lemma 1.38. *Let $P(x) \in \mathbb{Q}[x]$, $\deg(P) = d$. Then for all $n \in \mathbb{Z}$, $P(n) \in \mathbb{Z}$ if and only if there are integers $a_0, \ldots, a_d$ such that*

$$P(x) = \sum_{i=0}^{d} a_i \binom{x+i}{i}.$$

Proof. The implication ($\Rightarrow$) is trivial. Conversely, let $P(x) = b_d x^d + \cdots + b_0$. Note that $\binom{x+i}{i}$ is a polynomial of degree i, thus we may write

$$\begin{aligned}
P(x) &= b_d.d!\binom{x+d}{d} + c_{d-1}x^{d-1} + \cdots + c_0 \\
&= b_d.d!\binom{x+d}{d} + c_{d-1}.(d-1)!\binom{x+d-1}{d-1} + \text{(lower degree terms)} \\
&\ldots \\
&= \sum_{i=0}^{d} a_i \binom{x+i}{i},
\end{aligned}$$

for some $a_0, \ldots, a_d \in \mathbb{Q}$. Next, we claim that $a_j = (\Delta^j P)(-j-1)$ for all j, using the familiar identity

$$\binom{x+j+1}{j} - \binom{x+j}{j} = \binom{x+j}{j-1}.$$

Indeed, for $j = 0$,

$$(\Delta^j P)(-j-1) = P(-1) = a_0.$$

For $j \geq 1$, assume that a_j has the given form, we have

$$\begin{aligned}
\Delta P(x) &= P(x+1) - P(x) \\
&= \sum_{i=0}^{d} a_i \left(\binom{x+1+i}{i} - \binom{x+i}{i} \right) \\
&= \sum_{i=0}^{d} a_i \binom{x+i}{i-1} \\
&= \sum_{i=0}^{d-1} a_{i+1} \binom{x+i+1}{i} =: Q(x+1) \\
\implies Q(x) &= \sum_{i=0}^{d-1} a_{i+1} \binom{x+i}{i}.
\end{aligned}$$

By induction hypothesis, $a_{j+1} = (\Delta^j Q)(-j-1)$, continuing the computation

$$\begin{aligned}\Delta^j Q(-x-1) &= \Delta^{j-1}(Q(-x) - Q(-x-1)) \\ &= \Delta^{j-1}(\Delta P(-x-1) - \Delta P(-x-2)) \\ &= \Delta^{j-1}(\Delta(\Delta P(-x-2))) \\ &= \Delta^{j+1} P(-x-2) \\ &= \Delta^{j+1} P(-(x+1)-1).\end{aligned}$$

Replacing x by j gives $a_{j+1} = (\Delta^{j+1})P(-(j+1)-1)$, which is the desired form. $\square$

Lemma 1.39. *Let $H(t) = \sum_n a_n t^n \in \mathbb{Z}[[t, t^{-1}]]$ be a formal Laurent series with $a_n = 0$ for $n \ll 0$. Let d be a positive integer. The following are equivalent:*

(a) *the sequence $\{a_n\}_{n\in\mathbb{Z}}$ is of polynomial type of degree $d-1$;*

(b) *$H(t) = Q(t)/(1-t)^d$ where $Q(t) \in \mathbb{Z}[t, t^{-1}]$ and $Q(1) \neq 0$.*

Before turning to the proof, we need an observation.

Claim. *Let $F(n) = a_n$ for all $n \in \mathbb{Z}$, then*

$$(1-t)^d H(t) = \sum_n (\Delta^d F)(n-d)t^n.$$

Proof. Induction on d, for $d = 1$,

$$\begin{aligned}(1-t)H(t) &= \sum_n \left(a_n t^n - a_n t^{n+1}\right) \\ &= \sum_n \left(a_n t^n - a_{n-1} t^n\right) \\ &= \sum_n (\Delta F)(n-1)t^n\end{aligned}$$

Assume the equation holds for $d-1 > 0$, then by induction hypothesis, we have

$$\begin{aligned}(1-t)^d H(t) &= (1-t)\Big((1-t)^{d-1}H(t)\Big) \\ &= (1-t)\sum_n (\Delta^{d-1}F)(n-d+1)t^n \\ &= \sum_n (\Delta^{d-1}F)(n-d+1)t^n - \sum_n (\Delta^{d-1}F)(n-d+1)t^{n+1} \\ &= \sum_n (\Delta^{d-1}F)(n-d+1)t^n - \sum_n (\Delta^{d-1}F)(n-d)t^n \\ &= \sum_n (\Delta(\Delta^{d-1}F))(n-d)t^n \\ &= \sum_n (\Delta^d F)(n-d)t^n.\end{aligned}$$

This completes the claim. $\square$

Proof of Lemma. For (a) $\Rightarrow$ (b), since $F(n) = a_n$ is of polynomial type of degree $d-1$, then by Lemma 1.37, $(\Delta^{d-1}F)(n) = c$ for $n \gg 0$. Thus $(\Delta^d F)(n) = 0$ for all n large enough. Combining with our claim give

$$(1-t)^d H(t) \in \mathbb{Z}[t, t^{-1}].$$

Putting $Q(t) = (1-t)^d H(t) = \sum_n (\Delta^d F)(n-d)t^n$. We need to clarify that $Q(1) \neq 0$. Assume that $Q(1) = 0$, then

$$\begin{aligned}
0 &= \sum_n (\Delta^d F)(n-d) \\
&= \sum_n \big((\Delta^{d-1}F)(n-d+1) - (\Delta^{d-1}F)(n-d)\big) \\
&= (\Delta^{d-1}F)(N) - (\Delta^{d-1}F)(M) \text{ for } N \gg 0 \text{ and } M \ll 0 \\
&= (\Delta^{d-1}F)(N) \text{ for } N \gg 0.
\end{aligned}$$

this contradicts to Lemma 1.37.

For (b) $\Rightarrow$ (a), suppose (b), we have $(1-t)^d H(t) = Q(t) \in \mathbb{Z}[t, t^{-1}]$. The claim above implies

$$(1-t)^d H(t) = \sum_n (\Delta^d F)(n-d)t^n \in \mathbb{Z}[t, t^{-1}].$$

This implies

$$\begin{aligned}
& & (\Delta^d F)(n-d) &= 0 & &\text{for } n \gg 0 \\
&\Rightarrow & (\Delta^{d-1}F)(n-d+1) &= (\Delta^{d-1}F)(n-d) & &\text{for } n \gg 0 \\
&\Rightarrow & (\Delta^{d-1}F)(n) &= c & &\text{for } n \gg 0
\end{aligned}$$

Finally, we need to show $c \neq 0$. Now by $Q(1) \neq 0$, we have

$$0 \neq Q(1) = \sum_n (\Delta^d F)(n-d) = (\Delta^{d-1}F)(N) \text{ for } N \gg 0.$$

This concludes the proof. $\square$

1.4.3 Hilbert Functions and Hilbert Series

Definition 1.40. Let $R = \bigoplus_{n\in\mathbb{Z}} R_n$ be a graded ring and $M = \bigoplus_{n\in\mathbb{Z}} M_n$ a finitely generated graded R-module.

(a) The map $h_M(\cdot)\colon \mathbb{Z} \to \mathbb{Z}$ defined by $h_M(n) = l_{R_0}(M_n)$ is called the *Hilbert function* of M.

(b) If $h_M(n) < \infty$ for all n, we say that M *has a Hilbert series*, and the formal power series

$$H_M(t) = \sum_{n\in\mathbb{Z}} h_M(n)t^n$$

is called the *Hilbert series* of M.

Proposition 1.41. (a) *If* $0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0$ *is a short exact sequence of graded modules and homogeneous maps, then*

$$H_N(t) = H_M(t) + H_P(t).$$

(b) *If M is a graded R-module and $x \in R_d$, $d \geq 1$, is a non-zero divisor on M, then*

$$H_{M/xM}(t) = (1 - t^d)H_M(t).$$

Proof. (a) For each n, the exact sequence of R_0-modules $0 \to M_n \to N_n \to P_n \to 0$ implies

$$\begin{aligned} h_N(n) &= h_M(n) + h_P(n) \\ \implies h_N(n)t^n &= h_M(n)t^n + h_P(n)t^n. \end{aligned}$$

Taking sum over all n gives

$$\begin{aligned} \sum_{n\in\mathbb{Z}} h_N(n)t^n &= \sum_{n\in\mathbb{Z}} h_M(n)t^n + \sum_{n\in\mathbb{Z}} h_P(n)t^n \\ \implies H_N(t) &= H_M(t) + H_P(t). \end{aligned}$$

(b) Since x is a non-zero divisor on M, we have a short exact sequence

$$0 \longrightarrow M(-d) \xrightarrow{\cdot x} M \longrightarrow M/xM \longrightarrow 0\,,$$

where $\cdot x$ denotes the multiplication by x. Now (a) gives

$$H_{M(-d)}(t) + H_{M/xM}(t) = H_M(t).$$

Note that

$$\begin{aligned} H_{M(-d)}(t) &= \sum_{n\in\mathbb{Z}} h_{M(-d)}(n)t^n \\ &= \sum_{n\in\mathbb{Z}} h_M(n)t^{n+d} \\ &= t^d \cdot \sum_{n\in\mathbb{Z}} h_M(n)t^n \\ &= t^d \cdot H_M(t). \end{aligned}$$

Substituting back to the equation gives $H_{M/xM}(t) = (1 - t^d)H_M(t)$. □

For the subsequent discussion in this section, we impose the following standard conditions: the graded ring R is generated as an R_0-algebra by a finite set of elements $\{x_1, \dots, x_d\}$ residing in degree 1, so R takes the form $R_0[x_1, \dots, x_d]$; furthermore, the base ring R_0 is assumed to be an artinian local ring. We recall the fundamental characterization that a ring is artinian if and only if it is noetherian and has Krull dimension zero [6, Theorem 8.5]. Since R_0 being artinian implies it is noetherian, Hilbert's Basis Theorem ensures that the polynomial ring $R = R_0[x_1, \dots, x_d]$ is also noetherian.

Let M be a finitely generated graded R-module. Since R is a noetherian ring, M is a noetherian module.

Lemma 1.42. *With the above assumptions, there is a chain*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_k = M$$

such that $M_{i+1}/M_i \cong (R/P_i)(a_i)$ for some homogeneous prime ideals $P_i \in \operatorname{supp}(M)$ and integers a_i.

Proof. The proof relies on induction and the noetherian property of M. If $M = 0$, the chain is simply $0 = M_0 = M$, and the condition is trivially satisfied.

If $M \neq 0$, it has at least an associated prime $P = \operatorname{ann}(x)$ for some non-zero homogeneous element $x \in M$. Let $\deg(x) = -a$, consider the submodule $Rx \subseteq M$. Since x is homogeneous, Rx is a graded submodule. Define a map

$$\varphi : R \to Rx,\ \varphi(r) = rx.$$

This map is clearly a surjective R-module homomorphism. The kernel of φ is $\ker(\varphi) = \{r \in R \mid rx = 0\} = \operatorname{ann}(x) = P$. By the isomorphism theorem,

$$R/P = R/\ker(\varphi) \cong \operatorname{im}(\varphi) = Rx.$$

Now, consider the grading. If $r \in R$ has degree d, then $\varphi(r) = rx$ has degree $d + \deg(x) = d - a$. In order to make this a degree-preserving isomorphism, we need to shift the grading of R/P by a, which gives

$$Rx \cong (R/P)(a).$$

Clearly, $P \in \operatorname{supp}(Rx) \subseteq \operatorname{supp}(M)$.

Letting $M_1 = Rx$, consider the module M/M_1. This is a quotient of M hence also a finitely generated graded R-module. If $M/M_1 \neq 0$, then apply the inductive hypothesis to it; there is a submodule $M_1 \subset M_2 \subseteq M$ such that $M_2/M_1 \cong (R/P')(a')$ for some homogeneous prime ideal $P' \in \operatorname{supp}(M/M_1) \subseteq \operatorname{supp}(M)$ and integer a'. Keep doing this process, since M is noetherian, we must have a number $k > 0$ such that $M_k = M_{k+1}$. Choose the smallest k, we obtained the desired chain. □

Lemma 1.43. *Let M be a finitely generated graded R-module with $\dim M = 0$. Then there exists an integer n_0 such that for all $n \geq n_0$, the homogeneous component $M_n = 0$.*

Let $\mathfrak{m}_0$ denote the unique prime ideal of R_0. Denote $\mathfrak{m} := \mathfrak{m}_0 \oplus R_1 \oplus R_2 \oplus \cdots$. Clearly $\mathfrak{m}$ is an ideal of R and $R/\mathfrak{m} \cong R_0/\mathfrak{m}_0$ is a field, so $\mathfrak{m}$ is a graded maximal ideal of R. We begin the proof with a claim.

Claim. *$\mathfrak{m}$ is the unique graded maximal ideal of R.*

Proof of the claim. Let $\mathfrak{m}'$ be any graded maximal ideal of R. Since $\mathfrak{m}'$ is a prime ideal of R, $\mathfrak{m}' \cap R_0$ is a prime ideal of R_0, which forces $\mathfrak{m}' \cap R_0 = \mathfrak{m}_0$. Furthermore, any homogeneous element $y \in \mathfrak{m}'$ of positive degree $i \geq 1$ belongs to R_i. Therefore,

$$\mathfrak{m}' = (\mathfrak{m}' \cap R_0) \oplus \bigoplus_{i \geq 1} (\mathfrak{m}' \cap R_i) \subseteq \mathfrak{m}_0 \oplus \bigoplus_{i \geq 1} R_i = \mathfrak{m}.$$

Thus, $\mathfrak{m}$ is the unique graded maximal ideal of R. □

Proof of Lemma 1.43. Since $\dim M = 0$, $\operatorname{supp}(M)$ has Krull dimension zero. Thus, $\operatorname{supp}(M) = V(\operatorname{ann}(M)) \subseteq \{\mathfrak{m}\}$, where $\mathfrak{m}$ is the unique graded maximal ideal of R. This implies that any prime ideal containing $\operatorname{ann}(M)$ must contain $\mathfrak{m}$, and hence $\sqrt{\operatorname{ann}(M)} = \mathfrak{m}$. Thus there exists an integer $N > 0$ such that $\mathfrak{m}^N \subseteq \operatorname{ann}(M)$, implies that $\mathfrak{m}^N \cdot M = 0$.

A product of n elements from $\mathfrak{m}$ belongs to $\mathfrak{m}^n$, so every monomial of degree n is in $\mathfrak{m}^n$. Hence for $n \geq N$ we have $R_n \subseteq \mathfrak{m}^N$. Let D be the highest degree of a minimal homogeneous generator of M. For any $n \geq N + D$, any element in M_n can be written as a sum

$r_1g_1 + ... + r_kg_k$, where $g_1, ..., g_k$ are homogeneous generators of M with $\deg(g_i) = d_i \le D$, and $r_i \in R_{n-d_i}$.

Now $\deg(r_i) = n - d_i \ge n - D \ge N$, so $r_i \in \mathfrak{m}^N$ and $r_ig_i = 0$. Thus $M_n = 0$ for all $n \ge N + D$. Hence, the homogeneous components M_n must eventually become zero for sufficiently large n. □

Theorem 1.44. [7, Theorem 4.1.3] *Let M be a finitely generated graded R-module of dimension d. The Hilbert function of M is of polynomial type of degree $d-1$.*

Proof. Let $Q_M(x) \in \mathbb{Q}[x]$ be a polynomial such that $h_M(n) = Q_M(n)$ for $n \gg 0$. Using the filtration in Lemma 1.42, for each $i \in \{0, 1, \dots, k-1\}$, the quotient module M_{i+1}/M_i is isomorphic, as a graded R-module, to $(R/P_i)(a_i)$ for some homogeneous prime ideal $P_i \in \operatorname{Spec}(R)$ and some integer shift $a_i \in \mathbb{Z}$. This filtration induces short exact sequences

$$0 \longrightarrow M_i \longrightarrow M_{i+1} \longrightarrow M_{i+1}/M_i \longrightarrow 0.$$

For each $n \ge 0$, applying the additivity of length repeatedly yields

$$h_M(n) = \sum_{i=0}^{k-1} h_{M_{i+1}/M_i}(n) = \sum_{i=0}^{k-1} h_{(R/P_i)(a_i)}(n).$$

Thus, if for each i, $h_{(R/P_i)(a_i)}(n)$ agrees with a polynomial $Q_i(n) \in \mathbb{Q}[n]$ of degree $d_i - 1$ for sufficiently large n, where $d_i = \dim(R/P_i)$. Since $Q_i(n) \ge 0$ for all $n \gg 0$, each $Q_i(n)$ has non-negative highest coefficient. Then the degree of $Q_M(n)$ is

$$\deg(Q_M) = \max_{0 \le i \le k-1} \{\deg(Q_i)\}.$$

It is also clear that $d = \dim(M) = \max_{0 \le i \le k-1} \{\dim(M_{i+1}/M_i)\} = \max_{0 \le i \le k-1} \{\dim(R/P_i)(a_i)\}$. Hence we have

$$h_M(n) = \sum_{i=0}^{k-1} Q_i(n) \text{ for } n \gg 0.$$

Now we may assume that $M = (R/P)(a)$, $P \in \operatorname{Spec}(R)$ homogeneous, $a \in \mathbb{Z}$. We can further reduce to the case $M = R/P$, $P \in \operatorname{Spec}(R)$ homogeneous, since R/P and $(R/P)(a)$ differ only by degree shifts.

If $d = 0$, by Lemma 1.43, $M_n = 0$ for $n \gg 0$, so the Hilbert function of M is of polynomial type of degree -1.

Assume $d > 0$. Since $\dim(R/P) = d > 0$, P does not contain the ideal $\mathfrak{m} = \mathfrak{m}_0 \oplus R_1 \oplus R_2 \oplus \dots$ of R. Since $P \cap R_0 \in \operatorname{Spec}(R_0) = \{\mathfrak{m}_0\}$, we deduce $\mathfrak{m}_0 \subseteq P$. Hence $\exists x \in R_s$,

$s \geq 1$ such that $x \notin P$. Since R is generated over R_0 by R_1, we may assume that $s = 1$. Hence $\exists x \in R_1$ and $x \notin P$, and thus x is R/P-regular. Consider the exact sequence

$$0 \longrightarrow M(-1) \xrightarrow{\cdot x} M \longrightarrow M/xM \longrightarrow 0.$$

Using the additivity of Hilbert functions on short exact sequences, we have

$$h_M(n) - h_{M(-1)}(n) = h_{M/xM}(n).$$

Since $h_{M(-1)}(n) = h_M(n-1)$, we get

$$h_M(n) - h_M(n-1) = h_{M/xM}(n) \implies \Delta h_M(n) = h_{M/xM}(n).$$

By our inductive hypothesis, since $\dim(M/xM) = d-1$, the Hilbert function $h_{M/xM}(n)$ is of polynomial type of degree $\dim(M/xM) - 1 = d-2$. Therefore, $h_M(n)$ is of polynomial type of degree $(d-2) + 1 = d-1$. $\square$

Corollary 1.45 (Hilbert-Serre Theorem)**.** *Let $M \neq 0$ be a finitely generated graded R-module such that $\dim(M) = d$. There is a unique Laurent polynomial $Q(t) \in \mathbb{Z}[t, t^{-1}]$ with $Q(1) \neq 0$ such that*

$$H_M(t) = \frac{Q(t)}{(1-t)^d}.$$

Proof. In the case $d = 0$, by Lemma 1.43, $M_n = 0$ for $n \gg 0$. Hence the Hilbert function of M is of polynomial type of degree -1.

If $d > 0$, let $H_M(t) = \sum_n F(n) \cdot t^n$. Thanks to Theorem 1.44, $F(n)$ is of polynomial type of degree $d-1$. Then from Lemma 1.39 we obtain the required form of $H_M(t)$. $\square$

Lemma 1.46. *If $R = K$ is a field and M is a finitely generated K-vector space, then*

$$l_K(M) = \dim_K M.$$

Proof. Since M is a finitely generated vector space over a field K, it is both an artinian and noetherian K-module. Therefore, M has a composition series. Let

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_l = M$$

be a composition series of M, where $l_K(M) = l$.

Each factor module M_i/M_{i-1} in the composition series is isomorphic to K as a K-module. This means $\dim_K(M_i/M_{i-1}) = \dim_K K = 1$. We use the property that for any subspace $U \subseteq V$ of a K-vector space V, we have $\dim_K V = \dim_K U + \dim_K(V/U)$. Applying this property to our composition series, we have:

$$\begin{aligned}
\dim_K M = \dim_K M_l = \dim_K M_{l-1} + \dim_K(M_l/M_{l-1}) &= \dim_K M_{l-1} + 1, \\
\dim_K M_{l-1} = \dim_K M_{l-2} + \dim_K(M_{l-1}/M_{l-2}) &= \dim_K M_{l-2} + 1, \\
&\dots \\
\dim_K M_1 = \dim_K M_0 + \dim_K(M_1/M_0) = \dim_K\{0\} + \dim_K(M_1/M_0) &= 1.
\end{aligned}$$

By repeatedly substituting, we get:

$$\dim_K M = \dim_K M_{l-1} + 1 = (\dim_K M_{l-2} + 1) + 1 = \cdots = \dim_K M_0 + l = l.$$

Thus, $\dim_K M = l_K(M)$. ☐

Corollary 1.47. *Let $R = K[x_1, \dots, x_d]$ be a polynomial ring over a field K. Consider the $\mathbb{N}$-grading (not necessarily standard) $R = K \oplus R_1 \oplus R_2 \oplus \dots$. Then for all $n \geq 0$,*

$$h_R(n) = \dim_K(R_n),$$

where $\dim_K$ is considered as dimension of K-vector space.

Proof. Let $S_n = \{f_1, \dots, f_m\}$ be the set of monomials of degree n. Each R_n is an R_0-module, hence a K-vector space with a basis S_n. By the above lemma, $\dim_K(R_n) = l_{R_0}(R_n) = h_R(n)$. ☐

For each given degrees of x_i, the number m can be computed via n and d. This problem is called the Euler's candy division problem. For example, regarding the standard grading, we have $m = \binom{n+d-1}{d-1}$.

Example 1.48. Let K be a field. If $R = K[x_1, \dots, x_d]$ endowed with the standard grading, then

$$H_R(t) = \frac{1}{(1-t)^d}.$$

Proof. Induction on d. For $d = 1$, we have

$$\begin{aligned}
H_{K[x_1]}(t) &= 1 + t + t^2 + \dots \\
&= \frac{1}{1-t}.
\end{aligned}$$

Let $S = K[x_1, \ldots, x_{d-1}]$ and suppose that $H_S(t) = \frac{1}{(1-t)^{d-1}}$. By Proposition 1.41, we have

$$H_{R/\langle x_d \rangle}(t) = (1-t) \cdot H_R(t).$$

Note that $S \cong R/\langle x_d \rangle$, hence

$$(1-t) \cdot H_R(t) = H_S(t) = \frac{1}{(1-t)^{d-1}}.$$

Dividing both sides by $(1-t)$, we obtain the desired form. □

We prove a fundamental result of Hilbert series of ideals of leading terms.

Lemma 1.49. *Let $R = K[x_1, \ldots, x_d]$ be a polynomial ring over K and $I \subseteq R$ be a homogeneous ideal. For each monomial ordering $\preceq$, the quotient rings R/I and $R/\langle \mathrm{LT}(I) \rangle$ have the same Hilbert series.*

Proof. We will show that there is a K-vector space isomorphism between the graded components of R/I and $R/\langle \mathrm{LT}(I) \rangle$ in each degree, which implies the equality of their Hilbert series.

Let $J = \langle \mathrm{LT}(I) \rangle$ be the ideal of leading terms of I, and

$$B = \{m \in R \mid m \text{ is a monomial and } m \notin J\}.$$

We show that B induces a K-basis for both R/J and R/I.

1. B induces a K-basis for R/J: Consider any polynomial $f \in R$. If we perform monomial reduction of f modulo J (which is straightforward since J is a monomial ideal), we can express f as $f = r + g$, where $g \in J$ and r is a K-linear combination of monomials such that no monomial in r is divisible by any monomial generator of J. This means every monomial in r is in B. Thus, the residue class of f in R/J is $r + J$. Therefore, B spans R/J.

   Suppose $\sum_{i=1}^{k} c_i m_i = 0$ in R/J, where $c_i \in K$ and $m_i \in B$ are monomials. This means $\sum_{i=1}^{k} c_i m_i \in J$. However, since $\sum_{i=1}^{k} c_i m_i$ is a K-linear combination of monomials in B, and by definition no monomial in B is in J, this is only possible if all $c_i = 0$. Thus, B is linearly independent in R/J. Hence, B is a K-basis for R/J.

2. B induces a K-basis for R/I: Let $G = \{g_1, \ldots, g_m\}$ be a Gröbner basis of I with respect to $\preceq$. For any $f \in R$, the division algorithm gives $f = \sum_{i=1}^{m} a_i g_i + r$,

where $a_i \in R$ and r is the remainder. The remainder r has the property that no term in r is divisible by any $\mathrm{LT}(g_i)$. This means that every monomial in r is not in $\langle \mathrm{LT}(g_1), \ldots, \mathrm{LT}(g_m) \rangle = J$. Hence, every monomial in r is in B. Thus, r is a K-linear combination of elements in B, and the residue class of r in R/I is the same as the residue class of f. Therefore, B spans R/I.

Suppose $\sum_{i=1}^{k} c_i m_i = 0$ in R/I, where $c_i \in K$ and $m_i \in B$ are monomials. This means $h = \sum_{i=1}^{k} c_i m_i \in I$. If $h \neq 0$, then $\mathrm{LT}(h)$ must be in $\langle \mathrm{LT}(I) \rangle = J$. Note that $\mathrm{LT}(h) = m_i$ for some i, we must have $m_i \in J = \langle \mathrm{LT}(I) \rangle$, which is a contradiction. Thus, h must be 0, implying all $c_i = 0$. Therefore, B is linearly independent in R/I. Hence, B is a K-basis for R/I.

Ultimately, B induces a K-basis for both R/J and R/I. Thus for each degree j, the set $B_j = \{m \in B \mid \deg(m) = j\}$ is a basis for the degree j homogeneous components of both R/J and R/I, i.e.,

$$\dim_K (R/I)_j = |B_j| = \dim_K (R/J)_j.$$

This implies that their Hilbert series are equal. $\square$

Chapter 2
Inc($\mathbb{N}$)-Equivariant Gröbner Bases

Chapter 2 delves into the theory of Inc($\mathbb{N}$)-equivariant Gröbner bases, starting with the fundamental concept of well-partial-orders. It explores crucial properties of these orders, including the existence of infinite ascending subsequences and the behavior under component-wise ordering. In this chapter, we establish the connection to Dickson's Lemma and extends these ideas to infinite settings with Kruskal's Tree Theorem and Higman's Lemma. Then we introduce the monoid Inc($\mathbb{N}$) and its submonoids Inc$(\mathbb{N})^i$, crucial for studying symmetries in infinite dimensional polynomial rings. A key result is the proof for the existence of finite Inc($\mathbb{N}$)-equivariant Gröbner bases under specific monomial orderings. Finally, we define the Inc($\mathbb{N}$)-divisibility relation and demonstrates that the divisibility relation on Inc($\mathbb{N}$) and Inc$(\mathbb{N})^i$ are well-partial-orders, laying the background for the Hilbert-Serre theorem in the infinite dimensional context.

2.1 Well-Partial-Orders

The original references for Kruskal's Tree Theorem as well as Higman's Lemma are [9] and [10], respectively. The key concept for these results is that of a well-partial-order.

Definition 2.1. Let S be a set. A partial order $\preceq$ on S is called a *well-partial-order* if for any infinite sequence $s_1, s_2, \ldots$ of elements in S, there exist $i < j$ such that $s_i \preceq s_j$.

We call $(S, \preceq)$ a well-partially-ordered set if $\preceq$ is a well-partial-order on S.

Definition 2.2. Let S be a set with a partial order $\preceq$. An infinite sequence $s_1, s_2, \ldots$ of elements in S is called a *bad sequence* if $s_i \not\preceq s_j$ for all pairs of indices $i < j$.

From the above definitions, $\preceq$ is not a well-partial-order unless S has no bad sequence. The following give precisely examples.

Example 2.3. (a) The usual relation $\leq$ on the set of non-negative integers is a well-partial-order.

(b) The usual relation $\leq$ on $\mathbb{Q}$ is not a well-partial-order since the sequence $1, 1/2, 1/4, \ldots$ is bad.

We observe the following key properties of well-partial-orders.

Lemma 2.4. *Let $(S, \preceq)$ be a well-partially-ordered set and given an infinite sequence $s_1, s_2, \ldots$ of elements in S. Then there exists $i \geq 1$ such that there are infinitely many indices $j > i$ with the property that $s_i \preceq s_j$.*

Proof. Assume not, then for all $i \geq 1$ there exist $i < N(i)$ such that $s_i \npreceq s_j$ for all $j \geq N(i)$. We may check that the sequence $s_1, s_{N(1)}, s_{N(N(1))}, \ldots$ is bad, a contradiction. $\square$

Lemma 2.5. *Let $(S, \preceq)$ be a well-partially-ordered set, any infinite sequence $s_1, s_2, \ldots$ of elements in S has an ascending subsequence $s_{i_0} \preceq s_{i_1} \preceq \cdots$ with $i_0 < i_1 < \cdots$.*

Proof. By Lemma 2.4, there exists $i_0 \geq 1$ such that for a sequence of indices $i_0 < i_{01} < i_{02} < \cdots$ we have $s_{i_0} \preceq s_{i_{0j}}$ for all $j \geq 1$. Consider the sequence $s_{i_{01}}, s_{i_{02}}, \ldots$, again, by Lemma 2.4, there exists $i_1 \in \{i_{01}, i_{02}, \ldots\}$ such that for a subsequence of indices $i_1 < i_{11} < i_{12} < \cdots$ of $i_{01}, i_{02}, \ldots$ it holds that $s_{i_1} \preceq s_{i_{1j}}$ for all $j \geq 1$. Repeating this process, we get a subsequence

$$s_{i_0} \preceq s_{i_1} \preceq s_{i_2} \preceq \cdots$$

where $i_0 < i_1 < i_2 < \cdots$, as claimed. $\square$

The Cartesian product $S \times T$ can be equipped with a partial order $\preceq$, derived from the partial orders $\preceq_S$ on S and $\preceq_T$ on T. This order is defined component-wise, meaning $(s, t) \preceq (s', t')$ holds when both $s \preceq_S s'$ and $t \preceq_T t'$ hold.

Proposition 2.6. *Let $(S, \preceq_S)$ and $(T, \preceq_T)$ be two well-partially-ordered sets. Then the component-wise partial order $\preceq$ on $S \times T$ is also a well-partial-order.*

Proof. Consider an infinite sequence $(s_1, t_1), (s_2, t_2), \ldots$ in $S \times T$. Applying Lemma 2.5 for the set S, there is an infinite subsequence

$$s_{i_0} \preceq_S s_{i_1} \preceq_S \cdots .$$

Applying Lemma 2.5 again for the subsequence $t_{i_0}, t_{i_1}, \ldots$ in T, the proposition holds. $\square$

Corollary 2.7. *The component-wise partial order on the finite Cartesian product of well-partially-ordered sets is a well-partial-order.*

Since the usual relation $\leq$ on $\mathbb{N}_0$ is a well-partial-order, we now deduce a crucial property.

Corollary 2.8. *Fix an integer $n \geq 2$, the component-wise partial order on $\mathbb{N}_0^n$ is a well-partial-order.*

Corollary 2.8 and Lemma 1.18 are two equivalent versions of Dickson's Lemma. The following proof shows the equivalence.

Proof. Suppose that there is a monomial ideal $I \in K[x_1, \dots, x_n]$ such that I is not finitely generated. There are $x^{\alpha_1}, x^{\alpha_2}, \dots \in I$ such that $x^{\alpha_m} \notin \langle x^{\alpha_1}, \dots, x^{\alpha_{m-1}} \rangle$, this implies $\alpha_i \npreceq \alpha_m$ for all $i = 1, \dots, m-1$. Hence $\alpha_1, \alpha_2, \dots$ is a bad sequence, a contradiction.

Conversely, suppose that $(\mathbb{N}_0^n, \preceq)$ is not well-partially-ordered, then there exists a bad sequence $\alpha_1, \alpha_2, \dots \in \mathbb{N}_0^n$. Consider the ideal

$$I = \langle x^{\alpha_i} : i \geq 1 \rangle.$$

Since I is a finitely generated monomial ideal, there are $i_1, \dots, i_n \geq 1$ such that $I = \langle x^{\alpha_{i_1}}, \dots, x^{\alpha_{i_n}} \rangle$. Set $m = \max\{i_1, \dots, i_n\}$, since $x^{\alpha_{m+1}} \in I = \langle x^{\alpha_{i_1}}, \dots, x^{\alpha_{i_n}} \rangle$, there is $j \in \{i_1, \dots, i_n\}$ such that $x^{\alpha_j} | x^{\alpha_{m+1}}$. Hence $\alpha_j \preceq \alpha_{m+1}$, a contradiction. $\square$

2.2 Kruskal's Tree Theorem and Higman's Lemma

The infinite dimensional case of Dickson's Lemma is Kruskal's Tree Lemma, which is useful to prove the finiteness up to symmetry.

Definition 2.9. Let A be a set. The set B is called a *multi-subset* of A if every element of B is an element of A and elements of B need not be distinct. The ordering of elements in B is not important.

For example, let $A = \{1, 2, 3, 4, 5\}$, the set $B = \{1, 2, 2, 5\}$ is a 4-element multi-subset of A. Also the multi-subset $\{1, 2, 5, 2\}$ is the same as B.

Definition 2.10. Let $\preceq_S$ be a partial order on a set S and $\hat{S}$ be the set of finite multi-subsets of S. We define the partial order $\preceq$ on $\hat{S}$ by for any $A, B \in \hat{S}$, $A \preceq B$ if and only if there is an injective map $f : A \to B$ such that $a \preceq_S f(a)$ for all $a \in A$.

Remark 2.11. If there is a bad sequence $A_1, A_2, \dots$ of finite multi-subsets of S, then there are no $A_i = \emptyset$.

Proof. Indeed, if there exists $A_i = \emptyset$, the map $\emptyset \to A_j$ is injective for all j, thus $A_i \preceq A_j$, a contradiction. $\square$

Lemma 2.12. *If $(S, \preceq_S)$ is a well-partially-ordered set, then $(\hat{S}, \preceq)$ is a well-partially-ordered set.*

Proof. Proof by contradiction. Assume the set $\hat{S}$ contains at least one infinite bad sequence. We can select such a sequence $A_1, A_2, \dots,$ by making minimal choices iteratively based on set cardinality. Choose A_1 such that $|A_1|$ is minimized among all possible initial elements of infinite bad sequences; then, given A_1, choose A_2 such that $A_1, A_2, \dots$ is an infinite bad sequence and $|A_2|$ is minimized; continue this process, selecting A_k with minimal cardinality $|A_k|$ subject to the condition that $A_1, \dots, A_k, \dots$ remains an infinite bad sequence, given the previously chosen $A_1, \dots, A_{k-1}$.

By the definition of a bad sequence, the constructed sequence $A_1, A_2, \dots$ allows us to select an element $a_i \in A_i$ for each $i \geq 1$. For each i, define the set $B_i = A_i \setminus \{a_i\}$. By Lemma 2.5, there exist indices $i_0 < i_1 < \dots$ such that

$$a_{i_0} \preceq_S a_{i_1} \preceq_S \cdots .$$

Now consider the sequence $A_1, A_2, \dots, A_{i_0-1}, B_{i_0}, B_{i_1}, \dots,$ we prove that it is a bad sequence. Indeed, we have $A_i \not\preceq A_j$ for all $i < j \leq i_0 - 1$. As $A_i \not\preceq A_j$ for all $i < j$, it follows that $A_i \not\preceq B_j$ for all $i \leq i_0 - 1$, $j \in \{i_0, i_1, \dots\}$ (since the set of all the injective maps from A_i to B_j is just a subset of the set of all the injective maps from A_i to A_j). Finally, we have $B_i \not\preceq B_j$ with $i, j \in \{i_0, i_1, \dots\}$, $i < j$. Indeed, if $B_i \preceq B_j$, the injective map $f : B_i \to B_j$ could be extended to a map $g : A_i \to A_j$ by mapping $g(a) = f(a)$ for all $a \in B_i$ and $g(a_i) = a_j$, this implies that $A_i \preceq A_j$, which cannot happen. Hence, the sequence $A_1, A_2, \dots, A_{i_0-1}, B_{i_0}, B_{i_1}, \dots$ is a bad sequence, which contradicts the minimality of the cardinality $|A_{i_0}|$. $\square$

Definition 2.13. [11] Let S be a well-partially-ordered set. An *S-labelled trees* is the set of (isomorphism classes of) finite, rooted trees whose vertices are labelled with elements in S.

We now inductively define the partial order $\preceq$ (representing homeomorphic embedding) on the set of finite S-labelled trees. Given two S-labelled trees, T and T', we say $T \preceq T'$ if there exists a vertex v in T' such that:

1. The S-label of v is greater than or equal to the S-label of the root of T (with respect to the order on S).

2. Let $B_1, \ldots, B_p$ be the subtrees rooted at the children of the root of T, and let $B'_1, \ldots, B'_{p'}$ be the subtrees rooted at the children of v. There must exist an injective map $\pi : \{1, \ldots, p\} \to \{1, \ldots, p'\}$ such that $B_i \preceq B'_{\pi(i)}$ holds recursively for all $i \in \{1, \ldots, p\}$.

It is a consequence of this definition that $T \preceq T'$ corresponds to the existence of an injective, label-order-preserving, structure-preserving map from the nodes of T into the nodes of T'.

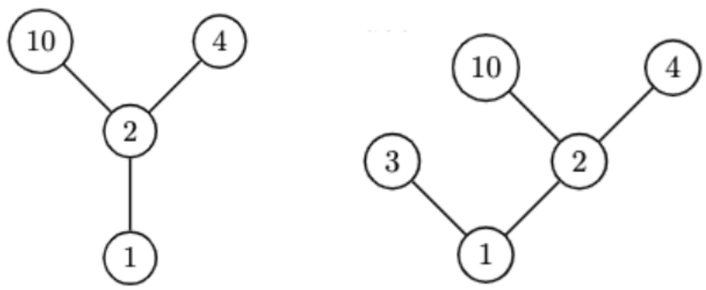

Figure 2.1: Two $\mathbb{N}$-labelled trees with the partial-order divides "|", rooted at 1.

In the figure, it is clear that the left tree is "less than" the right tree. Now by using the partial-order above, we discover the Kruskal's tree theorem.

Theorem 2.14. [11, Theorem 1.2] *If S is a well-partially-ordered set, then the set of S-labelled trees is well-partially-ordered by the partial order defined above.*

Proof. Proof by contradiction. Assume the set $\hat{S}$ contains at least one infinite bad sequence. We can select such a sequence $T_1, T_2, \ldots$ by making minimal choices iteratively based on set cardinality. Choose T_1 such that $|T_1|$ is minimized among all possible initial elements of infinite bad sequences; then, given T_1, choose T_2 such that $(T_1, T_2, \ldots)$ is an infinite bad sequence and $|T_2|$ is minimized; continue this process, selecting T_k with minimal cardinality $|T_k|$ subject to the condition that $(T_1, \ldots, T_k, \ldots)$ remains an infinite bad sequence, given the previously chosen $T_1, \ldots, T_{k-1}$. At its root, T_i branches into a finite multi-set R_i of smaller trees, which we shall called branches. Let

$$R = \bigcup_{i \geq 1} R_i.$$

We claim that R cannot contains a bad sequence. Indeed, assume that there is a bad sequence $B_{i_0}, B_{i_1}, \dots$ in R, with $B_i \in R_i$, $i_0 < i_1 < \dots$. Consider the sequence $T_1, \dots, T_{i_0-1}, B_{i_0}, B_{i_1} \dots$, proving as the above lemma, we deduce this sequence is bad. By taking the identity mapping, we have $B_i \preceq T_i$ for all $i \in \{i_0, i_1, \dots\}$, which contradicts to the minimality of T_{i_0} among all the sequences starting with $T_1, T_2, \dots, T_{i_0-1}$. Hence R is well-partially-ordered. Applying Lemma 2.12, the sequence $R_1, R_2, \dots$ of finite multi-subsets of R cannot be bad (with respect to the partial order defined in the lemma). Let s_i be the label of the root of T_i for each $i \geq 1$. Since S is well-partially-ordered, by Lemma 2.5, we may also assume that $s_1 < s_2 < s_3 < \cdots$. Hence there is an injective map $R_i \to R_j$ with $i < j$, mapping each branch B of T_i to a branch B' of T' with $B \preceq B'$. This means that $T_i \preceq T_j$, a contradiction. □

Let $\preceq_S$ be a partial order on the set S, define a partial order $\preceq$ on $S^* = \bigcup_{n \geq 1} S^n$ of finite sequences over S by saying that $(s_1, \dots, s_p) \preceq (s'_1, \dots, s'_q)$ if there exists a strictly increasing map $\pi : \{1, \dots, p\} \to \{1, \dots, q\}$, $s_i \preceq_S s'_{\pi(i)}$ for all i. We next prove the Higman's Lemma, which uses Kruskal's Tree Theorem.

Corollary 2.15 (Higman's Lemma)**.** *If S is a well-partially-ordered set, then $(S^*, \preceq)$ is a well-partially-ordered set, where the partial order $\preceq$ is defined above.*

Proof. We demonstrate Higman's Lemma by applying Kruskal's Tree Theorem. Define a mapping ϕ from the set of finite sequences S^* to the set of finite S-labelled trees: for a sequence $\sigma = (s_1, \dots, s_q) \in S^*$, let $\phi(\sigma)$ be the tree consisting of a single path of length $q - 1$, rooted at a node labelled s_1, whose child is labelled s_2, and so on, terminating at the leaf node labelled s_q.

Now, consider any infinite sequence $\sigma_1, \sigma_2, \dots$ of elements from S^*. This corresponds to an infinite sequence of trees $T_1 = \phi(\sigma_1), T_2 = \phi(\sigma_2), \dots$. According to Kruskal's Tree Theorem, the set of finite S-labelled trees is well-partial-ordered under the relation of homeomorphic embedding $\preceq$. Therefore, this infinite sequence of trees must contain an embedding, i.e., there exist indices $i < j$ such that $T_i \preceq T_j$. This embedding between the tree representations implies the divisibility condition required by Higman's Lemma, thus proving the lemma. □

2.3 The Monoid $\mathrm{Inc}(\mathbb{N})$ and Its Subsets $\mathrm{Inc}(\mathbb{N})^i$

Let K be a field and $X = \{x_{ij} \mid i \in [c], j \in \mathbb{N}\}$ an infinite countable collection of variables. We consider the associated polynomial ring $K[X]$. Interest in the ideal structure of such infinite polynomial rings arises from challenges in areas like algebraic statistics, tensor theory, and representation theory, especially when dealing with structures in indefinitely large dimensions.

A key aspect of recent research involves ideals in $K[X]$ that exhibit specific symmetries. Often, the focus is on ideals stable under the action of the symmetric group or under the action of submonoids derived from the monoid of strictly increasing functions on the positive integers. This latter monoid, denoted $\mathrm{Inc}(\mathbb{N})$, consists of all maps $\pi : \mathbb{N} \to \mathbb{N}$ such that $\pi(i) < \pi(i+1)$ holds for every $i \geq 1$:

$$\mathrm{Inc}(\mathbb{N}) = \{\pi : \mathbb{N} \to \mathbb{N} \mid \pi(i) < \pi(i+1) \text{ for all } i \geq 1\}.$$

Delving deeper into these monoids, we examine the submonoids $\mathrm{Inc}(\mathbb{N})^i$ that fix the initial elements

$$\mathrm{Inc}(\mathbb{N})^i = \{\pi \colon \mathbb{N} \to \mathbb{N} \mid \pi(j) = j \text{ for all } j \leq i\},$$

where $i \geq 0$ is an integer. By convention, $\mathrm{Inc}(\mathbb{N})^0 = \mathrm{Inc}(\mathbb{N})$.

An element $\sigma_i \in \mathrm{Inc}(\mathbb{N})^i$ for a given $i > 0$, is defined by

$$\sigma_i(j) = \begin{cases} j & \text{if } 1 \leq j \leq i, \\ j+1 & \text{if } j > i, \end{cases}$$

and $\sigma_0(j) = j + 1$. By definition, $\sigma_i \in \mathrm{Inc}(\mathbb{N})^j$ for all $j \leq i$.

We state some basic decompositions.

Proposition 2.16. *For $i \geq 0$ and $\pi \in \mathrm{Inc}(\mathbb{N})^i$, there exist $\tau \in \mathrm{Inc}(\mathbb{N})^{i+1}$ satisfying*

$$\sigma_i \circ \pi = \tau \circ \sigma_i.$$

Furthermore, if $\pi(m) \leq n$, then $\tau(m+1) \leq n+1$.

Proof. For $i = 0$, we may take

$$\tau(j) = \begin{cases} 1 & \text{if } j = 1, \\ \pi(j-1)+1 & \text{if } j \geq 2. \end{cases}$$

And for $i > 0$, we take

$$\tau(j) = \begin{cases} j & \text{if } 1 \le j \le i, \\ \pi(j-1)+1 & \text{if } j \ge i+1. \end{cases}$$

A straightforward computation implies the first part of the lemma. For the second part, consider 2 cases:

- If $m \le i$, then $\pi(m) = m \le n$, $\tau(m+1) = m+1 \le n+1$.
- If $m \ge i+1$, then $\tau(m+1) = \pi(m)+1 \le n+1$.

In conclusion, the lemma holds. □

Corollary 2.17. *$\sigma_i \circ \sigma_{j-1} = \sigma_j \circ \sigma_i$ for every $j > i \ge 0$.*

Proof. Apply the above lemma for $\pi = \sigma_{j-1} \in \mathrm{Inc}(\mathbb{N})^i$. Follow the proof of Proposition 2.16, for $i = 0$,

$$\begin{aligned} \tau(j) &= \begin{cases} 1 & \text{if } j = 1, \\ \sigma_{j-1}(j-1)+1 & \text{if } j \ge 2, \end{cases} \\ &= \begin{cases} 1 & \text{if } j = 1, \\ j & \text{if } j \ge 2, \end{cases} \\ &= j \text{ if } j \ge 1 \\ &= \sigma_j(j). \end{aligned}$$

For $i > 0$, the computation is identically the same. □

Lemma 2.18. *For any $\pi \in \mathrm{Inc}(\mathbb{N})^i \setminus \mathrm{Inc}(\mathbb{N})^{i+1}$, there exists $\tau \in \mathrm{Inc}(\mathbb{N})^{i+1}$ such that*

$$\pi = \tau \circ \sigma_i.$$

Proof. We may check that the map

$$\tau(j) = \begin{cases} j & \text{if } 1 \le j \le i+1, \\ \pi(j-1) & \text{if } j \ge i+2 \end{cases}$$

satisfies our lemma. Indeed, let us consider 3 cases:

- For $j \le i$, $\tau(\sigma_i(j)) = \tau(j) = j = \pi(j)$.
- For $j = i+1$, $\tau(\sigma_i(j)) = \tau(i+2) = \pi(i+1) = \pi(j)$.

- For $j \geq i+2$, $\tau(\sigma_i(j)) = \tau(j+1) = \pi(j)$.

Thus $\pi = \tau \circ \sigma_i$. $\square$

Definition 2.19. For integers $i \geq 0$, $m \leq n$, the set $\mathrm{Inc}(\mathbb{N})^i_{m,n}$ is defined by

$$\mathrm{Inc}(\mathbb{N})^i_{m,n} = \{\pi \in \mathrm{Inc}(\mathbb{N})^i \mid \pi(m) \leq n\}.$$

Furthermore, we define

$$\mathrm{Inc}(\mathbb{N})^j_{m,n} \circ \mathrm{Inc}(\mathbb{N})^i_{k,m} = \{\tau \circ \pi \mid \tau \in \mathrm{Inc}(\mathbb{N})^j_{m,n} \text{ and } \pi \in \mathrm{Inc}(\mathbb{N})^i_{k,m}\}$$

where $k \leq m \leq n$ and $i, j \geq 0$.

Proposition 2.20. *Consider integers $i \geq 0$ and $n > m \geq 1$. We have a decomposition*

$$\mathrm{Inc}(\mathbb{N})^i_{m,n} = \mathrm{Inc}(\mathbb{N})^{i+1}_{m+1,n} \circ \mathrm{Inc}(\mathbb{N})^i_{m,m+1}, \tag{2.1}$$

as subsets of $\mathrm{Inc}(\mathbb{N})$. *In particular,*

$$\mathrm{Inc}(\mathbb{N})^i_{m,n} = \mathrm{Inc}(\mathbb{N})^i_{m+1,n} \circ \mathrm{Inc}(\mathbb{N})^i_{m,m+1}.$$

Proof. Let $\pi \in \mathrm{Inc}(\mathbb{N})^{i+1}_{m+1,n}$ and $\tau \in \mathrm{Inc}(\mathbb{N})^i_{m,m+1}$. For any integer $j \leq m$, we have

$$\begin{aligned} &\tau(j) \leq m+1 \\ \implies\ &\pi(\tau(j)) \leq \pi(m+1) \leq n. \end{aligned}$$

Thus $\pi \circ \tau \in \mathrm{Inc}(\mathbb{N})^i_{m,n}$ and hence the inclusion ($\supseteq$) of (2.1) holds.

Conversely, let $\pi \in \mathrm{Inc}(\mathbb{N})^i_{m,n}$. If $\pi = \mathrm{id}$, the inclusion holds. If π is not the identity, we may find a $j \geq i$ such that $\pi \in \mathrm{Inc}(\mathbb{N})^j \setminus \mathrm{Inc}(\mathbb{N})^{j+1}$. From the proof of Lemma 2.18, the map τ given by

$$\tau(s) = \begin{cases} s & \text{if } 1 \leq s \leq j+1, \\ \pi(s-1) & \text{if } s \geq j+2 \end{cases}$$

satisfies $\pi = \tau \circ \sigma_j$. Note that

$$\tau \in \mathrm{Inc}(\mathbb{N})^{j+1}_{m+1,n} \subset \mathrm{Inc}(\mathbb{N})^{i+1}_{m+1,n} \text{ (since } j \geq i\text{)},$$

and

$$\sigma_j \in \mathrm{Inc}(\mathbb{N})^j_{m,m+1} \subset \mathrm{Inc}(\mathbb{N})^i_{m,m+1}.$$

Thus the Equation (2.1) holds. The last formula follows from the same proof, with the reminder that $\mathrm{Inc}(\mathbb{N})^{i+1}_{m,m+1} \subset \mathrm{Inc}(\mathbb{N})^i_{m,m+1}$. $\square$

2.4 The Existence of Finite $\mathrm{Inc}(\mathbb{N})$-Equivariant Gröbner Bases

We begin by recalling standard order-theoretic concepts. A *total order* $\preceq$ on a set S is a partial order satisfying the comparability condition: for any $x, y \in S$, either $x \preceq y$ or $y \preceq x$. With respect to such an order, an element $x \in S$ is called *minimal* if no element $y \in S$, distinct from x, satisfies $y \preceq x$. A total order $\preceq$ on S is called a *well order* if the property holds that every non-empty subset of S contains a minimal element with respect to $\preceq$.

Let Mon denote the set of monomials of variables in $X = \{x_{ij} | i \in [c], j \in \mathbb{N}\}$.

Definition 2.21. A monomial ordering $\preceq$ on Mon is a well order such that $1 \preceq u$ for all $u \in \mathrm{Mon}$ and $u \preceq v$ implies $uw \preceq vw$ for all $u, v, w \in \mathrm{Mon}$.

Proposition 2.22. *Every well-partially-ordered set $(S, \preceq)$ has only finitely many minimal elements.*

Proof. If $|S|$ is finite, we are done. If $|S|$ is infinite, suppose that there are infinitely many minimal elements in S. Consider the sequence which all the elements inside are minimal. Clearly, this is a bad sequence by definition of minimal element, a contradiction to the assumption that S is well-partially-ordered. $\square$

Fix a monomial order on Mon. Let Π be a monoid acting on Mon and assume that the action preserves strict inequalities, that is if $u \prec v$ then $\pi(u) \prec \pi(v)$ for all $\pi \in \Pi$ and $u, v \in \mathrm{Mon}$.

Note that $K[X] = K\mathrm{Mon}$ the polynomial ring in the variables in X, or equivalently, the K-algebra on Mon. The action of Π on $K[X]$ is additivity, that is for any $\alpha, \beta \in K$, $\pi \in \Pi$ and $u, v \in \mathrm{Mon}$, we have $\pi(\alpha u + \beta v) = \alpha\pi(u) + \beta\pi(v)$.

Definition 2.23. An ideal $I \subseteq K[X]$ is called a Π-*invariant ideal* if $\pi I \subseteq I$ for all $\pi \in \Pi$.

Definition 2.24. A subset B of a Π-invariant ideal $I \subseteq K[X]$ is called a Π-*Gröbner basis*, or *equivariant Gröbner basis* of I if for every non-zero polynomial $f \in I$, its leading monomial $\mathrm{LM}(f)$ must be divisible by $\mathrm{LM}(\pi(g))$ for some element $g \in B$ and some $\pi \in \Pi$.

Lemma 2.25. *If $I = \langle m_1, m_2, \dots \rangle \subseteq K[X]$ is an ideal generated by the monomials m_i, and $f \in K[X]$ is a monomial, then $f \in I$ if and only if $m_i | f$ for some $i \geq 1$.*

Proof. The "if" part is obvious. For the "only if" part, let $f \in K[X]$. Choose p, q such that every term of f only involves the variables $x_{11}, x_{12}, \dots, x_{1,q+1}, x_{21}, x_{22}, \dots, x_{p+1,q+1}$. Moreover, since f is a monomial, then

$$f = \alpha \prod_{i=1}^{p+1} \prod_{j=1}^{q+1} x_{ij}^{a_{ij}} \in K\big[x_{ij} | i \in [p+1], j \in [q+1]\big],$$

where $\alpha \in K$, $a_{ij} \in \mathbb{N}$. Now since $f \in I$, there are monomials $m_{n_1}, \dots, m_{n_k}$ which we may assume to involve only the variables $x_{ij}, i \in [p+1], j \in [q+1]$ such that $f = \sum_{i=1}^{k} g_i m_{n_i}$. Hence f is a monomial of the ideal $\langle m_{n_1}, \dots, m_{n_k} \rangle \subseteq K\big[x_{ij} | i \in [p+1], j \in [q+1]\big]$, where $\{m_{n_1}, \dots, m_{n_k}\}$ is a subset of $\{m_1, m_2, \dots\}$. This means that there is a m_s divides f, as claimed. $\square$

Proposition 2.26. *Let $I \subseteq K[X]$ be a Π-invariant ideal. If B is a Π-Gröbner basis of I, then $\langle \Pi B \rangle = I$.*

Proof. Since I is a Π-invariant ideal, ΠB is a subset of I, hence $\langle \Pi B \rangle \subseteq I$. Conversely, suppose that there is $f \in I \setminus \langle \Pi B \rangle$. Choose f such that $\mathrm{LT}(f)$ is minimal. We have

$$\mathrm{LT}(f) \in \langle \mathrm{LT}(I) \rangle = \langle \mathrm{LT}(g) | g \in \Pi B \rangle.$$

By Lemma 2.25, there is $g \in \Pi B$ such that $\mathrm{LT}(f) = \alpha m \mathrm{LT}(g)$, where m is a monomial and $\alpha \in K$. Thus $f - \alpha m g \in I \setminus \langle \Pi B \rangle$. But we also have $\mathrm{LT}(f - \alpha m g) < \mathrm{LT}(f)$, a contradiction to the minimality of $\mathrm{LT}(f)$. Hence $I = \langle \Pi B \rangle$. $\square$

A Π-Gröbner basis need not be finite. To determine the finiteness, we first define the relation Π-divisibility.

Definition 2.27. Let $u, v \in \mathrm{Mon}$. We define $u|_{\Pi} v$ if there is a $\pi \in \Pi$ such that $\pi(u)|v$.

This relation is well-defined. Indeed

- The reflexivity is obtained by taking $\pi = \mathrm{id}$;
- If $\pi(u)|v$ and $\sigma(v)|w$, then $(\sigma\pi)u|w$, then $|_{\Pi}$ is transitive;
- If $\pi(u)|v$ and $\sigma(v)|u$ then $u \preceq \pi(u) \preceq v \preceq \sigma(v) \preceq u$ so that $u = v$, then $|_{\Pi}$ is antisymmetric.

Proposition 2.28. [2, Theorem 2.12] *Given a monomial order on* Mon *and assume* Π *preserves the strict ordering of monomials. Every Π-invariant ideal $I \subseteq K[X]$ has a finite Π-Gröbner basis if and only if $|_{\Pi}$ is a well-partial-order.*

Proof. Suppose that $|_\Pi$ is not a well-partial-order, then there is a bad sequence $u_1, u_2, \ldots$ of monomials in $K[X]$. Consider the ideal Π-invariant ideal $I = \langle \Pi u_1, \Pi u_2, \ldots \rangle$. Suppose that I has a finite Π-Gröbner basis, say $v_1, \ldots, v_n$. If $v_1, \ldots, v_n$ are monomials, then $u_j \in I = \langle \pi v_i | \pi \in \Pi, i = 1, \ldots, n \rangle$. Therefore for all $j \geq 0$, there exists $v_{i_j} \in \{v_1, \ldots, v_n\}$ such that $v_{i_j} |_\Pi u_j$.

Since $\{v_1, \ldots, v_n\}$ is a finite set, for some $1 \leq i \leq n$, there exists an infinite sequence $i_1 < i_2 < \cdots$ such that v_i is Π-divisor of u_{i_j} for every j.

Since $v_i \in I = \langle \Pi u_1, \Pi u_2, \ldots \rangle$ there is an index r such that u_r is a Π-divisor of v_i. Hence u_r is a Π-divisor of u_{i_j} for every j. Choose j such that $r < i_j$, we get a contradiction to the assumption that the sequence $u_1, u_2, \ldots$ is bad.

Next assume that there is $v_i \in I$ such that v_i is not a monomial. Let J be the ideal generated by the Π-orbits of all the terms of $v_1, \ldots, v_n$. We have $I \subseteq J$. Since I is generated by monomials and $v_1, \ldots, v_n$ are in I, by Lemma 2.25, $J \subseteq I$. Hence $I = J$ and I is generated by the orbits of finitely many monomials. Repeating the above proof we will imply that Mon can't contain a bad sequence. Hence $|_\Pi$ is a well-partial-order.

For the converse direction, assume I is a Π-invariant ideal within $K[X]$. Consider the set of all leading monomials associated with non-zero elements of I, i.e., $L = \{\mathrm{LM}(f) \mid f \in I \setminus \{0\}\}$. Let M be the subset of L containing only the $|_\Pi$-minimal elements. By Proposition 2.22, M is a finite set; let us write $M = \{u_0, \ldots, u_{p-1}\}$. For each $u_i \in M$, we can select a corresponding polynomial $f_i \in I \setminus \{0\}$ such that $\mathrm{LM}(f_i) = u_i$. It then follows that the finite collection $\{f_0, \ldots, f_{p-1}\}$ forms a Π-Gröbner basis for the ideal I. $\square$

In Theorem 2.31 below we use the set of variables $X = \{x_{ij} \mid i \in [c], j \in \mathbb{N}\}$ with the lexicographic order on X: $x_{ij} \preceq x_{i'j'}$ if $i < i'$ or $i = i'$ and $j < j'$. And we use $\Pi := \mathrm{Inc}(\mathbb{N})$, the set of strictly increasing maps $\mathbb{N} \to \mathbb{N}$. The set $\mathrm{Inc}(\mathbb{N})$ acts on X by $\pi x_{ij} = x_{i\pi(j)}$.

A monomial order $\preceq$ for which $u \prec v$ implies $\pi u \prec \pi v$ for all $\pi \in \Pi$ is called a *monomial order preserved by* Π. For example, the afore-mentioned lexicographic order is preserved by $\mathrm{Inc}(\mathbb{N})$.

Consider the ring $K[x_{ij} | i \in [c], j \in \mathbb{N}]$. The following result employs Higman's Lemma in the case $S = \mathbb{N}^c$ with the component-wise partial order, which is a well-partial-order by Dickson's Lemma.

Lemma 2.29. $|_{\mathrm{Inc}(\mathbb{N})}$ *is a well-partial-order.*

Proof. Let $S = \mathbb{N}_0^c$. We define a mapping ϕ that encodes each monomial $u \in K[X]$ into a finite sequence in S^*. For $u = \prod_{i=1}^{c}\prod_{j=1}^{\infty} x_{ij}^{e_{ij}}$ (where only finitely many $e_{ij} > 0$), let $p = \max\{j \mid \exists i, e_{ij} > 0\}$ be the largest column index involved in u. The encoding is $\phi(u) = s = (s_1, \ldots, s_p)$, where each $s_j \in S$ is the vector $s_j = (e_{1j}, e_{2j}, \ldots, e_{cj}) \in \mathbb{N}_0^c$. For example, if $c = 3$ and $u = x_{11}^2 x_{12}^1 x_{13}^6 x_{22}^3 x_{32}^1$, then $p = 3$ and $s = ((2,0,0),(1,3,1),(6,0,0))$.

Now, take any infinite sequence of monomials $u_1, u_2, \ldots$. Applying the encoding yields an infinite sequence of words $\phi(u_1), \phi(u_2), \ldots$ in S^*. By Higman's Lemma, S^* is well-partial-ordered under the order $\preceq$. Therefore, there must exist indices $k < l$ such that $\phi(u_k) \preceq \phi(u_l)$. Let $s = \phi(u_k)$ with length p, and $s' = \phi(u_l)$ with length p'. The condition $s \preceq s'$ implies the existence of an injective, strictly order-preserving map $\pi : \{1, \ldots, p\} \to \{1, \ldots, p'\}$ such that $s_j \preceq s'_{\pi(j)}$ holds component-wise for all $j \in \{1, \ldots, p\}$.

We now show this implies $\pi(u_k)|u_l$, where $\pi(u_k)$ is the monomial obtained by replacing each x_{ij} in u_k with $x_{i,\pi(j)}$. Consider an arbitrary variable $x_{i,j'}$. We need to compare its exponent in $\pi(u_k)$ and u_l.

- Case 1: $j' = \pi(j)$ for some $j \in \{1, \ldots, p\}$. Then the exponent of $x_{i,\pi(j)}$ in $\pi(u_k)$ is, by definition of $\pi(u_k)$ and s, exactly $(s_j)_i$. The exponent of $x_{i,\pi(j)}$ in u_l is $(s'_{\pi(j)})_i$. The component-wise inequality $s_j \preceq s'_{\pi(j)}$ gives $(s_j)_i \le (s'_{\pi(j)})_i$, establishing the required exponent inequality.

- Case 2: j' is not in the image of π. Then the exponent of $x_{i,j'}$ in $\pi(u_k)$ is 0, which is less than or equal to its non-negative exponent in u_l. Since the exponent of every variable in $\pi(u_k)$ is less than or equal to its exponent in u_l, we conclude that $\pi(u_k)$ divides u_l.

Thus $\pi(u_k)|u_l$, as desired. □

By an argument similar to the proof of Lemma 2.29, we can show that the following is true.

Corollary 2.30. *For each integer $i > 0$, $|_{\mathrm{Inc}(\mathbb{N})^i}$ is a well-partial-order.*

Theorem 2.31. [11, Theorem 2.3] *Let I be an* Inc($\mathbb{N}$)*-invariant ideal in the polynomial ring $R = K[x_{ij} \mid i \in [c], j \in \mathbb{N}]$, where $c \ge 1$ is a fixed integer. If $\preceq$ is any monomial order on R that is preserved by the action of* Inc($\mathbb{N}$)*, then I possesses a finite* Inc($\mathbb{N}$)*-Gröbner basis with respect to $\preceq$. Additionally, every such* Inc($\mathbb{N}$)*-invariant ideal I is generated by finitely many* Inc($\mathbb{N}$)*-orbits of polynomials.*

Proof. By Proposition 2.28, we only need to show that $|_{\mathrm{Inc}(\mathbb{N})}$ is a well-partial-order, which is given in Lemma 2.29. $\square$

2.5 Hilbert's Basis Theorem for Infinite Dimensional Polynomial Rings

Let $\mathrm{Sym}(j)$ denote the symmetric group acting on the set $\{1, \ldots, j\}$. The infinite symmetric group is defined by $\mathrm{Sym}(\mathbb{N}) := \bigcup_{j\in\mathbb{N}} \mathrm{Sym}(j)$, using the standard inclusions $\mathrm{Sym}(j) \hookrightarrow \mathrm{Sym}(j+1)$ where permutations in $\mathrm{Sym}(j)$ are extended to fix the element $j+1$. This group $\mathrm{Sym}(\mathbb{N})$ acts naturally on the polynomial ring $K[X]$ (where $X = \{x_{ij} \mid i \in [c], j \in \mathbb{N}\}$) by permuting the column indices of the variables: for any $\pi \in \mathrm{Sym}(\mathbb{N})$, its action on a variable x_{ij} is defined as $\pi \cdot x_{ij} = x_{i,\pi(j)}$.

Lemma 2.32. *Let $f \in K[X]$ be any polynomial. Then the* $\mathrm{Inc}(\mathbb{N})$*-orbits of f is a subset of the* $\mathrm{Sym}(\mathbb{N})$*-orbits of f.*

Proof. Let $\pi \in \mathrm{Inc}(\mathbb{N})$ and n the maximal column index of f. Since $\pi x_{ij} = x_{i\pi(j)}$, $\pi(f)$ just involves the first $\pi(n)$ columns. Since the map $i \mapsto \pi(i)$ on $\{1, \ldots, \pi(n)\}$ is injective, there exists $\sigma \in \mathrm{Sym}(\pi(n))$ satisfying $\sigma(i) = \pi(i)$ for all $1 \leq i \leq n$. One checks that $\pi(f) = \sigma(f)$. $\square$

Consequently, if an ideal is $\mathrm{Sym}(\mathbb{N})$-invariant, it must be $\mathrm{Inc}(\mathbb{N})$-invariant. The following result is the infinite dimensional version for the Hilbert's basis theorem.

Corollary 2.33. *Every* $\mathrm{Sym}(\mathbb{N})$*-invariant ideal in $K[X]$ can be generated by a finite collection of* $\mathrm{Sym}(\mathbb{N})$*-orbits.*

Example 2.34. Let I be the ideal $\langle x_j \mid j \in \mathbb{N}\rangle$ of the ring $K[x_1, x_2, \ldots]$. The $\mathrm{Inc}(\mathbb{N})$-orbit of a variable x_i is $\mathrm{Inc}(\mathbb{N}) \cdot x_i = \{x_{\pi(i)} \mid \pi \in \mathrm{Inc}(\mathbb{N})\} = \{x_k \mid k \geq i\}$. In particular, the orbit of x_1 is $\mathrm{Inc}(\mathbb{N}) \cdot x_1 = \{x_k \mid k \geq 1\}$. Clearly, the orbit of x_1 generates I.

2.6 $\mathrm{Inc}(\mathbb{N})^i$-Invariant Chains of Ideals

We study $\mathrm{Inc}(\mathbb{N})^i$-invariant ideals following these construction. Consider the set of variables $X = \{x_{ij} \mid i \in [c], j \in \mathbb{N}\}$ as in Section 2.3. For each integer $n \geq 0$, put

$$X_n = \{x_{i,j} \mid i \in [c], j \in [n]\} \text{ if } n > 0, \text{ and } X_0 = \emptyset.$$

For each ideal $I_m \in K[X_m]$, we will write $\langle \mathrm{Inc}(\mathbb{N})^i_{m,n}(I_m)\rangle$ instead of $\langle \mathrm{Inc}(\mathbb{N})^i_{m,n}(I_m)\rangle_{K[X_n]}$ if the ring $K[X_n]$ is clear.

Definition 2.35. [4, Definition 5.1] Fix an integer i.

(a) An $\mathrm{Inc}(\mathbb{N})^i$-*invariant chain* is a chain $\mathcal{I} = (I_n)_{n\in\mathbb{N}}$ of ideals $I_n \subseteq K[X_n]$ such that, as subsets of $K[X]$, one has

$$\mathrm{Inc}(\mathbb{N})^i_{m,n}(I_m) \subseteq I_n \quad \text{whenever } m \leq n.$$

(b) An $\mathrm{Inc}(\mathbb{N})^i$-invariant chain $\mathcal{I} = (I_n)_{n\in\mathbb{N}}$ is said to *stabilize* if there exist an integer r such that

$$\langle \mathrm{Inc}(\mathbb{N})^i_{r,n}(I_r)\rangle_{K[X_n]} = I_n \quad \text{whenever } r \leq n.$$

The least integer $r \geq 1$ with this property is called the i-*stability index* $\mathrm{ind}^i(\mathcal{I})$ of $\mathcal{I}$.

The number $\mathrm{ind}(\mathcal{I}) = \mathrm{ind}^0(\mathcal{I})$ is called the *stability index* of $\mathcal{I}$.

Example 2.36. Let $X = \{x_i | i \in \mathbb{N}\}$ be a variable set; so $c = 1$ and we set $x_i := x_{1,i}$. Consider the chain $\mathcal{I} = (I_n)_{n\in\mathbb{N}}$, where $I_n = \langle x_1, x_2, \dots, x_n\rangle$. For each $\sigma \in \mathrm{Inc}(\mathbb{N})^0_{m,n} = \mathrm{Inc}(\mathbb{N})_{m,n}$, we have $\sigma(m) \in \{m, m+1, \dots, n\}$. Now

$$\mathrm{Inc}(\mathbb{N})_{m,n}(I_m) = \langle x_1, \dots, x_n\rangle = I_n.$$

Thus the $\mathrm{Inc}(\mathbb{N})$-invariant chain $\mathcal{I}$ has stability index 1.

Remark 2.37. Note that, for any $\mathrm{Inc}(\mathbb{N})^i$-invariant chain $\mathcal{I} = (I_n)_{n\in\mathbb{N}}$. As subsets of $K[X]$, we have

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots.$$

This chain is obtained by taking the identity element $\mathrm{id} \in \mathrm{Inc}(\mathbb{N})^i$.

Proposition 2.38. *Let $\mathcal{I} = (I_n)_{n\in\mathbb{N}}$ be an $\mathrm{Inc}(\mathbb{N})^i$-invariant chain. For any pair integers $n > m > 0$,*

$$\langle \mathrm{Inc}(\mathbb{N})^i_{m,n}(I_m)\rangle = \langle \mathrm{Inc}(\mathbb{N})^i_{m+1,n} \circ \mathrm{Inc}(\mathbb{N})^i_{m,m+1}(I_m)\rangle \subseteq \langle \mathrm{Inc}(\mathbb{N})^i_{m+1,n}(I_{m+1})\rangle.$$

Proof. By Proposition 2.20, the first equality is clear. For the inclusion, we only need to observe that $\mathrm{Inc}(\mathbb{N})^i_{m,m+1}(I_m) \subseteq I_{m+1}$. □

Lemma 2.39. *Let $\mathcal{I} = (I_n)_{n\in\mathbb{N}}$ be an $\mathrm{Inc}(\mathbb{N})^i$-invariant chain. For any integer $r > 0$, the following are equivalent:*

(a) $\mathcal{I}$ *stabilizes and* $\operatorname{ind}^i(\mathcal{I}) \leq r$*;*

(b) *For* $n \geq m \geq r$*, we have*

$$\langle \operatorname{Inc}(\mathbb{N})^i_{m,n}(I_m) \rangle_{K[X_n]} = I_n;$$

(c) *For any* $n \geq r$*, we have*

$$\bigcup_{j \leq r} \langle \operatorname{Inc}(\mathbb{N})^i_{j,n}(I_j) \rangle_{K[X_n]} = I_n.$$

Proof. The directions (b) ⇒ (a) and (a) ⇒ (c) are trivial by definition of the stability index. Now assume (c), by Proposition 2.38 we have

$$\begin{aligned}
I_n &= \bigcup_{j \leq r} \langle \operatorname{Inc}(\mathbb{N})^i_{j,n}(I_j) \rangle \\
&= \langle \operatorname{Inc}(\mathbb{N})^i_{1,n}(I_1) \rangle \cup \langle \operatorname{Inc}(\mathbb{N})^i_{2,n}(I_2) \rangle \cup \cdots \cup \langle \operatorname{Inc}(\mathbb{N})^i_{r-1,n}(I_{r-1}) \rangle \cup \langle \operatorname{Inc}(\mathbb{N})^i_{r,n}(I_r) \rangle \\
&\subseteq \langle \operatorname{Inc}(\mathbb{N})^i_{2,n}(I_2) \rangle \cup \langle \operatorname{Inc}(\mathbb{N})^i_{3,n}(I_3) \rangle \cup \cdots \cup \langle \operatorname{Inc}(\mathbb{N})^i_{r,n}(I_r) \rangle \\
&\dots \\
&\subseteq \langle \operatorname{Inc}(\mathbb{N})^i_{r,n}(I_r) \rangle \subseteq I_n \\
\Longrightarrow \ I_n &= \langle \operatorname{Inc}(\mathbb{N})^i_{r,n}(I_r) \rangle_{K[X_n]}.
\end{aligned}$$

Thus (c) ⇒ (a) holds. Now, if (a) holds, we have if $r \leq m \leq n$,

$$I_n = \langle \operatorname{Inc}(\mathbb{N})^i_{r,n}(I_r) \rangle \subseteq \langle \operatorname{Inc}(\mathbb{N})^i_{m,n}(I_m) \rangle \subseteq I_n.$$

Hence (a) ⇒ (b). □

The following corollary, which is based on the equivalence between (a) and (b) in the above lemma, implies a useful information on the stability index.

Corollary 2.40. *Let* $\mathcal{I} = (I_n)_{n \in \mathbb{N}}$ *be a stabilizes* $\operatorname{Inc}(\mathbb{N})^i$*-invariant chain. Then*

$$\operatorname{ind}^i(\mathcal{I}) = \inf\{r \mid \langle \operatorname{Inc}(\mathbb{N})^i_{m,n}(I_m) \rangle_{K[X_n]} = I_n \text{ whenever } r \leq m \leq n\}.$$

By Remark 2.37, an $\operatorname{Inc}(\mathbb{N})^i$-invariant chain $\mathcal{I} = (I_n)_{n \in \mathbb{N}}$ induces the following chain of ideals of $K[X]$

$$\langle I_1 \rangle_{K[X]} \subset \langle I_2 \rangle_{K[X]} \subset \dots. \tag{2.2}$$

These ideals are not necessarily $\operatorname{Inc}(\mathbb{N})^i$-invariant. But Remark 2.37 together with the chain (2.2) give

$$\bigcup_{n \geq 1} \langle I_n \rangle_{K[X]} = \bigcup_{n \geq 1} I_n =: I,$$

which implies the following invariant property.

Lemma 2.41. *The ideal I defined above is an $\mathrm{Inc}(\mathbb{N})^i$-invariant ideal.*

Proof. Let $f \in I = \bigcup_{n\geq 1} I_n$. Thus f must belong to I_m for some integer $m \geq 1$. Assume m is smallest. Let $\pi \in \mathrm{Inc}(\mathbb{N})^i$, then π can be viewed as an element of $\mathrm{Inc}(\mathbb{N})^i_{m,\pi(m)}$. Since $\mathcal{I}$ is an $\mathrm{Inc}(\mathbb{N})^i$-invariant chain, we have

$$\pi(f) \in \mathrm{Inc}(\mathbb{N})^i_{m,\pi(m)}(I_m) \subseteq I_{\pi(m)} \subseteq I.$$

Thus I is an $\mathrm{Inc}(\mathbb{N})^i$-invariant ideal. □

Lemma 2.42. *Let $i \geq 0$ be an integer and $\mathcal{I} = (I_n)_{n\in\mathbb{N}}$ be an $\mathrm{Inc}(\mathbb{N})^i$-invariant chain. Then the chain $\mathrm{LT}(\mathcal{I}) = (\mathrm{LT}(I_n))_{n\in\mathbb{N}}$ also is an $\mathrm{Inc}(\mathbb{N})^i$-invariant and the i-stability index of $\mathrm{LT}(\mathcal{I})$ is at least $\mathrm{ind}^i(\mathcal{I})$.*

Proof. Using Proposition 3.20, for $r \leq n$, we have

$$\langle \mathrm{Inc}(\mathbb{N})^i_{r,n}(\mathrm{LT}(I_r))\rangle \subseteq \langle \mathrm{LT}(\mathrm{Inc}(\mathbb{N})^i_{r,n}(I_r))\rangle \subseteq \mathrm{LT}(I_n).$$

Hence $\mathrm{LT}(\mathcal{I})$ is an invariant chain.

Let $r = \mathrm{ind}^i(\mathrm{LT}(\mathcal{I}))$, by definition, we have $\mathrm{LT}(I_n) = \langle \mathrm{Inc}(\mathbb{N})^i_{r,n}(\mathrm{LT}(I_r))\rangle$. To prove the second assertion, we only need to show that for all $r \geq n$, the equality below holds:

$$I_n = \langle \mathrm{Inc}(\mathbb{N})^i_{r,n}(I_r)\rangle.$$

The inclusion ($\supseteq$) is trivial. Conversely, take $f \in I_n \setminus \{0\}$. Then $\mathrm{LT}(f) \in \langle \mathrm{Inc}(\mathbb{N})^i_{r,n}(\mathrm{LT}(I_r))\rangle$, thus

$$\mathrm{LT}(f) = q_1 \cdot \mathrm{LT}(\pi_1(g_1)),$$

for some monomial $q_1 \in K[X_n]$ and for some $\pi_1 \in \mathrm{Inc}(\mathbb{N})^i_{r,n}$, $g_1 \in I_r$. Consider the polynomial $f' = f - q_1 \cdot \pi_1(g_1)$, there exist a monomial $q_2 \in K[X_n]$ and for some $\pi_2 \in \mathrm{Inc}(\mathbb{N})^i_{r,n}$, $g_2 \in I_r$ such that

$$\mathrm{LT}(f') = q_2 \cdot \mathrm{LT}(\pi_2(g_2)) \preceq \mathrm{LT}(f).$$

Continuing this procedure eventually leads to termination after finitely many steps. Thus we may find an $m > 0$ such that

$$f = q_1 \cdot \pi_1(g_1) + \cdots + q_m \cdot \pi_m(g_m),$$

for some monomials $q_k \in K[X_n]$ and for some $\pi_k \in \mathrm{Inc}(\mathbb{N})^i_{r,n}$ and $g_k \in I_r$. This means that $f \in \langle \mathrm{Inc}(\mathbb{N})^i_{r,n}(I_r))\rangle$, which implies the reverse inclusion. □

The above lemma is [4, Lemma 7.1]. The original proof of this lemma uses [4, Remark 5.5], which is a wrong property unless the chain $\mathcal{I}$ is saturated, i.e., $I_n = I \cap K[X_n]$ for all $n > 0$ and for some $\mathrm{Inc}(\mathbb{N})^i$-invariant ideal I of $K[X]$ (see Definition 2.44 below).

The fallacy of [4, Remark 5.5] can be exposed by a counterexample, inspired by [12, Example 6.5], but simpler: let $c = 1$, we use $x_j \equiv x_{1,j}$ for simplification. Consider the chain $\mathcal{I} = (I_n)_{n\in\mathbb{N}}$ where

$$I_n = \begin{cases} \langle x_1^2, \dots, x_n^2 \rangle & \text{for } n \leq 9, \\ \langle x_1, x_2^2, \dots, x_{10}^2 \rangle & \text{for } n = 10, \\ \langle \mathrm{Inc}(\mathbb{N})_{10,n}(I_{10}) \rangle & \text{for } n \geq 11. \end{cases}$$

Now we have $I = \bigcup_{n\geq 1} I_n = \langle x_1, x_2, \dots \rangle = \langle \mathrm{Inc}(\mathbb{N})(x_1) \rangle$. Clearly, the set $\{x_1\}$ is an $\mathrm{Inc}(\mathbb{N})$-equivariant Gröbner basis of I, hence $r = 1$, while the stability index of $\mathcal{I}$ is $\mathrm{ind}(\mathcal{I}) = 10$. Thus, the inequality $r \geq \mathrm{ind}(\mathcal{I})$ in this remark is wrong.

Corollary 2.43. *Each* $\mathrm{Inc}(\mathbb{N})^i$*-invariant chain stabilizes.*

Recall that Mon is the set of monomials in $K[X]$. In the proof, we sometimes write the monomial ordering $\preceq$ in place of $|_{\mathrm{Inc}(\mathbb{N})^i}$ for convenience. We will prove that $(\mathrm{Mon}, \preceq)$ is a well-partially-ordered set by using Higman's lemma, then imply the stability of the $\mathrm{Inc}(\mathbb{N})^i$-invariant chain. We consider the case $i = 0$, namely $\mathrm{Inc}^i(\mathbb{N}) = \mathrm{Inc}(\mathbb{N})$; the case $i > 0$ can be treated similarly.

Proof. Let $\mathrm{Mon}(K[X_n])$ be the set of monomials in $K[X_n]$. For each $n \in \mathbb{N}$, observe that there is a bijection between $\mathrm{Mon}(K[X_n])$ and the set $(\mathbb{N}_0^c)^n$ in which each monomial $x^\alpha \in \mathrm{Mon}(K[X_n])$ get mapped to

$$(\alpha_{1,1}, \alpha_{2,1}, \dots, \alpha_{c,1}, \dots, \alpha_{1,n}, \alpha_{2,n}, \dots, \alpha_{c,n}).$$

Thus it extended to a bijection between

$$M := \bigsqcup_{n\geq 1} \mathrm{Mon}(K[X_n]) \text{ and } (\mathbb{N}_0^c)^* = \bigcup_{n\geq 1} (\mathbb{N}_0^c)^n,$$

where the first union is disjoint union. By Dickson's lemma, $\mathbb{N}_0^c$ is well-partially-ordered by the standard component-wise partial order. Thus $(\mathbb{N}_0^c)^*$ is well-partially-ordered by Higman's lemma, or, equivalently, $(M, \preceq)$ is a well-partially-ordered set.

The partial-order of $(\mathrm{Mon}, \preceq)$ is as follows: for $m \leq n$, $b_m \in \mathrm{Mon}(K[X_m])$ and $b_n \in \mathrm{Mon}(K[X_n])$, we have $b_m \preceq b_n$ if and only if there exists an increasing map

$\pi : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, n\}$ such that $\pi(b_m)|b_n$. Clearly π can be viewed as an element of $\mathrm{Inc}(\mathbb{N})^i_{m,n}$. Here Mon can be considered as a subset of M since there is an injection from Mon to M, we must have (Mon, $\preceq$) is well-partially-ordered.

Let $\mathcal{I} = (I_n)_{n\in\mathbb{N}}$ be an $\mathrm{Inc}(\mathbb{N})^i$-invariant chain. By the above lemma, the chain $\mathrm{LT}(\mathcal{I}) = (\mathrm{LT}(I_n))_{n\in\mathbb{N}}$ is also $\mathrm{Inc}(\mathbb{N})^i$-invariant.

Now if the chain $\mathrm{LT}(\mathcal{I})$ stabilizes, then by Lemma 2.42, the i-index of $\mathcal{I}$ is bounded above by $\mathrm{ind}^i(\mathrm{LT}(\mathcal{I}))$. Thus $\mathcal{I}$ must stabilizes. By this observation, we may assume that $\mathcal{I}$ is a chain of monomial ideals, and we aim to prove that $\mathcal{I}$ is stabilizes.

Proof by contradiction. Assume that the chain $\mathcal{I}$ does not stabilize. Then for any integer m, there exists some $n > m$ such that $\langle \mathrm{Inc}(\mathbb{N})^i_{m,n}(I_m)\rangle \subsetneq I_n$. This allows us to construct an infinite sequence of indices $n_1 < n_2 < n_3 < \ldots$ and an infinite sequence of monomials $u_1, u_2, u_3, \ldots$ such that for all $k \geq 1$:

1. $u_k \in I_{n_k}$.

2. $u_k \notin \langle \mathrm{Inc}(\mathbb{N})^i_{n_{k-1},n_k}(I_{n_{k-1}})\rangle$. This means u_k is not divisible by any monomial $\pi(m)$ for any $m \in I_{n_{k-1}}$ and $\pi \in \mathrm{Inc}(\mathbb{N})^i_{n_{k-1},n_k}$.

Consider the infinite sequence $u_1, u_2, u_3, \ldots$. Since (Mon, $\preceq$) is well-partially-ordered, there must exist indices $j < k$ such that $u_j \preceq u_k$. This means $\exists \pi \in \mathrm{Inc}(\mathbb{N})^i_{n_j,n_k}$ satisfying $\pi(u_j)|u_k$. Now $\pi(u_j) \in \mathrm{Inc}(\mathbb{N})^i_{n_j,n_k}(I_{n_j})$ implies that $u_k \in \langle \mathrm{Inc}(\mathbb{N})^i_{n_j,n_k}(I_{n_j})\rangle$.

Since $j < k$, we have $n_j \leq n_{k-1}$, thus $I_{n_j} \subseteq I_{n_{k-1}}$. By Proposition 2.38

$$\langle \mathrm{Inc}(\mathbb{N})^i_{n_j,n_k}(I_{n_j})\rangle \subseteq \langle \mathrm{Inc}(\mathbb{N})^i_{n_{k-1},n_k}(I_{n_{k-1}})\rangle.$$

Therefore, $u_k \in \langle \mathrm{Inc}(\mathbb{N})^i_{n_{k-1},n_k}(I_{n_{k-1}})\rangle$. This contradicts the condition (2) in the construction of the sequence $\{u_k\}$. Thus the chain $\mathcal{I}$ must stabilize. $\square$

Definition 2.44. (a) Two $\mathrm{Inc}(\mathbb{N})^i$-invariant chains $\mathcal{I} = (I_n)_{n\in\mathbb{N}}$ and $\mathcal{J} = (J_n)_{n\in\mathbb{N}}$ are called *equivalent chains* if $\bigcup_{n\geq 1} I_n = \bigcup_{n\geq 1} J_n$.

(b) For an $\mathrm{Inc}(\mathbb{N})^i$-invariant ideal I of $K[X]$, the *saturated chain* of I is the $\mathrm{Inc}(\mathbb{N})^i$-invariant chain $(I \cap K[X_n])_{n\in\mathbb{N}}$.

Remark 2.45. If $(I_n)_{n\in\mathbb{N}}$ is an $\mathrm{Inc}(\mathbb{N})^i$-invariant chain, then it is a subchain of the saturated chain induced by the ideal $I = \bigcup_{n\geq 1} I_n$.

Given an arbitrary ideal $I_r \subset K[X_r]$, then the set

$$\bigcap_{\substack{I_r \subseteq J \subset K[X] \\ J \in \{\mathrm{Inc}(\mathbb{N})^i\text{-invariant ideals}\}}} J$$

is the smallest $\mathrm{Inc}(\mathbb{N})^i$-invariant ideal that contains I_r.

Lemma 2.46.

$$\bigcap_{\substack{I_r \subseteq J \subset K[X] \\ J \in \{\mathrm{Inc}(\mathbb{N})^i\text{-}\textit{invariant ideals}\}}} J = \langle \mathrm{Inc}(\mathbb{N})^i (I_r) \rangle_{K[X]}.$$

Proof. Let $I_r^* = \langle \mathrm{Inc}(\mathbb{N})^i (I_r) \rangle_{K[X]}$. Assume that there is an $\mathrm{Inc}(\mathbb{N})^i$-invariant ideal Q of $K[X]$ such that

$$I_r \subseteq Q \subseteq I_r^*.$$

Then we should have

$$\langle \mathrm{Inc}(\mathbb{N})^i (I_r) \rangle_{K[X]} \subseteq \langle \mathrm{Inc}(\mathbb{N})^i (Q) \rangle_{K[X]} \subseteq \langle \mathrm{Inc}(\mathbb{N})^i (I_r^*) \rangle_{K[X]}$$

Thus $I_r^* \subseteq Q \subseteq I_r^*$ or, equivalently, $Q = \langle \mathrm{Inc}(\mathbb{N})^i (I_r) \rangle_{K[X]}$. □

The above ideal I_r^* is called the $\mathrm{Inc}(\mathbb{N})^i$-*closure* of I_r. There are many smaller $\mathrm{Inc}(\mathbb{N})^i$-invariant chains, equivalent to the saturated chain of the $\mathrm{Inc}(\mathbb{N})^i$-closure. The construction below provides one such instance.

Lemma 2.47. *Let $i \geq 0$ be an integer and $0 \neq \tilde{I} \in K[X_r]$ be an ideal, consider two chains $\mathcal{I} = (I_n)_{n \in \mathbb{N}}$ and $\mathcal{J} = (J_n)_{n \in \mathbb{N}}$ defined by*

$$I_n = \begin{cases} \langle 0 \rangle & \textit{if } 1 \leq n < r, \\ \tilde{I} & \textit{if } n = r, \\ \langle \mathrm{Inc}(\mathbb{N})^i_{n-1,n} (I_{n-1}) \rangle & \textit{if } n > r, \end{cases}$$

and

$$J_n = \begin{cases} \langle 0 \rangle & \textit{if } 1 \leq n < r, \\ \langle \mathrm{Inc}(\mathbb{N})^i_{r,n} (\tilde{I}) \rangle & \textit{if } n \geq r. \end{cases}$$

Then

(a) *$\mathcal{I}$ and $\mathcal{J}$ are $\mathrm{Inc}(\mathbb{N})^i$-invariant chains,*

(b) *$I_n = J_n$ for all $n \geq 1$,*

(c) $\operatorname{ind}^i(\mathcal{I}) = \operatorname{ind}^i(\mathcal{J}) = r$,

(d) $J = \bigcup_{n\in\mathbb{N}} J_n$ *is the* $\operatorname{Inc}(\mathbb{N})^i$*-closure of* $\tilde{I}$.

Proof. (a) It suffices to prove that $\mathcal{J}$ is an $\operatorname{Inc}(\mathbb{N})^i$-invariant chain. Indeed, since $\operatorname{Inc}(\mathbb{N})^i_{m,n}(J_m) \subset \operatorname{Inc}(\mathbb{N})^i_{m,n}(\langle \operatorname{Inc}(\mathbb{N})^i_{r,m}(\tilde{I})\rangle) \subseteq J_n$ whenever $n \geq m \geq r$, the chain $(J_n)_n$ is $\operatorname{Inc}(\mathbb{N})^i$-invariant.

(b) We have $I_n = J_n = 0$ for $1 \leq n < r$; and for $n = r$, $I_n = J_n = \tilde{I}$. For $n > r$, applying Proposition 2.20 repeatedly, we get

$$J_n = \langle \operatorname{Inc}(\mathbb{N})^i_{r,n}(\tilde{I})\rangle = \langle \operatorname{Inc}(\mathbb{N})^i_{n-1,n} \circ \operatorname{Inc}(\mathbb{N})^i_{n-2,n-1} \circ \cdots \circ \operatorname{Inc}(\mathbb{N})^i_{r,r+1}(\tilde{I})\rangle \subseteq I_n.$$

Conversely, using induction on $n \geq r$. For $n = r$,

$$I_n = \tilde{I} = \langle \operatorname{Inc}(\mathbb{N})^i_{n,n}(\tilde{I})\rangle = J_n.$$

For $n > r$,

$$I_n = \langle \operatorname{Inc}(\mathbb{N})^i_{n-1,n}(I_{n-1})\rangle = \langle \operatorname{Inc}(\mathbb{N})^i_{n-1,n}(J_{n-1})\rangle \subseteq J_n.$$

Hence $J_n = I_n$ for all $n \geq r$.

(c) By definition of I_n, $\operatorname{ind}^i(\mathcal{I}) = r$, then by part (b), $\operatorname{ind}^i(\mathcal{I}) = \operatorname{ind}^i(\mathcal{J})$.

(d) Note that $\operatorname{ind}^i(\mathcal{J}) = r$, then we have

$$\begin{aligned} J = \bigcup_{n\in\mathbb{N}} J_n = \bigcup_{n\geq r} J_n &= \bigcup_{n\geq r} \langle \operatorname{Inc}(\mathbb{N})^i_{r,n}(\tilde{I})\rangle \\ &= \bigcup_{\pi\in\operatorname{Inc}(\mathbb{N})^i} \langle \operatorname{Inc}(\mathbb{N})^i_{r,\pi(r)}(\tilde{I})\rangle \\ &= \langle \operatorname{Inc}(\mathbb{N})^i(\tilde{I})\rangle_{K[X]} \\ &= \tilde{I}^*. \end{aligned}$$

Thus $J = \bigcup_{n\in\mathbb{N}} J_n$ is the $\operatorname{Inc}(\mathbb{N})^i$-closure of $\tilde{I}$. $\square$

Corollary 2.48. *With the notation of the above lemma, for all* $n \geq r$*, we have*

$$J \cap K[X_n] = I_n = J_n.$$

Chapter 3
Hilbert-Serre Theorem for Infinite Dimensional Polynomial Rings

In this chapter, we begin by introducing the q-invariant, a measure of complexity designed to prove the rationality of the equivariant Hilbert series for $\mathrm{Inc}(\mathbb{N})^i$-invariant chains of ideals. We then formally states and proves Theorem 3.6, demonstrating that under certain conditions, the equivariant Hilbert series is indeed a rational function of a specific form. Key techniques involve the use of certain chains involving colon and sum and the analysis of ideals of leading terms. Finally, Chapter 3 concludes with a detailed example to illustrate the computation of the equivariant Hilbert series for a concrete $\mathrm{Inc}(\mathbb{N})$-invariant chain.

3.1 The q-Invariant

The purpose of introducing the q-invariant is to prove the rationality of the bigraded Hilbert series. The induction using the q-invariant as a measure of complexity, is designed to show that the Hilbert series has the desired rational form when the q-invariant is finite. First, we define the equivariant Hilbert series.

Definition 3.1. The *equivariant Hilbert series* of a chain $\mathcal{I} = (I_n)_{n\in\mathbb{N}}$ where $I_0 = \langle 0 \rangle$, $I_n \subset K[X_n]$ is homogeneous for each n, is defined as

$$H_{\mathcal{I}}(s,t) = \sum_{n\geq 0, j\geq 0} \dim_K (K[X_n]/I_n)_j \cdot s^n t^j.$$

Since $I_0 = 0$ and $X_0 = \emptyset$, we have $K[X_0]/I_0 \cong K$. We examine two simple examples.

Example 3.2. (a) Consider the zero chain $\mathcal{I} = (I_n)_{n\in\mathbb{N}}$, where $I_n = 0$ for all n. By

Example 1.48, we have

$$\begin{aligned}
H_{\mathcal{I}}(s,t) &= \sum_{n\geq 0}\left(\sum_{j\geq 0}\dim_K(K[X_n])_j\cdot t^j\right)s^n\\
&= \sum_{n\geq 0} H_{K[X_n]}(t)\cdot s^n\\
&= \sum_{n\geq 0}\frac{1}{(1-t)^{cn}}\cdot s^n\\
&= \sum_{n\geq 0}\left(\frac{s}{(1-t)^c}\right)^n\\
&= \frac{(1-t)^c}{(1-t)^c-s}.
\end{aligned}$$

(b) Consider the chain $\mathcal{I}=(I_n)_{n\in\mathbb{N}}$, with $I_n=\langle X_n\rangle$. Since $K[X_n]/I_n\cong K$, we have

$$\dim_K(K[X_n]/I_n)_j=\dim_K(K)_j=\begin{cases}1 & \text{if } j=0,\\ 0 & \text{if } j>0.\end{cases}$$

Now its equivariant Hilbert series is

$$\begin{aligned}
H_{\mathcal{I}}(s,t) &= \sum_{n\geq 0}\left(\sum_{j\geq 0}\dim_K(K[X_n]/I_n)_j\cdot t^j\right)s^n\\
&= \sum_{n\geq 0}\left(\sum_{j\geq 0}\dim_K(K)_j\cdot t^j\right)s^n\\
&= \sum_{n\geq 0}s^n\\
&= \frac{1}{1-s}.
\end{aligned}$$

Let $J\subset K[X_n]$ be a monomial ideal, we know that J is generated by monomials. We now define the minimal system of monomial generators of J.

Definition 3.3. Let $J\subset K[X_n]$ be a monomial ideal. The *minimal system of monomial generators* of J is the set of monomials $G(J)=\{g_1,g_2,\ldots\}$ such that $J=\langle G(J)\rangle$; and

$$g_i\notin\langle G(J)\setminus\{g_i\}\rangle \text{ for all } g_i\in G(J).$$

Furthermore, we denote by $e^+(J)$ the *maximum degree of a minimal homogeneous generator* of a homogeneous ideal $J\in K[X_n]$.

Definition 3.4. The *q-invariant* of an $\mathrm{Inc}(\mathbb{N})^i$-invariant chain $\mathcal{I} = (I_n)_{n\in\mathbb{N}}$, denoted as $q(\mathcal{I})$, is defined by

$$q(\mathcal{I}) = \sum_{j=0}^{e^+(I_r)} \dim_K [K[X_r]/I_r]_j,$$

where $r = \mathrm{ind}^i(\mathcal{I})$.

Example 3.5. Let $c = 2$ be the row index of variables, thus $X_1 = \{x_{1,1}, x_{2,1}\}$. Define the chain $\mathcal{I} = (I_n)_{n\in\mathbb{N}}$, where

$$I_n = \begin{cases} \langle x_{1,1}^2, x_{2,1}\rangle & \text{if } n = 1, \\ \langle \mathrm{Inc}(\mathbb{N})_{1,n}(I_1)\rangle & \text{if } n > 1. \end{cases}$$

This creates an $\mathrm{Inc}(\mathbb{N})$-invariant chain with stability index $r = \mathrm{ind}^0(\mathcal{I}) = 1$. Since $x_{1,1}^2$ and $x_{2,1}$ are minimal monomial generators of I_1, it follows that $e^+(I_1) = \max\{2, 1\} = 2$. We compute dimensions $\dim_K(K[X_1]/I_1)_j$ for j up to $e^+(I_1) = 2$:

- Degree $j = 0$: $(K[X_1]/I_1)_0 \cong K$ because in degree 0, $K[X_1]_0 = K$ and $(I_1)_0 = 0$. So,

$$\dim_K(K[X_1]/I_1)_0 = 1.$$

- Degree $j = 1$: A basis for $(K[X_1])_1$ is $\{x_{1,1}, x_{2,1}\}$. Simplifying, we imply that a basis for $(K[X_1]/I_1)_1$ is given by the class of $\{x_{1,1}\}$. So,

$$\dim_K(K[X_1]/I_1)_1 = 1.$$

- Degree $j = 2$: A basis for $(K[X_1])_2$ is $\{x_{1,1}^2, x_{1,1}x_{2,1}, x_{2,1}^2\}$. In $K[X_1]/I_1$, we have $x_{1,1}^2 \equiv 0$ and $x_{2,1} \equiv 0$. Consequently, $x_{1,1}x_{2,1} \equiv 0$ and $x_{2,1}^2 \equiv 0$ as well. Thus, $(K[X_1]/I_1)_2 = 0$, and

$$\dim_K(K[X_1]/I_1)_2 = 0.$$

Hence, $q(\mathcal{I}) = 1 + 1 + 0 = 2$.

3.2 The Hilbert-Serre Theorem for $\mathrm{Inc}(\mathbb{N})^i$-Invariant Chains of Ideals

The following theorem is the main result of this thesis.

Theorem 3.6. [4, Theorem 7.2] *Assume $\mathcal{I} = (I_n)_{n\in\mathbb{N}}$ is an $\operatorname{Inc}(\mathbb{N})^i$-invariant chain of homogeneous ideals, where $i \geq 0$ is an integer. Then*

$$H_{\mathcal{I}}(s,t) = \frac{g(s,t)}{(1-t)^a \cdot \prod_{j=1}^{b}[(1-t)^{c_j} - s\cdot f_j(t)]},$$

where $a, b, c_j \geq 0$ are integers, $g(s,t) \in \mathbb{Z}[s,t]$, and each $f_j(t) \in \mathbb{Z}[t]$ such that $f_j(1) > 0$.

We will first prove Theorem 3.6 for invariant chains of monomial ideals by induction. The q-invariant is useful as an invariant to make sure that the induction process will terminate. Then we use the fact Hilbert series is invariant under Gröbner deformation, i.e., the Hilbert series of an arbitrary ideal and its initial ideal with respect to some monomial ordering are the same (Lemma 1.49). This will allow us to deduce the case of arbitrary invariant chains from the case of monomial invariant chains.

Lemma 3.7. *Let A be an $\mathbb{N}$-graded ring with $A_0 = K$, then*

$$H_{A[x_1,\dots,x_n]}(t) = \frac{1}{(1-t)^n}\cdot H_A(t)$$

Proof. Let $X = \{x_1,\dots,x_n\}$ and $R = A[x_1,\dots,x_n]$. We use the fact that

$$\frac{R/\langle x_1,\dots,x_d\rangle}{\langle x_1,\dots,x_{d+1}\rangle/\langle x_1,\dots,x_d\rangle} \cong \frac{R}{\langle x_1,\dots,x_{d+1}\rangle},$$

combining with Proposition 1.41, part (b), we get

$$H_R(t) = \frac{1}{1-t}\cdot H_{R/x_1R}(t) = \frac{1}{(1-t)^2}\cdot H_{R/\langle x_1,x_2\rangle R}(t)\cdots = \frac{1}{(1-t)^n}\cdot H_{R/\langle X\rangle R}(t).$$

This completes the proof since $H_{R/\langle X\rangle R}(t) = H_A(t)$. □

For the proof of Theorem 3.6, we firstly encounter a simpler case.

Lemma 3.8. *Let $i \geq 0$ be an integer and $\mathcal{I} = (I_n)_{n\in\mathbb{N}}$ be a non-trivial $\operatorname{Inc}(\mathbb{N})^i$-invariant chain with $r = \operatorname{ind}^i(\mathcal{I}) \leq i$. Then*

$$H_{\mathcal{I}}(s,t) = \frac{g(s,t)}{(1-t)^a\cdot[(1-t)^c - s]},$$

where $a = \max\{\dim K[X_n]/I_n \mid 1 \leq n < r\}$, $g(s,t) \in \mathbb{Z}[s,t]$, and $\deg(g(s,1)) \leq r$. Moreover, if $I_r = K[X_r]$, then $g(s,t)$ is a multiple of $(1-t)^c - s$.

Proof. Since $r = \operatorname{ind}^i(\mathcal{I}) \leq i$, $\operatorname{Inc}(\mathbb{N})^i_{r,n}(I_r) = I_r K[X_n]$. For $n \geq r$, we need a useful transformation:

$$\begin{aligned} K[X_n]/I_n &\cong K[X_n]/\langle \operatorname{Inc}(\mathbb{N})^i_{r,n}(I_r)\rangle_{K[X_n]} \\ &\cong K[X_n]/\langle I_r\rangle_{K[X_n]} \\ &\cong (K[X_r]/I_r)[X_n \setminus X_r]. \end{aligned}$$

Letting $A_n = K[X_n]/I_n$, by Lemma 3.7, we get

$$H_{A_n}(t) = \frac{1}{(1-t)^{c(n-r)}} \cdot H_{A_r}(t).$$

Hence, we have

$$\begin{aligned} H_{\mathcal{I}}(s,t) &= \sum_{n=0}^{r-1} H_{A_n}(t)\cdot s^n + \sum_{n\geq r} H_{A_n}(t)\cdot s^n \\ &= \sum_{n=0}^{r-1} H_{A_n}(t)\cdot s^n + \sum_{n\geq r} H_{A_r}(t)\cdot \frac{s^n}{(1-t)^{c(n-r)}} \\ &= \sum_{n=0}^{r-1} H_{A_n}(t)\cdot s^n + (1-t)^{cr}\cdot H_{A_r}(t)\cdot \sum_{n\geq r}\left(\frac{s}{(1-t)^c}\right)^n \\ &= \sum_{n=0}^{r-1} H_{A_n}(t)\cdot s^n + (1-t)^{cr}\cdot H_{A_r}(t)\cdot \left(\frac{s}{(1-t)^c}\right)^r \cdot \frac{(1-t)^c}{(1-t)^c - s} \\ &= \sum_{n=0}^{r-1} \frac{g_n(t)}{(1-t)^{d_n}}\cdot s^n + \frac{g_r(t)\cdot s^r}{(1-t)^{d_r-c}\cdot[(1-t)^c-s]}, \end{aligned} \tag{3.1}$$

where $H_{A_n}(t) = \dfrac{g_n(t)}{(1-t)^{d_n}}$ with $d_n = \dim A_n$. Letting the common denominator be $(1-t)^a\cdot[(1-t)^c - s]$ where $a = \max\{d_0,\dots,d_{r-1}\}$, we get

$$\begin{aligned} H_{\mathcal{I}}(s,t) &= \frac{(1-t)^{a+c-d_0}g_0(t) + \sum_{j=1}^r (1-t)^{a-d_{j-1}}\left[(1-t)^{c+d_{j-1}-d_j}g_j(t) - g_{j-1}(t)\right]s^j}{(1-t)^a\cdot[(1-t)^c-s]} \\ &= \frac{1}{(1-t)^a\cdot[(1-t)^c-s]}\sum_{j=0}^r p_j(t)s^j \\ &= \frac{g(s,t)}{(1-t)^a\cdot[(1-t)^c-s]} \end{aligned}$$

where $p_0(t) = (1-t)^{a+c-d_0}g_0(t)$, and $p_j(t) = (1-t)^{a-d_{j-1}}\left[(1-t)^{c+d_{j-1}-d_j}g_j(t) - g_{j-1}(t)\right]$, for $j \in [r]$.

Note that $c + d_{j-1} \geq d_j$ since the inclusion $I_{j-1}K[X_j] \subseteq I_j$ induces a surjection

$$A_{j-1}[X_j \setminus X_{j-1}] = \frac{K[X_j]}{I_{j-1}K[X_j]} \twoheadrightarrow \frac{K[X_j]}{I_j} = A_j,$$

hence $d_j = \dim A_j \le \dim A_{j-1}[X_j \setminus X_{j-1}] = d_{j-1} + c$.

Clearly, the degree in s of $g(s,t)$ is at most r. Finally, if $I_r = K[X_r]$ then $K[X_r]/I_r$ is the zero ring, thus $g_r(t) = 0$. Now look at (3.1), since then we choose $(1-t)^a[(1-t)^c - s]$ as the denominator, $g(s,t)$ must be a multiple of $(1-t)^c - s$. $\square$

Consider a ring R, an ideal $I \subset R$ and a subset $S \subseteq R$. We recall the colon ideal $(I : S)$ is an ideal of R such that $(I : S) = \{r \in R \mid rS \subseteq I\}$.

Lemma 3.9. *Let $\mathcal{I} = (I_n)_{n\in\mathbb{N}}$ be an $\operatorname{Inc}(\mathbb{N})^i$-invariant chain of monomial ideals such that $i < \operatorname{ind}^i(\mathcal{I})$. For any variable $x_{k,i} \in X_i \setminus X_{i-1}$ and any integer $e > 0$, consider two chains $(\mathcal{I}, x_{k,i})$ and $(\mathcal{I} : x_{k,i}^e)$ whose n-th ideals are $\langle I_n, x_{k,i}\rangle$ and $\langle I_n : x_{k,i}^e\rangle$, respectively, if $n \ge i$ and zero if $n < i$. Then*

(a) *$(\mathcal{I}, x_{k,i})$ is $\operatorname{Inc}(\mathbb{N})^i$-invariant, and $\operatorname{ind}^i(\mathcal{I}, x_{k,i}) \le \operatorname{ind}^i(\mathcal{I})$,*

(b) *$(\mathcal{I} : x_{k,i}^e)$ is $\operatorname{Inc}(\mathbb{N})^i$-invariant, and $\operatorname{ind}^i(\mathcal{I} : x_{k,i}^e) \le \operatorname{ind}^i(\mathcal{I})$.*

Proof. (a) Let $J_n = \langle I_n, x_{k,i}\rangle$. We first show that $(\mathcal{I}, x_{k,i}) = (J_n)_{n\in\mathbb{N}}$ is $\operatorname{Inc}(\mathbb{N})^i$-invariant. Consider $n \ge m \ge i$ and $\pi \in \operatorname{Inc}(\mathbb{N})^i_{m,n}$. Let $f \in J_m = \langle I_m, x_{k,i}\rangle$. Then $f = g + hx_{k,i}$ where $g \in I_m$ and $h \in K[X_m]$. Applying π, we get

$$\pi(f) = \pi(g + hx_{k,i}) = \pi(g) + \pi(h)\pi(x_{k,i}) = \pi(g) + \pi(h)x_{k,i}.$$

Since $\mathcal{I}$ is $\operatorname{Inc}(\mathbb{N})^i$-invariant, $\pi(g) \in I_n$. Also, $\pi(h) \in K[X_n]$. Thus, $\pi(f) = \pi(g) + \pi(h)x_{k,i} \in (I_n, x_{k,i}) = J_n$. Hence, $(\mathcal{I}, x_{k,i})$ is $\operatorname{Inc}(\mathbb{N})^i$-invariant.

Now we show the stability index inequality. Let $r = \operatorname{ind}^i(\mathcal{I})$. We need to show that for $n \ge r$, $\langle \operatorname{Inc}(\mathbb{N})^i_{r,n}(I_r, x_{k,i})\rangle_{K[X_n]} = \langle I_n, x_{k,i}\rangle_{K[X_n]}$. Note that $I_n = \langle \operatorname{Inc}(\mathbb{N})^i_{r,n}(I_r)\rangle$, we have

$$\begin{aligned}\langle I_n, x_{k,i}\rangle &= \langle \operatorname{Inc}(\mathbb{N})^i_{r,n}(I_r), x_{k,i}\rangle \\ &= \langle \operatorname{Inc}(\mathbb{N})^i_{r,n}(I_r), \operatorname{Inc}(\mathbb{N})^i_{r,n}(x_{k,i})\rangle \\ &= \langle \operatorname{Inc}(\mathbb{N})^i_{r,n}(I_r, x_{k,i})\rangle.\end{aligned}$$

Hence $\operatorname{ind}^i(\mathcal{I}, x_{k,i}) \le r = \operatorname{ind}^i(\mathcal{I})$.

(b) Let $J_n = \langle I_n : x_{k,i}^e\rangle$. We first show that $(\mathcal{I} : x_{k,i}^e) = (J_n)_{n\in\mathbb{N}}$ is $\operatorname{Inc}(\mathbb{N})^i$-invariant. Consider $n \ge m \ge i$ and $\pi \in \operatorname{Inc}(\mathbb{N})^i_{m,n}$. Let $f \in J_m = \langle I_m : x_{k,i}^e\rangle$. Then $fx_{k,i}^e \in I_m$. We want to show $\pi(f) \in J_n$, i.e., $\pi(f)x_{k,i}^e \in I_n$. Consider $\pi(f)x_{k,i}^e = \pi(f)\pi(x_{k,i}^e) =$

$\pi(fx_{k,i}^e)$. Since $fx_{k,i}^e \in I_m$ and $\mathcal{I}$ is $\mathrm{Inc}(\mathbb{N})^i$-invariant, $\pi(fx_{k,i}^e) \in I_n$. Thus $\pi(f)x_{k,i}^e \in I_n$, so $\pi(f) \in \langle I_n : x_{k,i}^e \rangle = J_n$. Hence, $(\mathcal{I} : x_{k,i}^e)$ is $\mathrm{Inc}(\mathbb{N})^i$-invariant.

Now we show the stability index inequality. Let $r = \mathrm{ind}^i(\mathcal{I})$. We need to prove that for $n \geq r$, $\langle \mathrm{Inc}(\mathbb{N})_{r,n}^i (I_r : x_{k,i}^e)\rangle_{K[X_n]} = \langle I_n : x_{k,i}^e \rangle_{K[X_n]}$.

($\subseteq$): Let $f \in \langle \mathrm{Inc}(\mathbb{N})_{r,n}^i (I_r : x_{k,i}^e)\rangle$. Then $f = \sum_j \pi_j(g_j)m_j$ where $g_j \in \langle I_r : x_{k,i}^e \rangle$, $\pi_j \in \mathrm{Inc}(\mathbb{N})_{r,n}^i$, $m_j \in K[X_n]$. We need to show $f \in \langle I_n \colon x_{k,i}^e \rangle$, i.e., $fx_{k,i}^e \in I_n$. We have

$$
\begin{aligned}
fx_{k,i}^e &= \left(\sum_j \pi_j(g_j)m_j\right) x_{k,i}^e \\
&= \sum_j \pi_j(g_j)m_j x_{k,i}^e \\
&= \sum_j \pi_j(g_j)\pi_j(x_{k,i}^e)m_j \\
&= \sum_j \pi_j(g_j x_{k,i}^e)m_j.
\end{aligned}
$$

Since $g_j \in (I_r : x_{k,i}^e)$, $g_j x_{k,i}^e \in I_r$, hence $\pi_j(g_j x_{k,i}^e) \in I_n$. Thus $fx_{k,i}^e \in I_n$.

($\supseteq$): Let $u \in \langle I_n : x_{k,i}^e \rangle$ be a monomial. Then $ux_{k,i}^e \in I_n = \langle \mathrm{Inc}(\mathbb{N})_{r,n}^i (I_r)\rangle$. Then $ux_{k,i}^e = \pi(v)m$ for some monomial $v \in I_r$, $\pi \in \mathrm{Inc}(\mathbb{N})_{r,n}^i$, and monomial $m \in K[X_n]$. Write $v = wx_{k,i}^a$ where $x_{k,i} \nmid w$. Then

$$
ux_{k,i}^e = \pi(wx_{k,i}^a)m = \pi(w)x_{k,i}^a m.
$$

Since π fixes $[i]$ and $x_{k,i} \nmid w$, $\pi(w)$ is not divisible by $x_{k,i}$. Thus $x_{k,i}^e | (x_{k,i}^a m)$, which implies $\pi(w)|u$. If $a \leq e$, then $wx_{k,i}^e \in \langle v \rangle \subseteq I_r$, so $w \in I_r \colon x_{k,i}^e$, therefore $u \in \langle \pi(w) \rangle \subseteq \langle \mathrm{Inc}(\mathbb{N})_{r,n}^i (I_r : x_{k,i}^e)\rangle$.

If $a > e$, then $u = \pi(w)x_{k,i}^{a-e}m = \pi(wx_{k,i}^{a-e})m$. Again, $wx_{k,i}^{a-e} \cdot x_{k,i}^e = v \in I_r$, so $wx_{k,i}^{a-e} \in I_r \colon x_{k,i}^e$, and u is a multiple of $\pi(wx_{k,i}^{a-e}) \in \langle \mathrm{Inc}(\mathbb{N})_{r,n}^i (I_r : x_{k,i}^e)\rangle$.

Thus, the equality holds. Therefore, $\mathrm{ind}^i(\mathcal{I} : x_{k,i}^e) \leq r = \mathrm{ind}^i(\mathcal{I})$. □

The following is a direct consequence.

Corollary 3.10. *Keep the assumptions as in Lemma 3.9. Let $e_1, \ldots, e_c \geq 0$ be integers, consider a sequence*

$$
\mathcal{J} = \langle \mathcal{I} : x_{1,i}^{e_1} \cdots x_{c,i}^{e_c}, x_{1,i}, \ldots, x_{c,i} \rangle
$$

with the n-th ideal

$$J_n = \begin{cases} \langle 0 \rangle & \text{if } n < i, \\ \langle I_n : x_{1,i}^{e_1} \cdots x_{c,i}^{e_c}, x_{1,i}, \ldots, x_{c,i} \rangle & \text{if } n \geq i. \end{cases}$$

Then $\mathcal{J}$ is an $\operatorname{Inc}^i(\mathbb{N})$-invariant chain, and $\operatorname{ind}^i(\mathcal{J}) \leq \operatorname{ind}^i(\mathcal{I})$.

Lemma 3.11. *Assume $\mathcal{I} = (I_n)_{n\in\mathbb{N}}$ is an $\operatorname{Inc}(\mathbb{N})^i$-invariant chain of monomial ideals, where $i \geq 0$ is an integer. Then for $n > m \geq 1$, we have the following inclusions of ideals of $K[X_n]$:*

$$\langle \operatorname{Inc}(\mathbb{N})^i_{m,n}(I_m) \rangle \subset \langle \operatorname{Inc}(\mathbb{N})^{i+1}_{m+1,n}(I_{m+1}) \rangle \subset \langle \operatorname{Inc}(\mathbb{N})^i_{m+1,n}(I_{m+1}) \rangle.$$

Proof. By Proposition 2.20, the first inclusion is true. The second inclusion is also clear since $\operatorname{Inc}(\mathbb{N})^{i+1} \subset \operatorname{Inc}(\mathbb{N})^i$. □

Below are two consequences about the stability index of invariant chains.

Corollary 3.12. *If $\mathcal{I} = (I_n)_{n\in\mathbb{N}}$ is an $\operatorname{Inc}(\mathbb{N})^i$-invariant chain of monomial ideals, then $\mathcal{I}$ is also $\operatorname{Inc}(\mathbb{N})^{i+1}$-invariant, and the $(i+1)$-index of $\mathcal{I}$ is at most $1 + \operatorname{ind}^i(\mathcal{I})$.*

Proof. Since $\operatorname{Inc}(\mathbb{N})^{i+1}_{m,n} \subseteq \operatorname{Inc}(\mathbb{N})^i_{m,n}$ for every $m < n$, then the first claim is true.

Assume $n > m \geq \operatorname{ind}^i(\mathcal{I})$, the above lemma gives

$$I_n = \langle \operatorname{Inc}(\mathbb{N})^i_{m,n}(I_m) \rangle \subseteq \langle \operatorname{Inc}(\mathbb{N})^{i+1}_{m+1,n}(I_{m+1}) \rangle \subseteq \langle \operatorname{Inc}(\mathbb{N})^i_{m+1,n}(I_{m+1}) \rangle \subseteq I_n.$$

This implies the equality $I_n = \operatorname{Inc}(\mathbb{N})^{i+1}_{m+1,n}(I_{m+1})$, which gives

$$\operatorname{ind}^{i+1}(\mathcal{I}) \leq m + 1,$$

hence $\operatorname{ind}^{i+1}(\mathcal{I}) \leq \operatorname{ind}^i(\mathcal{I}) + 1$. □

Corollary 3.13. *Assume $\mathcal{I} = (I_n)_{n\in\mathbb{N}}$ is an $\operatorname{Inc}(\mathbb{N})^i$-invariant chain of monomial ideals, where $i \geq 0$ is an integer. Let $x_{k,i}$ be a variable such that $x_{k,i} \in X_i \setminus X_{i-1}$. Then the chain $(\mathcal{I}, x_{k,i})$ whose n-th ideal is $(I_n, x_{k,i})$ is an $\operatorname{Inc}(\mathbb{N})^{i+1}$-invariant chain and*

$$\operatorname{ind}^{i+1}(\mathcal{I}, x_{k,i}) \leq 1 + \operatorname{ind}^i(\mathcal{I}).$$

Proof. Combining Lemma 3.9 and Corollary 3.12, the inequality holds. □

Recall that, a linear form $\ell \in K[X]$ is a finite sum $\ell = \sum_{i,j} a_{ij} x_{ij}$ where $a_{ij} \in K$.

Lemma 3.14. *Let $I \subseteq R = K[X_n]$ be a homogeneous ideal, and let $\ell \in K[X_n]$ be a linear form satisfying $I : \ell^d = I : \ell^{d+1}$ for some integer $d > 0$. Then*

$$H_{R/I}(t) = \sum_{e=0}^{d-1} H_{R/\langle I:\ell^e,\ell\rangle}(t) \cdot t^e + H_{R/\langle I:\ell^d,\ell\rangle}(t) \cdot \frac{t^d}{1-t}.$$

Proof. Consider $d+1$ sequences as follows, where $\cdot\ell$ is the multiplication by ℓ:

$$\begin{array}{lcccccr}
0 \to & [R/I : \ell](-1) & \xrightarrow{\cdot\ell} & R/I & \xrightarrow{g} & R/\langle I, \ell\rangle & \to 0 \\
0 \to & [R/I : \ell^2](-1) & \xrightarrow{\cdot\ell} & R/I : \ell & \to & R/\langle I : \ell, \ell\rangle & \to 0 \\
\vdots & & & \vdots & & & \vdots \\
0 \to & [R/I : \ell^d](-1) & \xrightarrow{\cdot\ell} & R/I : \ell^{d-1} & \to & R/\langle I : \ell^{d-1}, \ell\rangle & \to 0 \\
0 \to & [R/I : \ell^{d+1}](-1) & \xrightarrow{\cdot\ell} & R/I : \ell^d & \to & R/\langle I : \ell^d, \ell\rangle & \to 0
\end{array}$$

For the first sequence, we have

$$\mathrm{Im}(\cdot\ell) = \{x\ell + I \mid x \in R\} = \ell R + I,$$

while

$$\mathrm{Ker}(g) = \{x + I \mid x \in R,\, x \in \langle I, \ell\rangle\} = \ell R + I.$$

Moreover, $\cdot\ell$ is injective and g is surjective. Thus this is an exact sequence. Using the same method, all the above sequences are exact. Now by exactness of the first d sequences, we obtain

$$\begin{aligned}
H_{R/I}(t) &= t \cdot H_{R/I:\ell}(t) + H_{R/\langle I,\ell\rangle}(t) \\
H_{R/I:\ell}(t) &= t \cdot H_{R/I:\ell^2}(t) + H_{R/\langle I:\ell,\ell\rangle}(t) \\
&\cdots \\
H_{R/I:\ell^{d-1}}(t) &= t \cdot H_{R/I:\ell^d}(t) + H_{R/\langle I:\ell^{d-1},\ell\rangle}(t).
\end{aligned}$$

Substituting inductively into the first equation, we deduce that

$$H_{R/I}(t) = \sum_{e=0}^{d-1} H_{R/\langle I:\ell^e,\ell\rangle}(t) \cdot t^e + H_{R/I:\ell^d}(t) \cdot t^d.$$

The last exact sequence combining with the assumption $I : \ell^d = I : \ell^{d+1}$ gives

$$\begin{aligned}
H_{R/I:\ell^d}(t) &= t \cdot H_{R/I:\ell^{d+1}}(t) + H_{R/\langle I:\ell^d,\ell\rangle}(t) \\
&= t \cdot H_{R/I:\ell^d}(t) + H_{R/\langle I:\ell^d,\ell\rangle}(t) \\
\Longrightarrow\ H_{R/\langle I:\ell^d,\ell\rangle}(t) &= (1-t) \cdot H_{R/I:\ell^d}(t).
\end{aligned}$$

Replacing this formula to the earlier equation, we imply our claim. ∎

Corollary 3.15. *Let $i \geq 0$ be an integer and $\mathcal{I} = (I_n)_{n\in\mathbb{N}}$ be an $\operatorname{Inc}(\mathbb{N})^{i+1}$-invariant chain of homogeneous ideals. Assume that there are two integers $r \geq i+1$, $d > 0$ and a linear form $\ell \in K[X_{i+1}]$ satisfying*

$$I_n : \ell^d = I_n : \ell^{d+1} \text{ for every } n \geq r.$$

For each $e \in \{0, \ldots, d\}$, we define a chain $(\mathcal{I} : \ell^e, \ell) = (J_{n,e})_{n\in\mathbb{N}}$ by

$$J_{n,e} = \begin{cases} \langle 0 \rangle & \text{if } n < r, \\ \langle I_n : \ell^e, \ell \rangle & \text{if } n \geq r. \end{cases}$$

Then each $(J_{n,e})$ is an $\operatorname{Inc}(\mathbb{N})^{i+1}$-invariant chain, and there is $g(s,t) \in \mathbb{Z}[s,t]$, $g(s,1) = -s^{r-1}$ such that

$$H_{\mathcal{I}}(s,t) = \sum_{e=0}^{d-1} H_{(\mathcal{I}:\ell^e,\ell)}(s,t) \cdot t^e + H_{(\mathcal{I}:\ell^d,\ell)}(s,t) \cdot \frac{t^d}{1-t} + \frac{g(s,t)}{(1-t)^{(r-1)c+1}}.$$

Proof. Since $\ell \in K[X_{i+1}]$, for each $\pi \in \operatorname{Inc}(\mathbb{N})^{i+1}$ we have $\pi(\ell) = \ell$. Thus $\ell\pi(f) = \pi(\ell f)$ for any polynomial $f \in K[X]$. Any element $\gamma \in \langle I_n : \ell^e, \ell \rangle$ has the form

$$\gamma = a_1 y + a_2 \ell,$$

where $a_1, a_2 \in K[X_n]$, $y \in \langle I_n : \ell^e \rangle$. Applying $\pi \in \operatorname{Inc}^{i+1}_{n,m}(\mathbb{N})$ on both sides gives

$$\pi(\gamma) = \pi(a_1 y) + \ell\pi(a_2) \in K[X_m].$$

Since $y\ell^e \in I_n$, $\pi(y)\ell^e = \pi(y\ell^e) \in I_{n+1}$. Therefore $(J_{n,e})$ is an $\operatorname{Inc}(\mathbb{N})^{i+1}$-invariant chain.

Applying Lemma 3.14 to each ideal I_n with $n \geq r$. We have

$$\begin{aligned} H_{\mathcal{I}}(s,t) = {} & \sum_{n=0}^{r-1} H_{K[X_n]/I_n}(t) \cdot s^n + \sum_{n\geq r} H_{K[X_n]/I_n}(t) \cdot s^n \\ = {} & \sum_{n\geq r} \left(\sum_{e=0}^{d-1} H_{K[X_n]/\langle I_n:\ell^e,\ell\rangle}(t) \cdot t^e + H_{K[X_n]/\langle I_n:\ell^d,\ell\rangle}(t) \cdot \frac{t^d}{1-t} \right) \cdot s^n + \sum_{n=0}^{r-1} H_{K[X_n]/I_n}(t) \cdot s^n \\ = {} & \sum_{e=0}^{d-1} H_{(\mathcal{I}:\ell^e,\ell)}(s,t) \cdot t^e + H_{(\mathcal{I}:\ell^d,\ell)}(s,t) \cdot \frac{t^d}{1-t} \\ & - \sum_{n=0}^{r-1} \sum_{e=0}^{d-1} H_{K[X_n]/J_{n,e}}(t) \cdot s^n \cdot t^e - \sum_{n=0}^{r-1} H_{K[X_n]/J_{n,d}}(t) \cdot s^n \cdot \frac{t^d}{1-t} \\ & + \sum_{n=0}^{r-1} H_{K[X_n]/I_n}(t) \cdot s^n. \end{aligned}$$

For every $1 \leq n \leq r-1$, $\dim(K[X_n]/I_n)$ and $\dim(K[X_n]/J_{n,e}) = \dim(K[X_n]/\langle 0 \rangle) = \dim(K[X_n])$ is at most $c(r-1)$, then we have the following equality:

$$
\begin{aligned}
& -\sum_{n=0}^{r-1}\sum_{e=0}^{d-1} H_{K[X_n]/J_{n,e}}(t) \cdot s^n \cdot t^e - \sum_{n=0}^{r-1} H_{K[X_n]/J_{n,d}}(t) \cdot s^n \cdot \frac{t^d}{1-t} + \sum_{n=0}^{r-1} H_{K[X_n]/I_n}(t) \cdot s^n \\
&= -\sum_{n=0}^{r-1}\sum_{e=0}^{d-1} \frac{s^n \cdot t^e}{(1-t)^{cn}} - \sum_{n=0}^{r-1} \frac{s^n}{(1-t)^{cn}} \cdot \frac{t^d}{1-t} + \sum_{n=0}^{r-1} H_{K[X_n]/I_n}(t) \cdot s^n \\
&= \frac{(1-t)\cdot g_1(s,t) - s^{r-1}t^d + (1-t)\cdot g_2(s,t)}{(1-t)^{c(r-1)+1}},
\end{aligned}
$$

for some $g_1(s,t), g_2(s,t) \in \mathbb{Z}[s,t]$. Putting $t = 1$, the numerator is equal to $-s^{r-1}$. Now our assertion follows. $\square$

Lemma 3.16. *Let $a, x \in K[X_n]$ be monomials and $I_n \subset K[X_n]$ a monomial ideal. Assume that $1 \leq i \leq c$, $1 \leq j \leq n$, $\gcd(x, x_{1j}x_{2j}\ldots x_{ij}) = 1$. The following equality holds*

$$\langle I_n : ax, x_{1,j}, \ldots, x_{i,j} \rangle = \langle I_n : a, x_{1,j}, \ldots, x_{i,j} \rangle : x.$$

Proof. Since I_n is a monomial ideal, both sides are monomial ideals. We only need to prove: for any monomial $f \in K[X_n]$, f belongs to the right hand side (RHS) if and only if it belongs to the left hand side (LHS).

If $f \in$ RHS, $fx \in \langle I_n\colon a, x_{1,j}, \ldots, x_{i,j} \rangle$. If $fx \in \langle x_{1,j}, \ldots, x_{i,j} \rangle$, since $\gcd(x, x_{1j}x_{2j}\ldots x_{ij}) = 1$, we have $f \in \langle x_{1,j}, \ldots, x_{i,j} \rangle$. If $fx \in I_n\colon a$, then $f \in I_n\colon ax$. Hence $f \in$ LHS.

If $f \in$ LHS, so $f \in \langle I_n\colon ax, x_{1,j}, \ldots, x_{i,j} \rangle$. If f is a multiple of some $x_{t,j}$, $1 \leq t \leq i$, then clearly $f \in$ RHS. If not, $f \in I_n\colon ax$, so $fx \in I_n\colon a$, namely $f \in$ RHS. This concludes the proof. $\square$

The following is [4, Lemma 6.10]. We correct the following typos:

1. Change the condition $r > \operatorname{ind}^i(\mathcal{I}) \geq i$ to $r \geq \operatorname{ind}^i(\mathcal{I}) > i$.

2. Correct the index of the first non-zero ideal of the colon-sum chain from r to $r+1$.

3. Change $\operatorname{ind}^{i+1}(\mathcal{I}_{\mathbf{e}}) = r$ to $\operatorname{ind}^{i+1}(\mathcal{I}_{\mathbf{e}}) = r+1$.

Lemma 3.17. *Assume $\mathcal{I} = (I_n)_{n\in\mathbb{N}}$ is an $\operatorname{Inc}(\mathbb{N})^i$-invariant chain of monomial ideals, where $i \geq 0$ is an integer. Select an integer r such that $r \geq \operatorname{ind}^i(\mathcal{I}) > i$. Let $G(I_r)$ be a minimal generating set for the monomial ideal I_r. Choose a positive integer d with* *the property that for all* $k \in [c]$, *the monomial* $x_{k,i+1}^{d+1}$ *does not divide any element of* $G(I_r)$. *Given these prerequisites, for an arbitrary sequence of non-negative integers* $\mathbf{e} = (e_1, \dots, e_c)$, *we define a sequence*

$$\mathcal{I}_{\mathbf{e}} = \langle \mathcal{I} : x_{1,i+1}^{e_1} \cdots x_{c,i+1}^{e_c}, x_{1,i+1}, \dots, x_{c,i+1} \rangle$$

with its n*-th ideal is*

$$\mathcal{I}_{\mathbf{e},n} = \begin{cases} \langle 0 \rangle & \text{if } 1 \le n \le r, \\ \langle I_n : x_{1,i+1}^{e_1} \cdots x_{c,i+1}^{e_c}, x_{1,i+1}, \dots, x_{c,i+1} \rangle & \text{if } n \ge r+1. \end{cases}$$

Then

(a) $\mathcal{I}_{\mathbf{e}}$ *is an* $\operatorname{Inc}(\mathbb{N})^{i+1}$*-invariant chain with* $\operatorname{ind}^{i+1}(\mathcal{I}_{\mathbf{e}}) = r+1$.

(b) *there exist* $g(s,t) \in \mathbb{Z}[s,t]$, $g(s,1) = -d^{c-1}s^{r-1}$ *satisfying*

$$H_{\mathcal{I}}(s,t) = \frac{1}{(1-t)^c} \cdot \sum_{\substack{\mathbf{e}=(e_1,\dots,e_c)\in\mathbb{Z}^c \\ 0 \le e_l \le d}} f_{\mathbf{e}}(t) \cdot H_{\mathcal{I}_{\mathbf{e}}}(s,t) + \frac{g(s,t)}{(1-t)^{rc}},$$

where

$$f_{\mathbf{e}}(t) = t^{|\mathbf{e}|}(1-t)^{\delta(\mathbf{e})} \quad \text{with} \quad |\mathbf{e}| = e_1 + \cdots + e_c, \quad \delta(\mathbf{e}) = \#\{l \mid e_l = d \text{ and } 1 \le l \le c\}.$$

Proof. (a) By Corollary 3.12, the sequence $\mathcal{I}$ is an $\operatorname{Inc}(\mathbb{N})^{i+1}$-invariant chain with index $\operatorname{ind}^{i+1}(\mathcal{I}) \le r+1$. Now by Corollary 3.10, the ideal $\mathcal{I}_{\mathbf{e}}$ is an $\operatorname{Inc}(\mathbb{N})^{i+1}$-invariant chain with $\operatorname{ind}^{i+1}(\mathcal{I}_{\mathbf{e}}) \le \operatorname{ind}^{i+1}(\mathcal{I}) \le r+1$. By definition, $J_n = \langle 0 \rangle$ for all $n < r+1$, thus $\operatorname{ind}^{i+1}(\mathcal{I}_{\mathbf{e}}) \ge r+1$, which implies (a).

(b) By assumption, the monomial $x_{k,i+1}^{d+1}$ does not divide any element of $G(I_r)$. Thus for all $k \in [c]$ and $n \ge r \ge \operatorname{ind}^{i+1}(\mathcal{I})$, $x_{k,i+1}^{d+1}$ does not divide any element of $G(I_n)$. Hence

$$\langle I_n : x_{k,i+1}^{d} \rangle = \langle I_n : x_{k,i+1}^{d+1} \rangle,$$

which implies

$$\begin{aligned} &\langle I_n : x_{1,i+1}^{e_1} \cdots x_{k-1,i+1}^{e_{k-1}} x_{k,i+1}^{d}, x_{1,i+1}, \dots, x_{k-1,i+1} \rangle \\ &\qquad = \langle I_n : x_{1,i+1}^{e_1} \cdots x_{k-1,i+1}^{e_{k-1}} x_{k,i+1}^{d+1}, x_{1,i+1}, \dots, x_{k-1,i+1} \rangle, \end{aligned} \tag{3.2}$$

where $0 \leq e_j \leq d$ and $1 \leq k \leq c$. Next, Lemma 3.16 gives

$$\begin{aligned}&\langle I_n : x_{1,i+1}^{e_1} \dots x_{k-1,i+1}^{e_{k-1}} x_{k,i+1}^{e_k}, x_{1,i+1}, \dots, x_{k-1,i+1}\rangle \\ &\qquad = \langle I_n : x_{1,i+1}^{e_1} \dots x_{k-1,i+1}^{e_{k-1}}, x_{1,i+1}, \dots, x_{k-1,i+1}\rangle : x_{k,i+1}^{e_k}. \end{aligned} \tag{3.3}$$

For each $1 \leq k \leq c$, consider chains

$$\langle \mathcal{I} : x_{1,i+1}^{e_1} \dots x_{k-1,i+1}^{e_{k-1}} x_{k,i+1}^{e_k}, x_{1,i+1}, \dots, x_{k-1,i+1}\rangle,$$

where the n-th ideal is $\langle I_n : x_{1,i+1}^{e_1} \dots x_{k-1,i+1}^{e_{k-1}} x_{k,i+1}^{e_k}, x_{1,i+1}, \dots, x_{k-1,i+1}\rangle$ if $n \geq r+1$ and $\langle 0 \rangle$ if $n \leq r$.

We proceed by induction on k, where $1 \leq k \leq c$. We will prove the statement $P(k)$: For any non-negative integers $e_1, \dots, e_k$,

$$\begin{aligned} H_{\mathcal{I}}(s,t) =& \frac{1}{(1-t)^k} \cdot \sum_{\substack{e=(e_1,\dots,e_k)\in\mathbb{Z}^k \\ 0 \leq e_l \leq d}} f_e(t) \cdot H_{\langle \mathcal{I}:x_{1,i+1}^{e_1} \dots x_{k-1,i+1}^{e_{k-1}} x_{k,i+1}^{e_k}, x_{1,i+1},\dots,x_{k-1,i+1}\rangle}(s,t) \\ &+ \frac{g_k(s,t)}{(1-t)^{(r-1)c+k}}, \end{aligned}$$

where $f_{\mathbf{e},k}(t) = t^{|\mathbf{e}|}(1-t)^{\delta_k(\mathbf{e})}$ with $|\mathbf{e}|_k = \sum_{j=1}^k e_j$, $\delta_k(\mathbf{e}) = \#\{j \in [k] \mid e_j = d\}$, and $g_k(s,t) \in \mathbb{Z}[s,t]$ with $g_k(s,1) = -d^{k-1}s^{r-1}$.

Base Case $k=1$: Consider the chain $\langle \mathcal{I} : x_{1,i+1}^{e_1}\rangle$. By assumption,

$$\langle I_n : x_{1,i+1}^{d}\rangle = \langle I_n : x_{1,i+1}^{d+1}\rangle$$

for $n \geq r$. Thus, Corollary 3.15 implies that $\exists g_1(s,t) \in \mathbb{Z}[s,t]$, $g_1(s,1) = -s^{r-1}$ such that

$$\begin{aligned} H_{\mathcal{I}}(s,t) &= \sum_{e=0}^{d-1} H_{\langle \mathcal{I}:x_{1,i+1}^{e}, x_{1,i+1}\rangle}(s,t) \cdot t^d + H_{\langle \mathcal{I}:x_{1,i+1}^{d}, x_{1,i+1}\rangle}(s,t) \cdot \frac{t^d}{1-t} + \frac{g_1(s,t)}{(1-t)^{(r-1)c+1}}. \\ &= \frac{1}{1-t} \cdot \left(\sum_{e=0}^{d} f_{e,1}(t) \cdot H_{\langle \mathcal{I}:x_{1,i+1}^{e}, x_{1,i+1}\rangle}(s,t) \right) + \frac{g_1(s,t)}{(1-t)^{(r-1)c+1}}, \end{aligned}$$

where $f_{e,1}(t) = t^e(1-t)$ for $0 \leq e \leq d-1$ and $f_{d,1}(t) = t^d$. This establishes the base case $P(1)$.

Inductive Step: Let $1 < k < c$. Note that from (3.2) we have

$$\begin{aligned}&\langle I_n : x_{1,i+1}^{e_1} \cdots x_{k-1,i+1}^{e_{k-1}} x_{k,i+1}^{d}, x_{1,i+1}, \dots, x_{k-1,i+1}\rangle \\ &\qquad = \langle I_n : x_{1,i+1}^{e_1} \cdots x_{k-1,i+1}^{e_{k-1}} x_{k,i+1}^{d+1}, x_{1,i+1}, \dots, x_{k-1,i+1}\rangle. \end{aligned}$$

Applying Corollary 3.15 to the chain $\langle \mathcal{I} \colon x_{1,i+1}^{e_1}\cdots x_{k-1,i+1}^{e_{k-1}}\rangle$ deduce that there is $\tilde{g}_{e,k-1}(s,t) \in \mathbb{Z}[s,t]$ with $\tilde{g}_{e,k-1}(s,1) = -s^{r-1}$ such that

$$\begin{aligned}
&H_{\langle \mathcal{I}: x_{1,i+1}^{e_1}\cdots x_{k-1,i+1}^{e_{k-1}}, x_{1,i+1},\ldots,x_{k-1,i+1}\rangle}(s,t) \\
&= \sum_{e_k=0}^{d-1} H_{\langle \mathcal{I}: x_{1,i+1}^{e_1}\cdots x_{k,i+1}^{e_k}, x_{1,i+1},\ldots,x_{k,i+1}\rangle}(s,t)\cdot t^{e_k} \\
&\quad + H_{\langle \mathcal{I}: x_{1,i+1}^{e_1}\cdots x_{k,i+1}^{d}, x_{1,i+1},\ldots,x_{k,i+1}\rangle}(s,t)\cdot \frac{t^d}{1-t} + \frac{\tilde{g}_{e,k-1}(s,t)}{(1-t)^{(r-1)c+1}} \\
&= \frac{1}{(1-t)}\cdot \sum_{e_k=0}^{d} \tilde{f}_{e_k}(t)\cdot H_{\langle \mathcal{I}: x_{1,i+1}^{e_1}\cdots x_{k,i+1}^{e_k}, x_{1,i+1},\ldots,x_{k,i+1}\rangle}(s,t) + \frac{\tilde{g}_{e,k-1}(s,t)}{(1-t)^{(r-1)c+1}},
\end{aligned} \tag{3.4}$$

where $\tilde{f}_{e_k}(t) = t^{e_k}(1-t)$ for $0 \le e_k \le d-1$ and $\tilde{f}_d(t) = t^d$.

The induction hypothesis $P(k-1)$ yields

$$\begin{aligned}
H_{\mathcal{I}}(s,t) = \frac{1}{(1-t)^{k-1}}\cdot &\sum_{\substack{e=(e_1,\ldots,e_{k-1})\in\mathbb{Z}_{\ge 0}^{k-1}\\ 0\le e_i\le d}} f_{e,k-1}(t)\cdot H_{\langle \mathcal{I}: x_{1,i+1}^{e_1}\cdots x_{k-1,i+1}^{e_{k-1}}, x_{1,i+1},\ldots,x_{k-1,i+1}\rangle}(s,t) \\
&+ \frac{g_{k-1}(s,t)}{(1-t)^{(r-1)c+k-1}},
\end{aligned}$$

where

$$f_{e,k-1}(t) = t^{|e|_{k-1}}(1-t)^{\delta_{k-1}(e)}$$

and $g_{k-1}(s,t) \in \mathbb{Z}[s,t]$ with $g_{k-1}(s,1) = -d^{k-2}s^{r-1}$.

Replacing (3.4) into the last equation gives

$$\begin{aligned}
H_{\mathcal{I}}(s,t) = \frac{1}{(1-t)^{k}}\cdot &\sum_{\substack{e=(e_1,\ldots,e_{k-1})\in\mathbb{Z}^{k-1}\\ 0\le e_i\le d}} \left(\sum_{e_k=0}^{d} f_{e,k-1}(t)\tilde{f}_{e_k}(t)\cdot H_{\langle \mathcal{I}: x_{1,i+1}^{e_1}\cdots x_{k,i+1}^{e_k}, x_{1,i+1},\ldots,x_{k,i+1}\rangle}(s,t)\right) \\
&+ \frac{1}{(1-t)^{(r-1)c+k}}\cdot \sum_{\substack{e=(e_1,\ldots,e_{k-1})\in\mathbb{Z}^{k-1}\\ 0\le e_i\le d}} f_{e,k-1}(t)\tilde{g}_{e,k-1}(s,t) + \frac{g_{k-1}(s,t)}{(1-t)^{(r-1)c+k-1}} \\
= \frac{1}{(1-t)^{k}}\cdot &\sum_{\substack{e=(e_1,\ldots,e_{k})\in\mathbb{Z}^{k}\\ 0\le e_i\le d}} f_{e,k}(t)\cdot H_{\langle \mathcal{I}: x_{1,i+1}^{e_1}\cdots x_{k,i+1}^{e_k}, x_{1,i+1},\ldots,x_{k,i+1}\rangle}(s,t) \\
&+ \frac{g_k(s,t)}{(1-t)^{(r-1)c+k}},
\end{aligned}$$

where

$$f_{(e_1,\ldots,e_k),k}(t) = f_{(e_1,\ldots,e_{k-1}),k-1}(t)\cdot \tilde{f}_{e_k}(t) = t^{|e|_k}(1-t)^{\delta_k(e)}$$

and

$$g_k(s,t) = \sum_{\substack{e=(e_1,\dots,e_{k-1})\in\mathbb{Z}_{\geq 0}^{k-1}\\ 0\leq e_i\leq d}} f_{e,k-1}(t)\tilde{g}_{e,k-1}(s,t) + g_{k-1}(s,t)\cdot(1-t) \in \mathbb{Z}[s,t].$$

Finally, we must check $g_k(s,t)$ at $t=1$:

$$\begin{aligned} g_k(s,1) &= \sum_{\substack{e=(e_1,\dots,e_{k-1})\in\mathbb{Z}_{\geq 0}^{k-1}\\ 0\leq e_i\leq d}} f_{e,k-1}(1)\tilde{g}_{e,k-1}(s,1) \\ &= \sum_{\substack{e=(e_1,\dots,e_{k-1})\in\mathbb{Z}_{\geq 0}^{k-1}\\ 0\leq e_i\leq d,\, \delta_{k-1}(e)=0}} (-s^{r-1}) \\ &= -d^{k-1}s^{r-1}. \end{aligned}$$

We complete the proof. □

Lemma 3.18. *Assume $\mathcal{I}=(I_n)_{n\in\mathbb{N}}$ is an $\mathrm{Inc}(\mathbb{N})^i$-invariant chain of monomial ideals, where $i\geq 0$ is an integer. Assume further that $\mathrm{ind}^i(\mathcal{I})\geq i+1$. Let $r\geq \mathrm{ind}^i(\mathcal{I})$ be an integer, $\sigma_i\in\mathrm{Inc}(\mathbb{N})^i$ and set*

$$J_{r+1} = (\sigma_i(I_r), x_{1,i+1},\dots,x_{c,i+1}).$$

Let $\mathcal{J}=(J_n)_{n\in\mathbb{N}}$ be a chain satisfying

$$J_n = \begin{cases} \langle 0\rangle & \text{if } 1\leq n\leq r,\\ \langle \mathrm{Inc}(\mathbb{N})^{i+1}_{r+1,n}(J_{r+1})\rangle & \text{if } n\geq r+1. \end{cases}$$

Then $\mathcal{J}$ is an $\mathrm{Inc}(\mathbb{N})^{i+1}$-invariant chain, and there is $g(s,t)\in\mathbb{Z}[s,t]$, $g(s,1)=s^r$ such that

$$H_{\mathcal{J}}(s,t) = s\cdot H_{\mathcal{I}}(s,t) + \frac{g(s,t)}{(1-t)^{rc}}.$$

Proof. By Lemma 2.47, $\mathcal{J}$ is an $\mathrm{Inc}(\mathbb{N})^{i+1}$-invariant chain with $\mathrm{ind}^{i+1}(\mathcal{J})=r+1$.

We first prove that for any $n\geq r$, we have

$$J_{n+1} = \langle \sigma_i(I_n), x_{1,i+1},\dots,x_{c,i+1}\rangle.$$

Induction on n. If $n=r$, the statement $J_{n+1} = \langle \sigma_i(I_n), x_{1,i+1},\dots,x_{c,i+1}\rangle$ becomes

$$J_{r+1} = \langle \sigma_i(I_r), x_{1,i+1},\dots,x_{c,i+1}\rangle,$$

which is exactly the definition of J_{r+1} given in the lemma.

Let $n > r$. We have

$$\begin{aligned}\langle \sigma_i(I_n), x_{1,i+1}, \ldots, x_{c,i+1}\rangle &= \langle \sigma_i(\mathrm{Inc}(\mathbb{N})^i_{r,n}(I_r)), x_{1,i+1}, \ldots, x_{c,i+1}\rangle \\ &\subseteq \langle \mathrm{Inc}(\mathbb{N})^{i+1}_{r+1,n+1}(\sigma_i(I_r)), x_{1,i+1}, \ldots, x_{c,i+1}\rangle \\ &\qquad\qquad\qquad\qquad\text{(by Proposition 2.16)} \\ &= \langle \mathrm{Inc}(\mathbb{N})^{i+1}_{r+1,n+1}(J_{r+1})\rangle = J_{n+1},\end{aligned}$$

For the reverse inclusion. Since $n \geq r+1 = \mathrm{ind}^{i+1}(\mathcal{I})$, the induction hypothesis yields

$$\begin{aligned}J_{n+1} &= \langle \mathrm{Inc}(\mathbb{N})^{i+1}_{n,n+1}(J_n)\rangle \\ &= \langle \mathrm{Inc}(\mathbb{N})^{i+1}_{n,n+1}(\langle \sigma_i(I_{n-1}), x_{1,i+1}, \ldots, x_{c,i+1}\rangle)\rangle \\ &= \langle \mathrm{Inc}(\mathbb{N})^{i+1}_{n,n+1}(\sigma_i(I_{n-1})), x_{1,i+1}, \ldots, x_{c,i+1}\rangle.\end{aligned}$$

We note that $\mathrm{Inc}(\mathbb{N})^{i+1}_{n,n+1}(\sigma_i(I_{n-1})) = \{\sigma_{i+1}, \ldots, \sigma_{n+1}\}(\sigma_i(I_{n-1}))$ since for every $f \in K[X_n]$ and every $\pi \in \mathrm{Inc}(\mathbb{N})_{n,n+1}$, there exists $\sigma_j \in \mathrm{Inc}(\mathbb{N})^j$ with $i+1 \leq j \leq n+1$ satisfying $\pi(f) = \sigma_j(f)$. By Corollary 2.17, we imply that

$$\begin{aligned}J_{n+1} &\subseteq \langle \sigma_i(\mathrm{Inc}(\mathbb{N})_{n-1,n}(I_{n-1})), x_{1,i+1}, \ldots, x_{c,i+1}\rangle \\ &\subseteq \langle \sigma_i(I_n), x_{1,i+1}, \ldots, x_{c,i+1}\rangle.\end{aligned}$$

Hence $J_{n+1} = (\sigma_i(I_n), x_{1,i+1}, \ldots, x_{c,i+1})$, which completes the induction.

Observe that $\sigma_i(I_n)$ is obtained from I_n by replacing $x_{k,l}$ by $x_{k,l+1}$ for $1 \leq k \leq c$ and $i+1 \leq l$. In particular, no element in $G(\sigma_i(I_n))$ is divisible by any $x_{k,i+1}$ for $1 \leq k \leq c$. Thus, for $n \geq r$, the folowing map

$$K[X_n]/I_n \longrightarrow K[X_{n+1}]/J_{n+1}, \quad \overline{x_{k,l}} \mapsto \begin{cases} \overline{x_{k,l}} & \text{if } l \leq i, \\ \overline{x_{k,l+1}} & \text{if } i+1 \leq l \leq n, \end{cases}$$

is a graded K-algebra isomorphism, implying $H_{K[X_n]/I_n}(t) = H_{K[X_{n+1}]/J_{n+1}}(t)$ for $n \geq r$.

Thus,

$$\begin{aligned}
H_{\mathcal{J}}(s,t) &= \sum_{n=0}^{r} H_{K[X_n]/J_n}(t)s^n + \sum_{n=r+1}^{\infty} H_{K[X_n]/J_n}(t)s^n \\
&= \sum_{n=0}^{r} H_{K[X_n]}(t)s^n + \sum_{n=r}^{\infty} H_{K[X_n]/I_n}(t)s^{n+1} \qquad \text{(by the above isomorphism)} \\
&= \sum_{n=0}^{r} H_{K[X_n]}(t)s^n + s\sum_{n=r}^{\infty} H_{K[X_n]/I_n}(t)s^n \\
&= \sum_{n=0}^{r} H_{K[X_n]}(t)s^n + s\left(H_{\mathcal{I}}(s,t) - \sum_{n=0}^{r-1} H_{K[X_n]/I_n}(t)s^n\right) \\
&= s\cdot H_{\mathcal{I}}(s,t) + \sum_{n=0}^{r} H_{K[X_n]}(t)s^n - s\sum_{n=0}^{r-1} H_{K[X_n]/I_n}(t)s^n \\
&= s\cdot H_{\mathcal{I}}(s,t) + \sum_{n=0}^{r} \frac{s^n}{(1-t)^{cn}} - \sum_{n=0}^{r-1} \frac{s^{n+1}g_n(t)}{(1-t)^{d_n}},
\end{aligned}$$

where $H_{K[X_n]/I_n}(t) = \dfrac{g_n(t)}{(1-t)^{d_n}}$, $\dim K[X_n]/I_n = d_n \leq (r-1)c$. Letting the common denominator of the last two sums be $(1-t)^{rc}$, we have

$$H_{\mathcal{J}}(s,t) = s\cdot H_{\mathcal{I}}(s,t) + \frac{g(s,t)}{(1-t)^{rc}},$$

where

$$g(s,t) = \sum_{n=0}^{r}(1-t)^{c(r-n)}s^n - \sum_{n=0}^{r-1} s^{n+1}g_n(t)\cdot(1-t)^{cr-d_n} \in \mathbb{Z}[s,t].$$

and $g(s,1) = s^r$. □

We have established all the necessary preliminaries to prove Theorem 3.6 when $\mathcal{I}$ is a chain of monomial ideals.

Theorem 3.19. [4, Theorem 6.2] *Assume $\mathcal{I} = (I_n)_{n\in\mathbb{N}}$ is an $\operatorname{Inc}(\mathbb{N})^i$-invariant chain of monomial ideals, where $i \geq 0$ is an integer. Let $r = \operatorname{ind}^i(\mathcal{I})$ be the i-index of $\mathcal{I}$ and $q = q(\mathcal{I}) = \sum_{j=0}^{e^+(I_r)} \dim_K(K[X_r]/I_r)_j$ be the q-invariant of $\mathcal{I}$. Then*

$$H_{\mathcal{I}}(s,t) = \frac{g(s,t)}{(1-t)^a\cdot\prod_{j=1}^{b}[(1-t)^{c_j} - s\cdot f_j(t)]},$$

where $a,b,c_j \geq 0$ are integers, $g(s,t)\in\mathbb{Z}[s,t]$, and each $f_j(t)\in\mathbb{Z}[t]$ such that $f_j(1) > 0$.

Proof. The proof uses double induction. Firstly, we show by the outer induction on $p \geq 0$ that, for an $\operatorname{Inc}(\mathbb{N})^i$-invariant chain $\mathcal{I} = (I_n)_{n\in\mathbb{N}}$, such that $\operatorname{ind}^i(\mathcal{I}) - i \leq p$, $H_{\mathcal{I}}(s,t)$ is of the desired form. Then for each p, the inner induction on the q-invariant of $\mathcal{I}$ implies that $H_{\mathcal{I}}(s,t)$ is rational, as we need.

Outer Induction on p:

1. **Base Case $p = 0$:** $\operatorname{ind}^i(\mathcal{I}) \leq i$. This is Lemma 3.8.

2. **Inductive Step:** Let $p \geq 1$. Now we use a second induction on $q \geq 0$ to show: if an $\operatorname{Inc}(\mathbb{N})^i$-invariant chain $\mathcal{I} = (I_n)_{n\in\mathbb{N}}$ satisfies $r - i = \operatorname{ind}^i(\mathcal{I}) - i \leq p$ and $q(\mathcal{I}) \leq q$, then $H_{\mathcal{I}}(s,t)$ has the necessary form as in the theorem. By the first induction, we assume that $\operatorname{ind}^i(\mathcal{I}) - i = p$.

   Inner Induction on q:

   - **Base case $q = 0$:** $\dim_K [K[X_r]/I_r]_j = 0$ for all $0 \leq j \leq e^+(I_r)$. Thus $I_r = K_r$, then $K[X_n]/I_n = 0$ for all $n \geq r$. Hence

   $$\begin{aligned} H_{\mathcal{I}}(s,t) &= \sum_{n=0}^{r-1} H_{K[X_n]/I_n}(t) \cdot s^n \\ &= \sum_{n=0}^{r-1} \frac{g_n(t)}{(1-t)^{d_n}} s^n \\ &= \frac{\sum_{n=0}^{r-1} g_n(t)(1-t)^{a-d_n} \cdot s^n}{(1-t)^a}, \end{aligned}$$

     where $d_n = \dim K[X_n]/I_n$, $g_n(1) > 0$ and $a = \max\{d_n \mid 1 \leq n \leq r-1\}$. Thus in this case, $H_{\mathcal{I}}(s,t)$ has the desired form as in the theorem.

   - **Inductive step:** Let $q \geq 1$, assume that $q(\mathcal{I}) = q$. Thanks to Corollary 3.12, the sequence $\mathcal{I}$ is an $\operatorname{Inc}(\mathbb{N})^{i+1}$-invariant chain with $\operatorname{ind}^{i+1}(\mathcal{I}) \leq r+1$. If $\operatorname{ind}^{i+1}(\mathcal{I}) \leq r$, then $H_{\mathcal{I}}(s,t)$ has the given form by the outer induction hypothesis.

     Assume $\operatorname{ind}^{i+1}(\mathcal{I}) = r+1$. For each non-negative c-tuples $\mathbf{e} = (e_1, \ldots, e_c) \in \mathbb{N}_0^c$, consider a chain

   $$\mathcal{I}_{\mathbf{e}} = (\mathcal{I} : x_{1,i+1}^{e_1} \cdots x_{c,i+1}^{e_c}, x_{1,i+1}, \ldots, x_{c,i+1}) = (I_{\mathbf{e},n})_{n\in\mathbb{N}}$$

     where

   $$I_{\mathbf{e},n} = \begin{cases} \langle 0 \rangle & \text{if } n \leq r, \\ \langle I_n : x_{1,i+1}^{e_1} \cdots x_{c,i+1}^{e_c}, x_{1,i+1}, \ldots, x_{c,i+1} \rangle & \text{if } n \geq r+1. \end{cases}$$

By Lemma 3.17, $\mathcal{I}_{\mathbf{e}}$ is an $\operatorname{Inc}(\mathbb{N})^{i+1}$-invariant chain with $\operatorname{ind}^{i+1}(\mathcal{I}_{\mathbf{e}}) = r+1$. Suppose that $0 \leq e_k \leq d$, where

$$d = \max\{e \mid x_{k,i+1}^{e} \text{ divides some element of } G(I_r) \text{ for some } k \in [c]\}.$$

Because

$$\sigma_i(I_r) \subset I_{r+1} \subset I_{r+1} : x_{1,i+1}^{e_1} \cdots x_{c,i+1}^{e_c}$$

and, note that $r+1 = p+i+1 > i+1$, we have

$$K[X_{r+1}]/\langle \sigma_i(I_r), x_{1,i+1}, \ldots, x_{c,i+1}\rangle \cong K[X_r]/I_r.$$

Then for each $j \geq 0$, we have the dimension inequality

$$\dim_K[K[X_{r+1}]/I_{\mathbf{e},r+1}]_j \leq \dim_K[K[X_r]/I_r]_j.$$

Since $r = \operatorname{ind}^i(\mathcal{I})$, we have $e^+(I_{r+1}) = e^+(I_r)$, thus $e^* = e^+(I_{\mathbf{e},r+1}) \leq e^+(I_r)$. Hence, we get

$$q(\mathcal{I}_{\mathbf{e}}) = \sum_{j=0}^{e^*} \dim_K[K[X_{r+1}]/I_{\mathbf{e},r+1}]_j \leq \sum_{j=0}^{e^+(I_r)} \dim_K[K[X_r]/I_r]_j = q. \tag{3.5}$$

If $q(\mathcal{I}_{\mathbf{e}}) < q$, then by the inner induction hypothesis, we implies that $\mathcal{I}_{\mathbf{e}}$ is an $\operatorname{Inc}(\mathbb{N})^{i+1}$-invariant chain which has a Hilbert series of rational form, as desired. If (3.5) becomes equality, it implies

$$I_{\mathbf{e},r+1} = \langle \sigma_i(I_r), x_{1,i+1}, \ldots, x_{c,i}\rangle.$$

Applying Lemma 3.18 to the ideal in the right hand side, we have

$$H_{\mathcal{I}_{\mathbf{e}}}(s,t) = s \cdot H_{\mathcal{I}}(s,t) + \frac{g_{\mathbf{e}}(s,t)}{(1-t)^{rc}}, \tag{3.6}$$

where $g_{\mathbf{e}}(s,t) \in \mathbb{Z}[s,t]$, $g_{\mathbf{e}}(s,1) = s^r$.

Now we use Lemma 3.17. Using equation (3.6) for all chains with the q-invariant equal to q, it gives

$$\begin{aligned} H_{\mathcal{I}}(s,t) = \frac{h(s,t)}{(1-t)^{(r+1)c}} &+ \frac{1}{(1-t)^c} \cdot \sum_{\substack{\mathbf{e}=(e_1,\ldots,e_c)\in\mathbb{Z}^c \\ 0\leq e_l\leq d \\ q(\mathcal{I}_{\mathbf{e}})<q}} f_{\mathbf{e}}(t) \cdot H_{\mathcal{I}_{\mathbf{e}}}(s,t) \\ &+ \frac{1}{(1-t)^c} \cdot \sum_{\substack{\mathbf{e}=(e_1,\ldots,e_c)\in\mathbb{Z}^c \\ 0\leq e_l\leq d \\ q(\mathcal{I}_{\mathbf{e}})=q}} f_{\mathbf{e}}(t) \cdot \left[s \cdot H_{\mathcal{I}}(s,t) + \frac{g_{\mathbf{e}}(s,t)}{(1-t)^{rc}} \right], \end{aligned} \tag{3.7}$$

where $h(s,1) = -d^{c-1}s^r$ and $f_{\mathbf{e}}(t) = t^{|\mathbf{e}|}(1-t)^{\delta(\mathbf{e})}$. Bringing the term $H_{\mathcal{I}}(s,t)$ in one side gives

$$H_{\mathcal{I}}(s,t)\cdot\left[1-\frac{s}{(1-t)^c}\tilde{f}(t)\right] = \frac{1}{(1-t)^c}\cdot\sum_{\substack{\mathbf{e}=(e_1,\dots,e_c)\in\mathbb{Z}^c\\ 0\le e_l\le d\\ q(\mathcal{I}_{\mathbf{e}})<q}} f_{\mathbf{e}}(t)\cdot H_{\mathcal{I}_{\mathbf{e}}}(s,t) + \frac{\tilde{g}(s,t)}{(1-t)^{(r+1)c}}, \tag{3.8}$$

where

$$\tilde{f}(t) = \sum_{\substack{\mathbf{e}=(e_1,\dots,e_c)\in\mathbb{Z}^c\\ 0\le e_l\le d\\ q(\mathcal{I}_{\mathbf{e}})=q}} t^{|\mathbf{e}|}(1-t)^{\delta(\mathbf{e})} = (1-t)^{\tilde{c}}f(t)$$

with $0 \le \tilde{c} \le c$, $f(t) \in \mathbb{Z}[t]$ with $f(1) \ge 0$.

By inductive hypothesis, all the equivariant Hilbert series $H_{\mathcal{I}_{\mathbf{e}}}(s,t)$ in (3.8) with $q(\mathcal{I}_{\mathbf{e}}) < q$ is rational. Thus we must have

$$H_{\mathcal{I}}(s,t) = \frac{1}{(1-t)^c - s\cdot\tilde{f}(t)}\cdot\left(\frac{\tilde{g}(s,t)}{(1-t)^{rc}} + \sum_{\substack{\mathbf{e}=(e_1,\dots,e_c)\in\mathbb{Z}^c\\ 0\le e_l\le d\\ q(\mathcal{I}_{\mathbf{e}})<q}} f_{\mathbf{e}}(t)\cdot H_{\mathcal{I}_{\mathbf{e}}}(s,t)\right),$$

which is the desired form of $H_{\mathcal{I}}(s,t)$.

Thus, we proved the inductive step of the outer induction, which ends the proof. $\square$

Next, we use the lexicographic ordering as in Theorem 2.31, that is $x_{pj} \preceq x_{p'j'}$ if $p < p'$ or $p = p'$ and $j < j'$. Since $\mathrm{Inc}(\mathbb{N})$ preserves this monomial ordering, we must have $\mathrm{LT}(\pi(g)) = \pi(\mathrm{LT}(g))$, for all $\pi \in \mathrm{Inc}^i(\mathbb{N})$ and for all $g \in K[X]$. The following results takes ideals of leading terms with respect to $\preceq$.

Proposition 3.20. *Let $I_r \subseteq K[X_r]$ be an ideal. For any integers $i \ge 0$ and $r \le n$, the following inclusion holds:*

$$\langle \mathrm{Inc}(\mathbb{N})^i_{r,n}(\mathrm{LT}(I_r))\rangle \subseteq \langle \mathrm{LT}(\mathrm{Inc}(\mathbb{N})^i_{r,n}(I_r))\rangle.$$

Proof. Consider a generator $v \in \langle \mathrm{Inc}(\mathbb{N})^i_{r,n}(\mathrm{LT}(I_r))\rangle$. Such a generator is of the form $v = \pi(u)$, where $u \in \mathrm{LT}(I_r)$ is a monomial, $\pi \in \mathrm{Inc}(\mathbb{N})^i_{r,n}$. Since $u \in \mathrm{LT}(I_r)$, there exists $g \in I_r$ such that $\mathrm{LT}(g) = u$. Now, consider the polynomial $f = \pi(g) \in \langle \mathrm{Inc}(\mathbb{N})^i_{r,n}(I_r)\rangle$, the leading monomial of f is

$$\mathrm{LT}(f) = \mathrm{LT}(\pi(g)) = \pi(\mathrm{LT}(g)) = \pi(u) = v.$$

The second equality is since $\mathrm{Inc}(\mathbb{N})^i$ respects the given monomial order. Now, since $f = \pi(g) \in \langle \mathrm{Inc}(\mathbb{N})^i_{r,n}(I_r)\rangle$, its leading monomial $\mathrm{LT}(f) = v$ must be in the ideal of leading terms of $\langle \mathrm{Inc}(\mathbb{N})^i_{r,n}(I_r)\rangle$, i.e., $v \in \langle \mathrm{LT}(\mathrm{Inc}(\mathbb{N})^i_{r,n}(I_r))\rangle$. Thus, the inclusion holds. □

Below is Theorem 3.6, which is the main result of this thesis.

Theorem 3.6. [4, Theorem 7.2] *Assume $\mathcal{I} = (I_n)_{n\in\mathbb{N}}$ is an $\mathrm{Inc}(\mathbb{N})^i$-invariant chain of homogeneous ideals, where $i \geq 0$ is an integer. Then*

$$H_{\mathcal{I}}(s,t) = \frac{g(s,t)}{(1-t)^a \cdot \prod_{j=1}^{b}[(1-t)^{c_j} - s \cdot f_j(t)]},$$

where $a, b, c_j \geq 0$ are integers, $g(s,t) \in \mathbb{Z}[s,t]$, and each $f_j(t) \in \mathbb{Z}[t]$ such that $f_j(1) > 0$.

Proof. Consider the chain of ideals of leading terms $\mathrm{LT}(\mathcal{I}) = (\mathrm{LT}(I_n))_{n\in\mathbb{N}}$. By Lemma 2.42, $\mathrm{LT}(\mathcal{I})$ is also an $\mathrm{Inc}(\mathbb{N})^i$-invariant chain. Furthermore, $\mathrm{LT}(\mathcal{I})$ is an $\mathrm{Inc}(\mathbb{N})^i$-invariant chain of monomial ideals. Thus the equivariant Hilbert series $H_{\mathrm{LT}(\mathcal{I})}(s,t)$ is a rational function of the same form as in Theorem 3.19. Now, by Lemma 1.49

$$H_{\mathcal{I}}(s,t) = H_{\mathrm{LT}(\mathcal{I})}(s,t),$$

which finally implies the theorem. □

3.3 An Example

Before turning to the last example, we need a useful isomorphism.

Lemma 3.21. *For $c = 2$ and $I_n = \langle x_{1,1}^2, x_{2,1}, x_{1,2}^2, x_{2,2}, \ldots, x_{1,n}^2, x_{2,n}\rangle$. The following isomorphism holds:*

$$K[X_n]/I_n \cong \bigotimes_{j=1}^{n} K[x_{1,j}, x_{2,j}]/\langle x_{1,j}^2, x_{2,j}\rangle,$$

where tensor product is taken over K.

This follows from a simple lemma.

Lemma 3.22. *Let $A = K[x_1, \ldots, x_d]$ and $B = K[y_1, \ldots, y_e]$ be polynomial rings over K, $I \subseteq A$, $J \subseteq B$ be ideals. Let $R = A \otimes_K B \cong K[x_1, \ldots, x_d, y_1, \ldots, y_e]$. Then*

$$\frac{A}{I} \otimes_K \frac{B}{J} \cong \frac{R}{IR + JR}.$$

Proof. Tensor product over a field is exact, so $M \otimes_K \dfrac{U}{V} \cong \dfrac{M \otimes_K U}{M \otimes_K V}$ for K-modules M, U, V. Thus

$$\frac{A}{I} \otimes_K \frac{B}{J} \cong \frac{\frac{A}{I} \otimes_K B}{\frac{A}{I} \otimes_K J} \cong \frac{\frac{A \otimes_K B}{I \otimes_K B}}{\frac{A \otimes_K J}{I \otimes_K J}} \cong \frac{\frac{A \otimes_K B}{I \otimes_K B}}{\frac{A \otimes_K J}{(I \otimes_K B) \cap (A \otimes_K J)}}$$
$$\cong \frac{\frac{A \otimes_K B}{I \otimes_K B}}{\frac{A \otimes_K J + I \otimes_K B}{I \otimes_K B}} \cong \frac{A \otimes_K B}{I \otimes_K B + A \otimes_K J} \cong \frac{R}{IR + JR}$$

This completes the proof. □

Applying this lemma repeatedly, we get the conclusion of Lemma 3.21.

Proposition 3.23. *Let $R = \bigoplus_{n=0}^{\infty} R_n$ and $S = \bigoplus_{m=0}^{\infty} S_m$ be two graded K-algebras. Consider their tensor product $T = R \otimes_K S$, which is also a graded K-algebra with grading $(R \otimes_K S)_l = \bigoplus_{i+j=l} (R_i \otimes_K S_j)$. Then the Hilbert series of the tensor product is*

$$H_{R \otimes_K S}(t) = H_R(t) \cdot H_S(t)$$

Proof. Let $H_R(t) = \sum_{n=0}^{\infty} \dim_K(R_n) t^n$ and $H_S(t) = \sum_{m=0}^{\infty} \dim_K(S_m) t^m$ be the Hilbert series of R and S, respectively. Then

$$\begin{aligned} H_{R \otimes_K S}(t) &= \sum_{l=0}^{\infty} \dim_K((R \otimes_K S)_l) t^l \\ &= \sum_{l=0}^{\infty} \dim_K \left(\bigoplus_{i+j=l} (R_i \otimes_K S_j) \right) t^l \\ &= \sum_{l=0}^{\infty} \left(\sum_{i+j=l} \dim_K(R_i \otimes_K S_j) \right) t^l \\ &= \sum_{l=0}^{\infty} \left(\sum_{i+j=l} \dim_K(R_i) \cdot \dim_K(S_j) \right) t^l. \end{aligned}$$

Now consider the product

$$\begin{aligned} H_R(t) \cdot H_S(t) &= \left(\sum_{n=0}^{\infty} \dim_K(R_n) t^n \right) \cdot \left(\sum_{m=0}^{\infty} \dim_K(S_m) t^m \right) \\ &= \sum_{l=0}^{\infty} \left(\sum_{n+m=l} \dim_K(R_n) \cdot \dim_K(S_m) \right) t^l. \end{aligned}$$

Therefore, $H_{R \otimes_K S}(t) = H_R(t) \cdot H_S(t)$. □

We compute the equivariant Hilbert series of the chain defined in Example 3.5.

Example 3.24. Let $X_n = \{X_{i,j} \mid i \in [2], j \in [n]\}$ and define the chain $\mathcal{I} = (I_n)_{n\in\mathbb{N}}$ by

$$I_n = \begin{cases} \langle x_{1,1}^2, x_{2,1} \rangle & \text{if } n = 1, \\ \langle \mathrm{Inc}(\mathbb{N})_{1,n}(I_1) \rangle & \text{if } n > 1. \end{cases}$$

Thus, for $n \geq 1$, $I_n = \langle x_{1,j}^2, x_{2,j} \mid j \in [n] \rangle \subset K[X_n]$.

For $n = 0$, $H_{K[X_0]/I_0}(t) = H_K(t) = 1$. For $n = 1$, $H_{K[X_1]/I_1}(t) = 1 + t$. For $n \geq 1$, by Lemma 3.21 we have the isomorphism:

$$K[X_n]/I_n \cong \bigotimes_{j=1}^{n} (K[X_1]/I_1).$$

Therefore by Proposition 3.23, we have

$$H_{K[X_n]/I_n}(t) = \left(H_{K[X_1]/I_1}(t)\right)^n = (1+t)^n.$$

Finally, the equivariant Hilbert series of $\mathcal{I}$ is

$$\begin{aligned} H_{\mathcal{I}}(s,t) &= H_{K[X_0]/I_0}(t)s^0 + \sum_{n=1}^{\infty} H_{K[X_n]/I_n}(t)s^n \\ &= 1 \cdot s^0 + \sum_{n=1}^{\infty} (1+t)^n s^n \\ &= 1 + \frac{(1+t)s}{1-(1+t)s} \\ &= \frac{1}{1-s-st}. \end{aligned}$$

Conclusion

We have presented the following in this thesis:

1. In the preliminary part, we presented fundamental properties of graded rings and graded modules, especially the theory of Gröbner bases and monomial ideals, and the classical Hilbert-Serre theorem.

2. We introduced the concept of well-partial-orders and presented Kruskal's tree theorem and Higman's lemma. We then explored the monoid $\mathrm{Inc}(\mathbb{N})$ and its subsets $\mathrm{Inc}(\mathbb{N})^i$, demonstrating the existence of finite $\mathrm{Inc}(\mathbb{N})$-equivariant Gröbner bases and the Hilbert's basis theorem for infinite dimensional polynomial rings of the type $K\left[x_{i,j} | 1 \leq i \leq c, j \geq 1\right]$ with the action of $\mathrm{Inc}(\mathbb{N})$ on the variables given by $\pi(x_{i,j}) = x_{i,\pi(j)}$. We also discussed chains of $\mathrm{Inc}(\mathbb{N})^i$-invariant ideals.

3. Finally, we established the Hilbert-Serre theorem for infinite dimensional polynomial rings, proving the rationality of the equivariant Hilbert series for $\mathrm{Inc}(\mathbb{N})^i$-invariant chains of monomial ideals. We introduced the q-invariant as a measure of complexity and employed induction on this invariant to demonstrate the rationality. We also provided a detailed example to illustrate the computation of the equivariant Hilbert series.

In summary, this thesis provides a significant step towards extending classical results in commutative algebra concerning Noetherian ring to the infinite dimensional setting, specifically for ideals invariant under the action of the monoid $\mathrm{Inc}(\mathbb{N})$. The use of equivariant Gröbner bases and the q-invariant offers a new framework for studying these infinite dimensional polynomial rings.

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